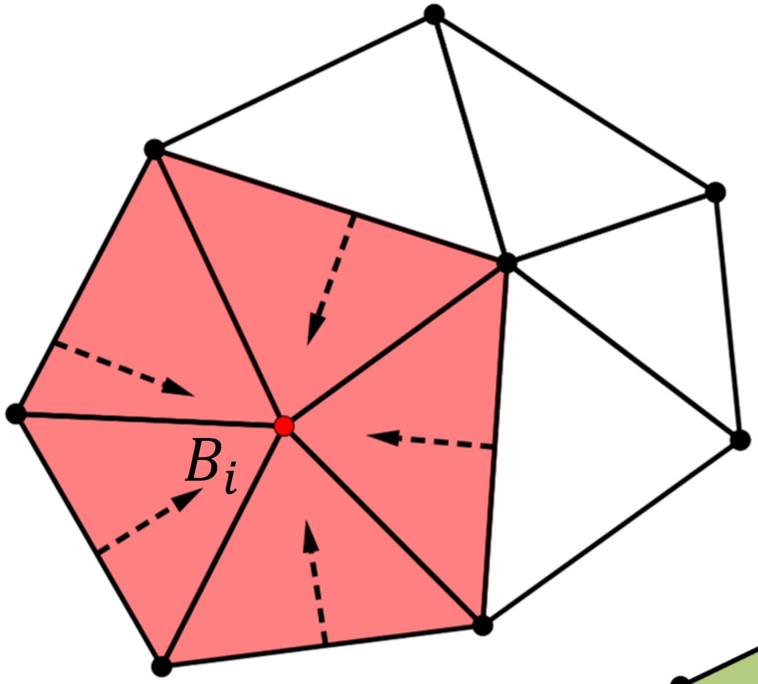
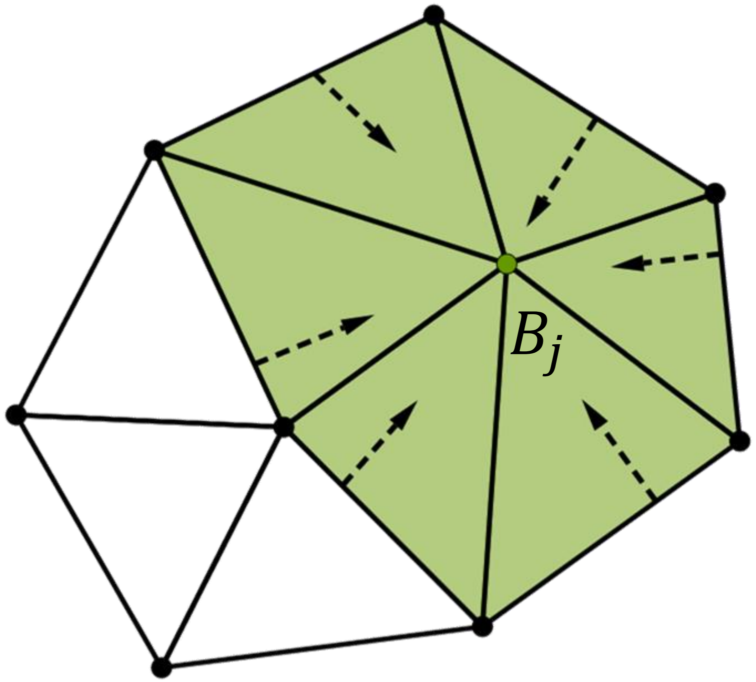
Lecture 8

