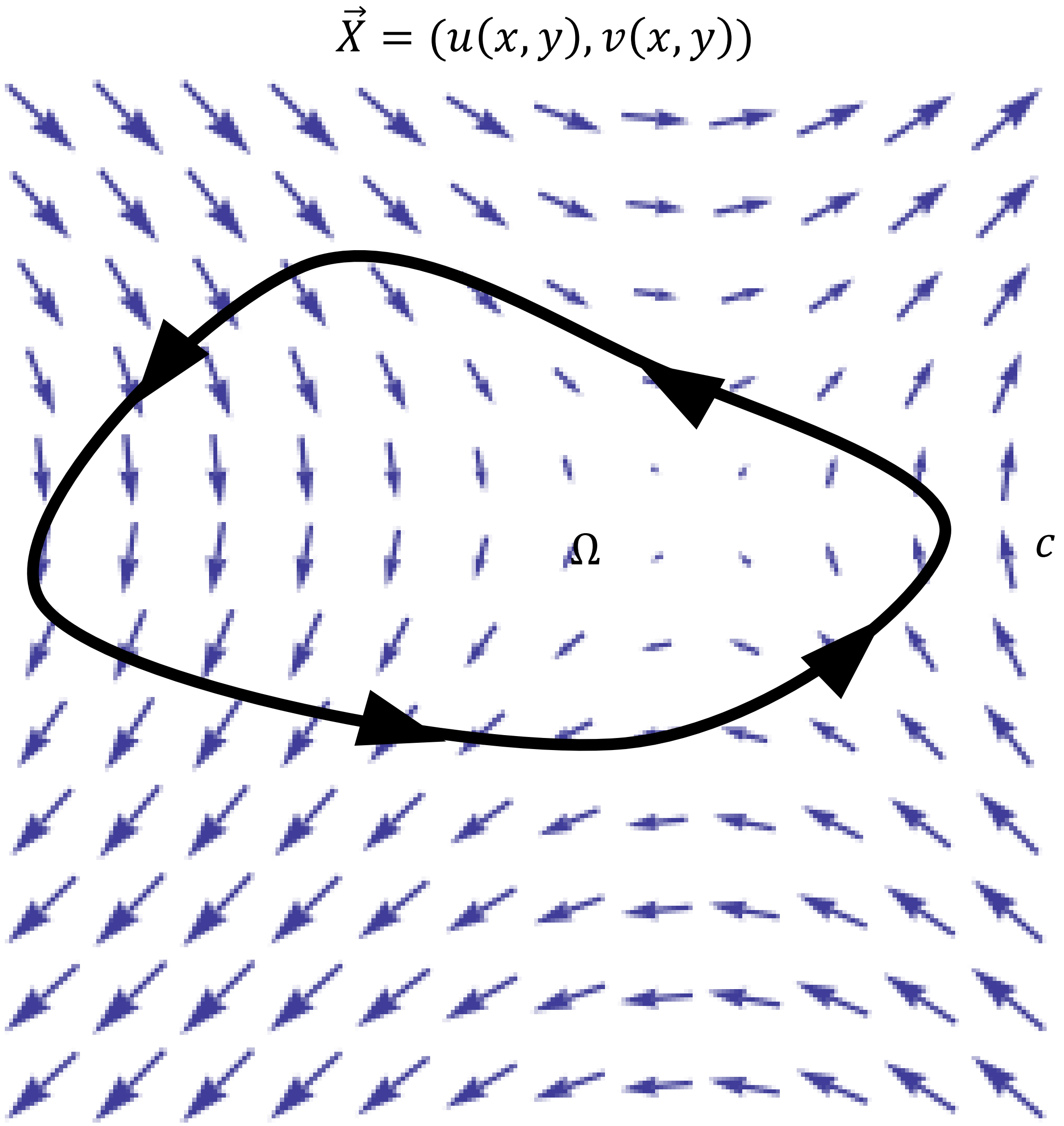
Lecture 6

Introduction to Geometry Processing

Spring 2017

Johns Hopkins University

Green Theorem ($\nabla \times$)



Given a curve c in anticlockwise orientation enclosing a region Ω, we have:

$$\int_c \vec{X} \cdot ds = \int_\Omega \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy$$

The operator $\nabla \times \vec{X} := \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, is called curl.