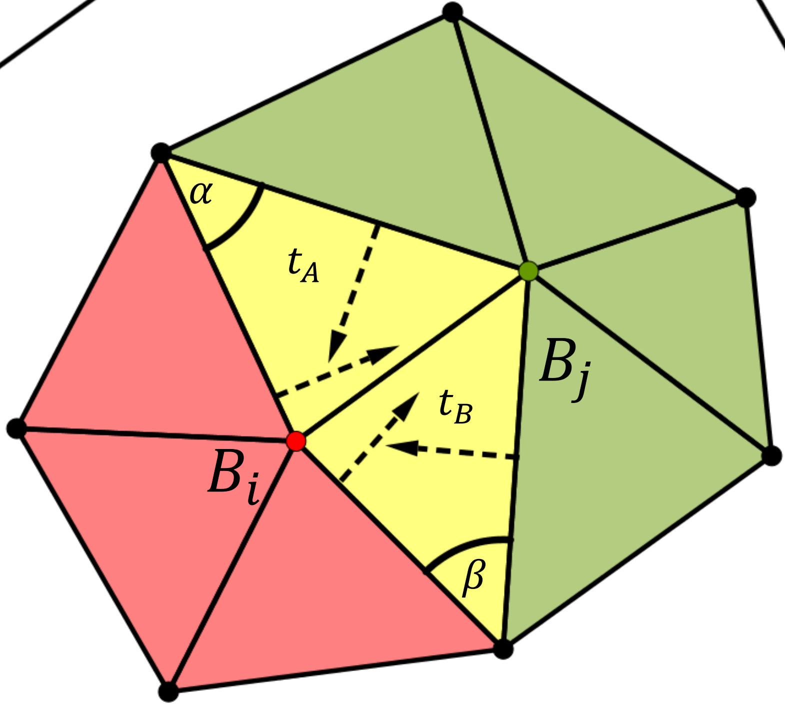
Introduction to Geometry Processing

Spring 2017

Johns Hopkins University

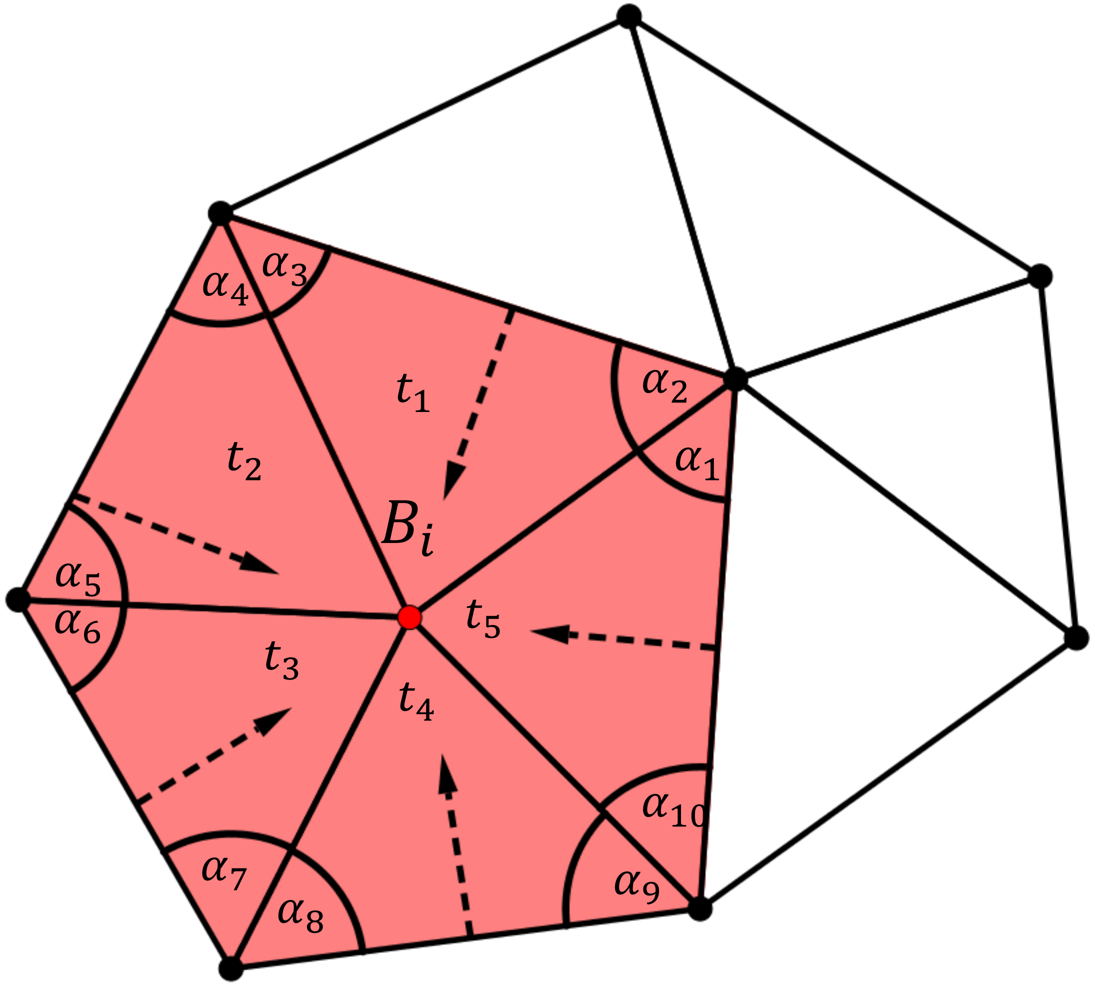






$$\int_T B_i B_j = \frac{|t_A| + |t_B|}{12}$$

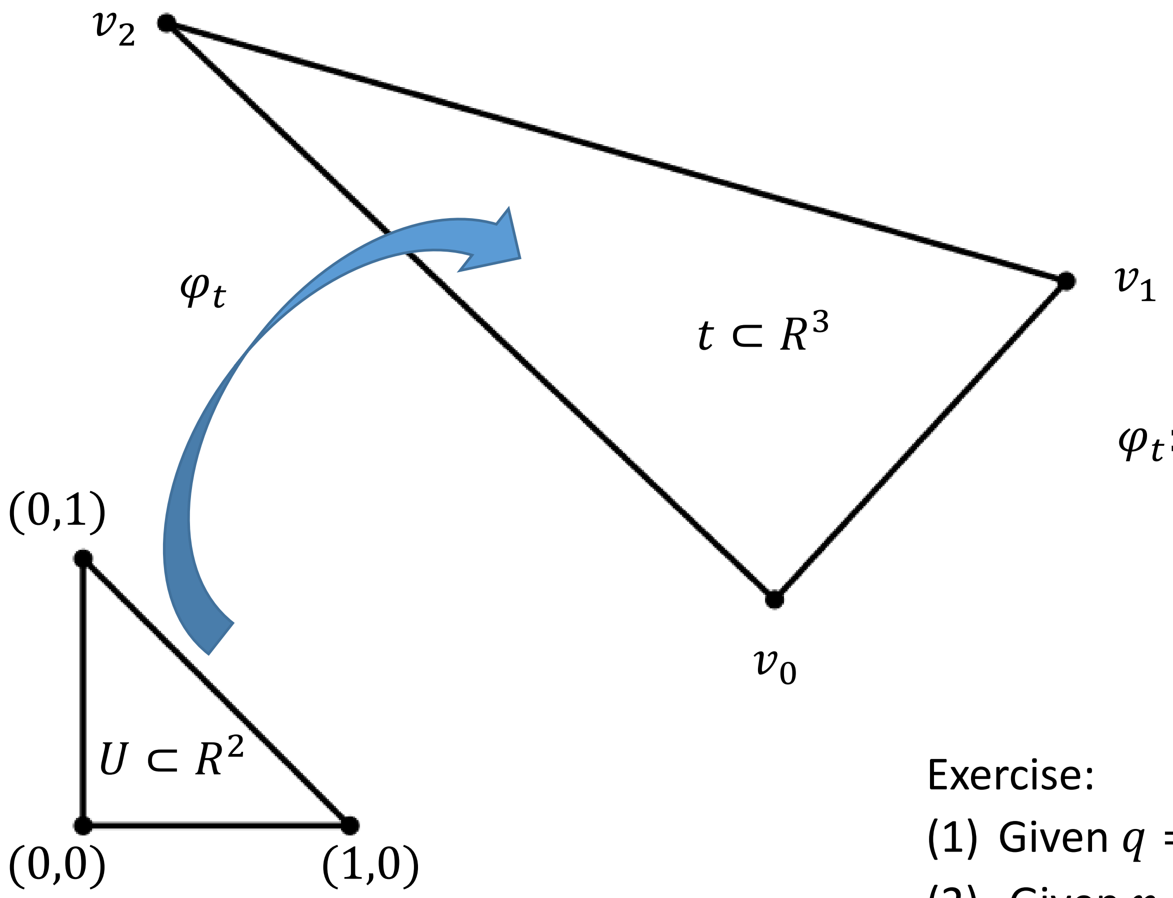
$$\int_T \langle \nabla B_i, \nabla B_j \rangle = -\frac{\cot(\alpha) + \cot(\beta)}{2}$$