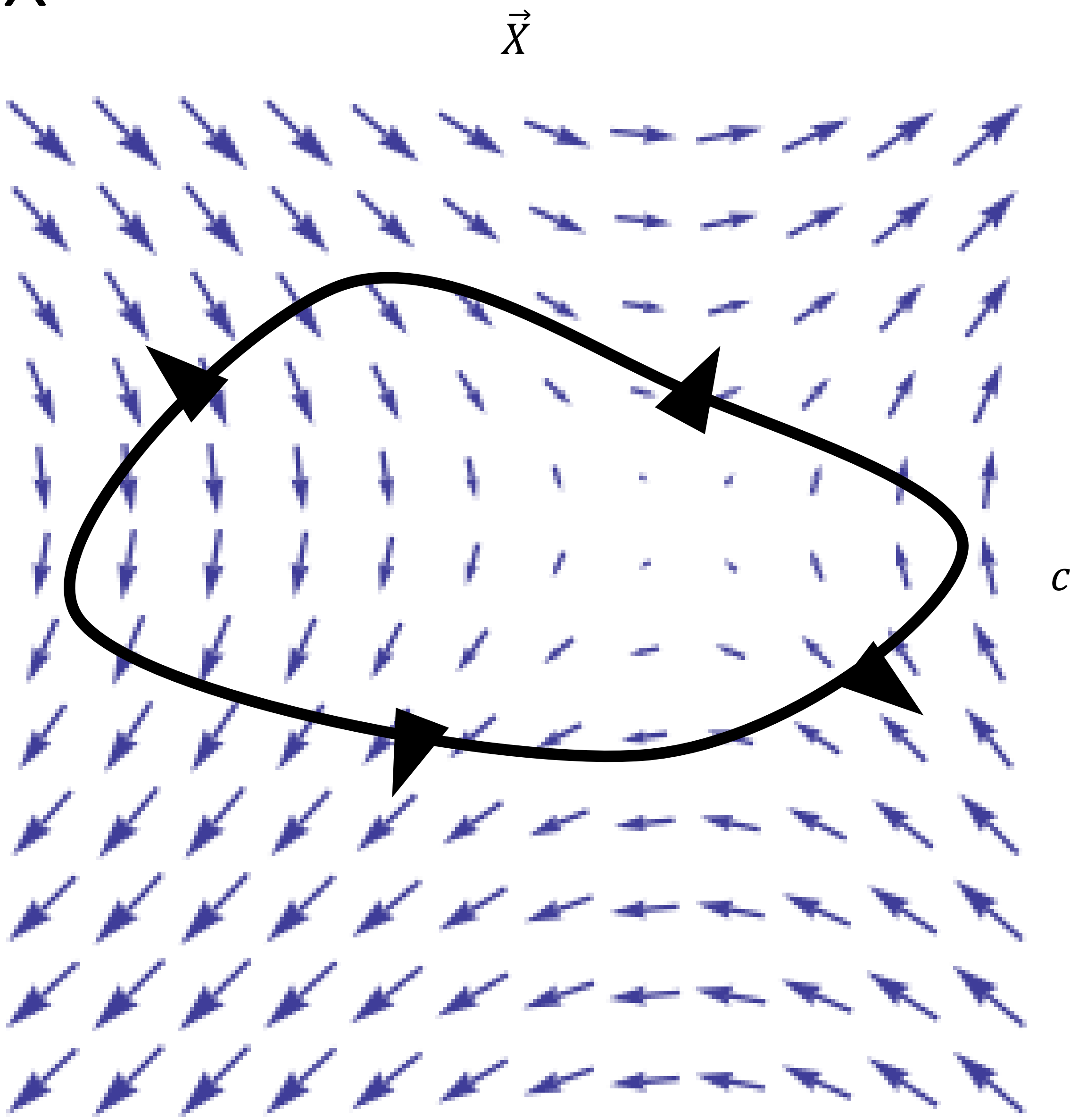
Using this notation we get,

$$\int_c \vec{X} \cdot ds = \int_\Omega \nabla \times \vec{X}\, dxdy$$

Proof:

https://en.wikipedia.org/wiki/Green's_theorem

Flux



Given any anticlockwise parametrization $\gamma: I \to c$, the flux of field $\vec{X}$ along c is given by:

$$\int_c \vec{X} \cdot dn := \int_I \langle -J\gamma'(t), \vec{X}(\gamma(t)) \rangle \, dt$$

Here, J denotes the 90 degree rotation of a vector.

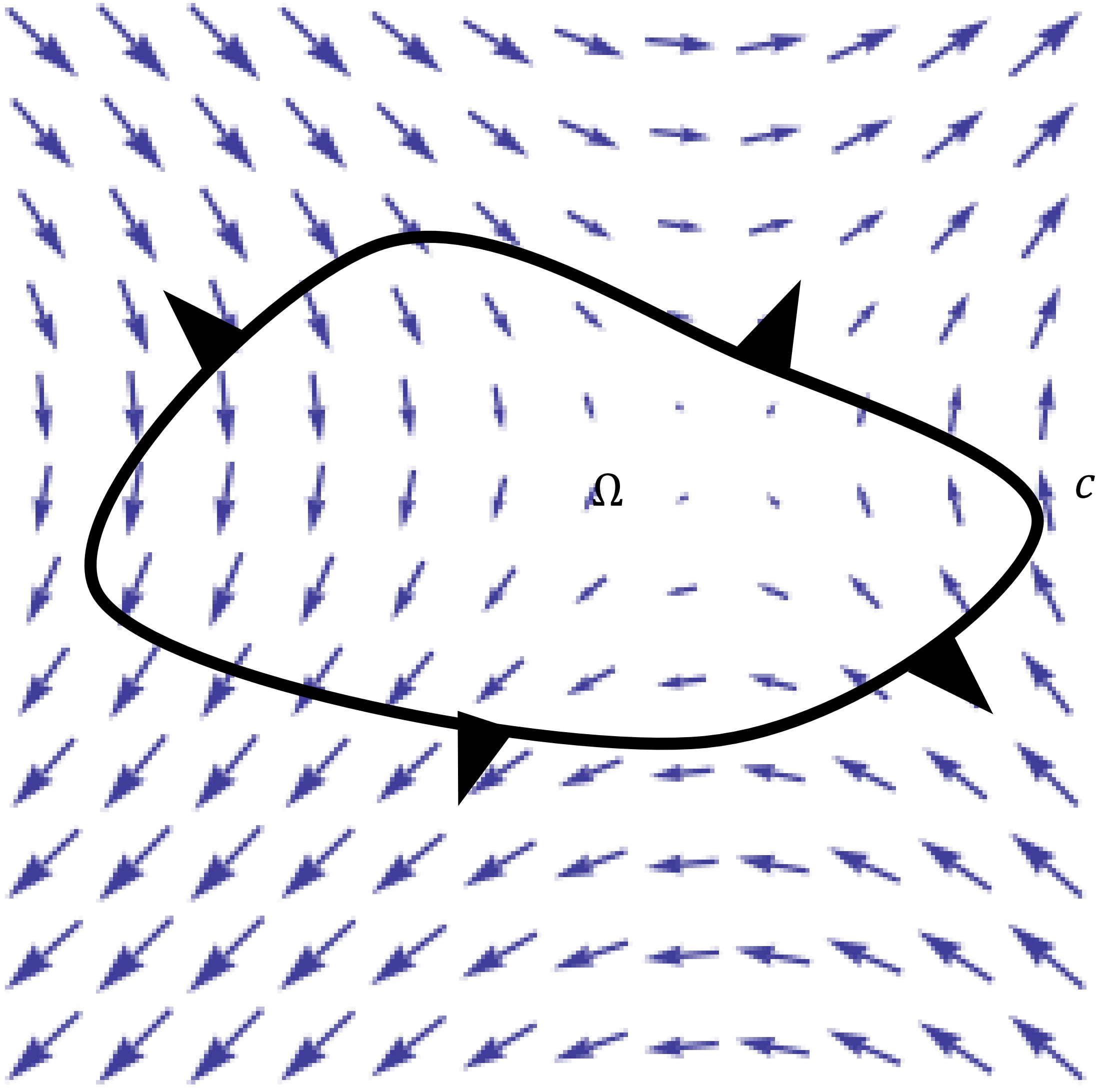
Exercise

Let $\vec{X}(x,y) = (2x + y, x)$
Let c be the unit circle traversed in anti-clockwise orientation.
Compute the flux of $\vec{X}$ across c.

Let $\vec{X}(x,y) = (-(2y + x), y)$
Let c be the unit circle traversed in anti-clockwise orientation.
Compute the flux of $\vec{X}$ across c.

Divergence Theorem ($\nabla \cdot$)

$\vec{X} = (u(x,y), v(x,y))$



Given a curve c enclosing a region Ω, we have:

$$\int_c \vec{X} \cdot dn = \int_\Omega \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dxdy$$

The operator $\nabla \cdot \vec{X} := \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, is called divergence. Using this notation we get,

$$\int_c \vec{X} \cdot dn = \int_\Omega \nabla \cdot \vec{X}\, dxdy$$

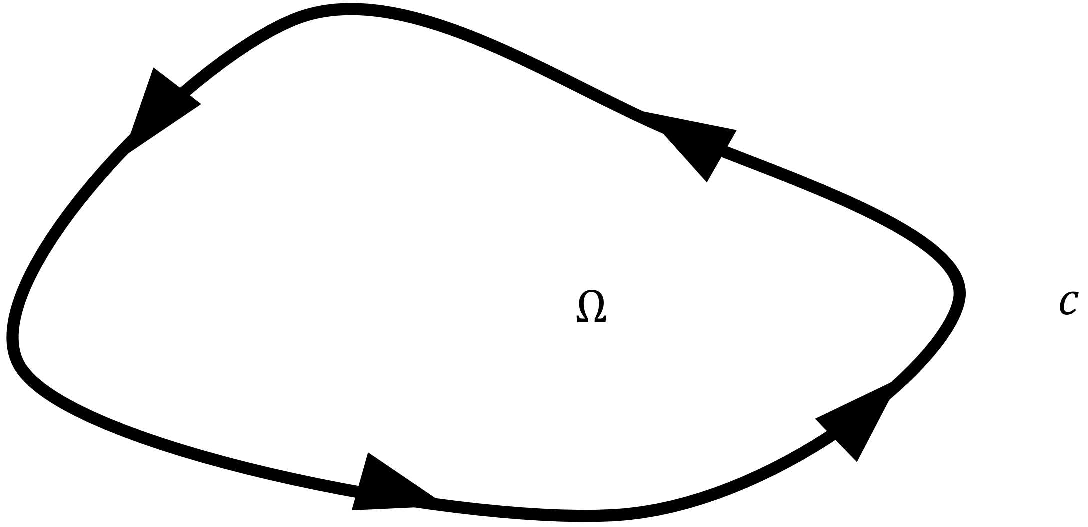
Exercise:

(1) Prove $\nabla \cdot \vec{X} = \nabla \times J\vec{X}$

(2) Prove Divergence theorem from Green theorem.

Applications:

Area computation:



$$\vec{X} = (x, y)$$

$$\text{Area}(\Omega) = \frac{1}{2}\int_c \overrightarrow{X_i} \cdot dn$$

Exercise:
Given a polygonal curve $p_0, p_0, \dots, p_n$ (anticlockwise), prove that the area of the enclosed polygon is given by,