$$\int_T B_i^2 = \sum_{t_k \in N(i)} \frac{|t_k|}{6}$$

$$\int_T \langle \nabla B_i, \nabla B_i \rangle = \sum_{\alpha_k \in \partial N(i)} \frac{\cot \alpha_k}{2}$$

Right triangle parametrization



$\varphi_t : U \subset R^2 \to t \subset R^3$

$$\varphi_t(0,0) = v_0$$

$$\varphi_t(1,0) = v_1$$

$$\varphi_t(0,1) = v_2$$

$d\varphi_t : TU \subset R^2 \to Tt \subset R^3$

$$d\varphi_t \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v_1 - v_0$$

$$d\varphi_t \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v_2 - v_0$$

Exercise:

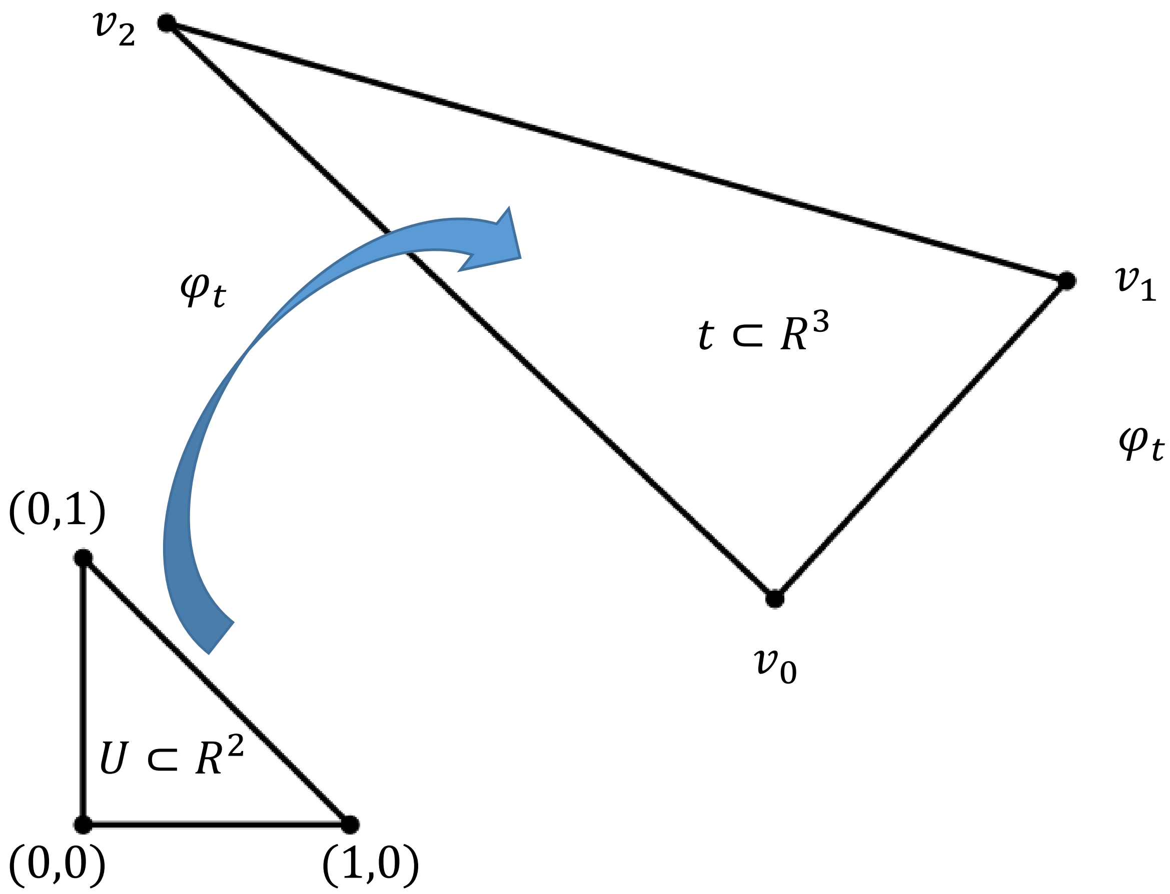
(1) Given $q = (q_x, q_y) \in U$ compute $\varphi_t(q)$.

(2) Given $p = (p_x, p_y, p_z) \in t$ compute $\varphi_t^{-1}(p)$.

(3) Given $v = (v_x, v_y) \in TU$ compute $d\varphi_t(v)$.

(4) Given $w = (w_x, w_y, w_z) \in Tt$ compute $d\varphi_t^{-1}(w)$.

Right triangle parametrization



$$\varphi_t\colon U \subset R^2 \to t \subset R^3$$

$$\varphi_t(0,0) = v_0$$

$$\varphi_t(1,0) = v_1$$

$$\varphi_t(0,1) = v_2$$

$$d\varphi_t\colon TU \subset R^2 \to Tt \subset R^3$$

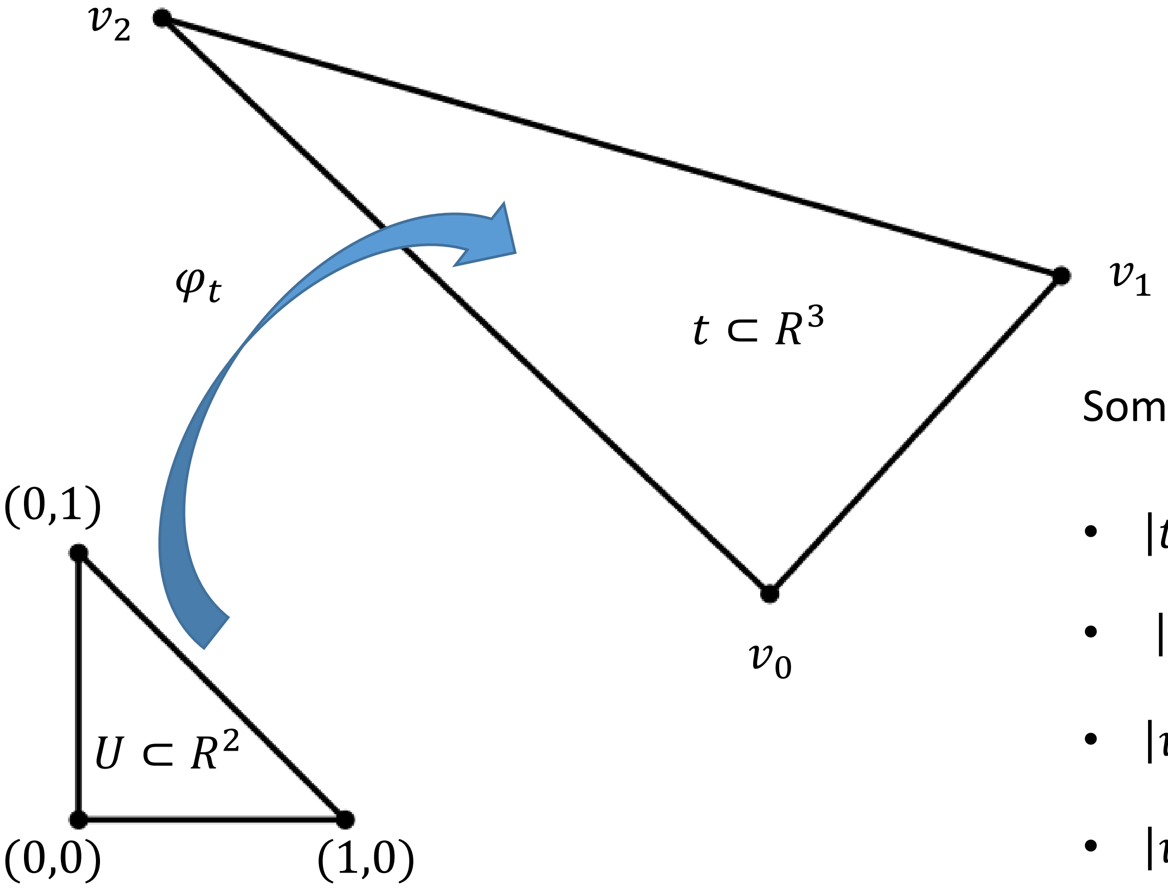
$$d\varphi_t \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v_1 - v_0$$

$$d\varphi_t \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v_2 - v_0$$

Definition: The metric tensor on the right unit triangle induced by the triangle embedding is given by,

$$g_t = d\varphi_t^{\top} d\varphi_t$$

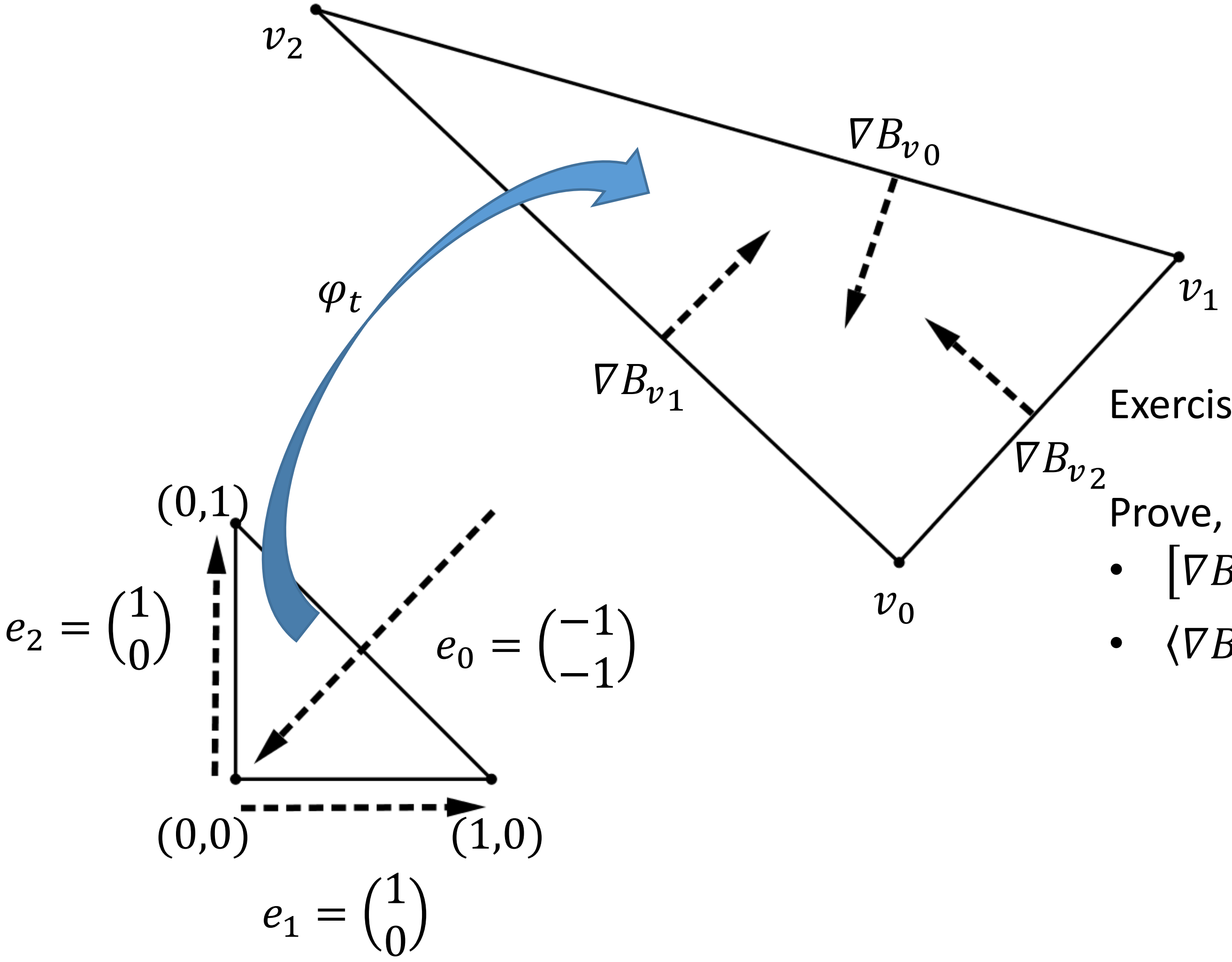
Right triangle parametrization



Some properties of $g_t = d\varphi_t^\top d\varphi_t$:

- $|t| = \frac{1}{2}\sqrt{\det g_t}$
- $|v_1 - v_0|^2 = \langle \binom{1}{0}, \binom{1}{0} \rangle_{g_t} = \binom{1}{0}^\top g_t \binom{1}{0}$
- $|v_2 - v_0|^2 = \langle \binom{0}{1}, \binom{0}{1} \rangle_{g_t} = \binom{0}{1}^\top g_t \binom{0}{1}$
- $|v_2 - v_1|^2 = \langle \binom{-1}{1}, \binom{-1}{1} \rangle_{g_t} = \binom{-1}{1}^\top g_t \binom{-1}{1}$

Right triangle parametrization

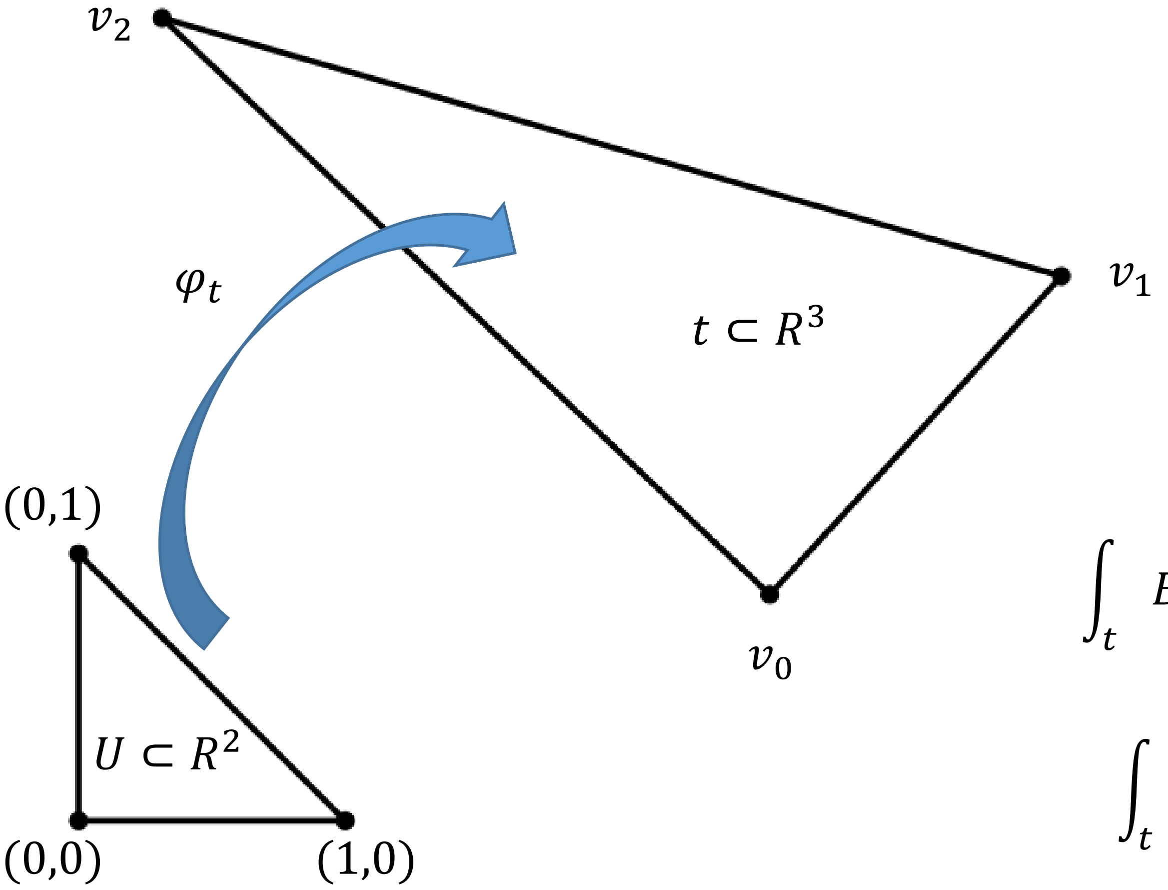


Exercise:

Prove,

- $[\nabla B_{v_i}] \coloneqq d\varphi_t^{-1}(\nabla B_{v_i}) = g_t^{-1} e_i$
- $\langle \nabla B_{v_i}, \nabla B_{v_j} \rangle = \langle [\nabla B_{v_i}], [\nabla B_{v_j}] \rangle_{g_t} = e_i^\top g_t^{-1} e_j$

Right triangle parametrization



$$\int_t B_{v_i} B_{v_j} = \frac{\sqrt{\det g_t}}{24} \qquad \int_t B_{v_i} B_{v_i} = \frac{\sqrt{\det g_t}}{12}$$

$$\int_t \langle \nabla B_{v_i}, \nabla B_{v_j} \rangle = e_i^\top g_t^{-1} e_j \frac{\sqrt{\det g_t}}{2}$$

$$M_{ij} = \int_T B_i B_j \rightarrow \text{Mass matrix}$$

$$S_{ij} = \int_T \langle \nabla B_i, \nabla B_j \rangle \rightarrow \text{Stiffness matrix}$$

Exersice:

Let $\vec{1}$, be the vector of ones.

(a) Prove $\vec{1}^\top M \vec{1} = \sum_{t \in T} |t|$.
(b) Prove $S\vec{1} = \vec{0}$
(c) Whats the null space of S?

Filtering

Given, $g = \sum_i g_i B_i$, find $f = \sum_i f_i B_i$, that minimize:

$$\min_f ||f - g||^2 + \alpha \left||\nabla f - \beta \nabla g|\right|^2$$

$$\left||f - g|\right|^2 := \int_T (f-g)^2 = ([f] - [g])^\top M([f] - [g])$$

$$\left||\nabla f - \beta \nabla g|\right|^2 := \int_T \langle \nabla f - \beta \nabla g, \nabla f - \beta \nabla g \rangle = ([f] - \beta[g])^\top S([f] - \beta[g])$$

$$\min_{[f]} [f]^\top (M + \alpha S)[f] - [f]^\top (M + \alpha\beta S)[g]$$

$$[f] = (M + \alpha S)^{-1}(M + \alpha\beta S)[g]$$

Example

$\alpha = 0, \beta = 0$

$\alpha = 10^{-4}, \beta = 0$

$\alpha = 10^{-4}, \beta = 2$

Poisson Problem

Compute the closest gradient field to a vector field v:

$$\min_f ||\nabla f \ - v||^2$$

Q. How many degrees of freedom has v?

$$G: R^V \to R^{2T}, G_{i\in V, t\in T} \ = \begin{cases} g_t^{-1} e_{c(i)} & \text{If } i \text{ is the } c(i)\text{-corner of } t \\ 0 & \text{Otherwise} \end{cases}$$

Gradient matrix

$$[\nabla f] = G[f]$$

$$N: R^{2T} \to R^{2T} = \begin{pmatrix} \frac{g_1^{-1}\sqrt{\det g_1}}{2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{g_N^{-1}\sqrt{\det g_N}}{2} \end{pmatrix}$$

Vector product matrix

$$\langle u, v \ \rangle = [u]N[v]$$

Prove: $S = G^\top N \ G$

Prove: $\min_f ||\nabla f \ - v||^2$, is given by $S \ [\nabla f] = (G^\top N[v])$. How many solution does this equation have?

Application: Approximate Geodesic Distances

Algorithm:
Input: $i \leftarrow$ Vertex index
Output: $d \leftarrow$ approximate geodesic distance to v_i.

1. Solve for

$$\min_f ||f - \delta_i||^2 + \alpha \left||\nabla f|\right|^2$$

2. For each triangle let $v_t = \frac{\nabla f|_t}{||\nabla f|_t||}$
3. Solve for

$$\min_d \left||\nabla d - v|\right|^2$$

4. Offset d by a constant to have $d[i] = 0$.