$$\text{A} = \frac{-1}{2} \sum_i p_i J p_{i+1}$$

Exercise

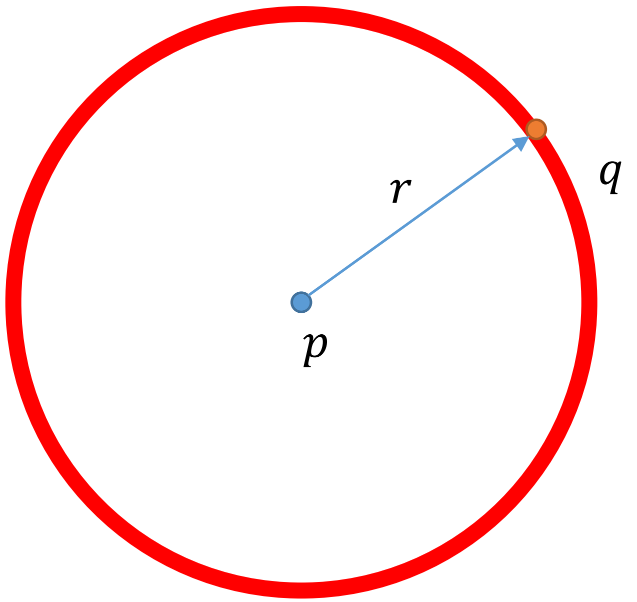
Prove:

- $\nabla \times \nabla f = 0$
- $\nabla \cdot J\nabla f = 0$
- $\nabla \cdot (fv) = f\nabla \cdot v + \langle \nabla f, v \rangle$

Laplacian (Δ)

$$\Delta f := \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$\int_{B_r(p)} \Delta f \, dxdy = \int_{B_r(p)} \nabla \cdot \nabla f \, dxdy = \int_{\partial B_r(p)} \nabla f \cdot dn$$



$$\int_{B_r(p)} \Delta f \, dxdy = \pi r^2 \Delta f(p) + O(r^3)$$

$$\bar{f}_{\partial B_r(p)} := \frac{\int_{\partial B_r(p)} f}{2\pi r}$$

$$\nabla f(q) = \nabla f(p) + \nabla f^2(p)(p-q) + O(r^2)$$

$$\nabla f(q) \cdot dn = \frac{1}{r}(\nabla f(p) \cdot (p-q) + O(r^3))$$

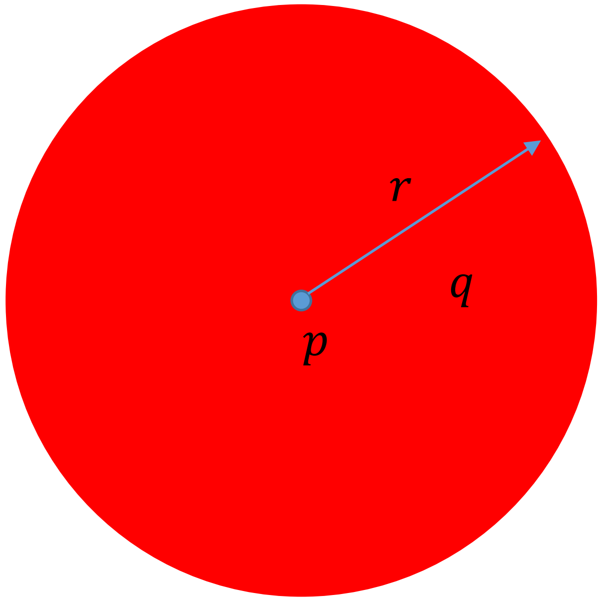
$$\nabla f(q) \cdot dn = \frac{1}{r}(f(q) - f(p) + O(r^3))$$

$$\int_{\partial B_r(p)} \nabla f \cdot dn = 2\pi \left(\bar{f}_{\partial B_r(p)} - f(p) \right) + O(r^3)$$

$$\Delta f(p) = \lim_{r \to 0} \frac{2\left(\bar{f}_{\partial B_r(p)} - f(p) \right)}{r^2}$$

Laplacian (Δ)

$$\Delta f := \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$



$$\bar{f}_{B_r(p)} := \frac{\int_{B_r(p)} f}{\pi r^2} = \frac{\int_0^s 2\pi s \bar{f}_{\partial B_s(p)} ds}{\pi r^2}$$

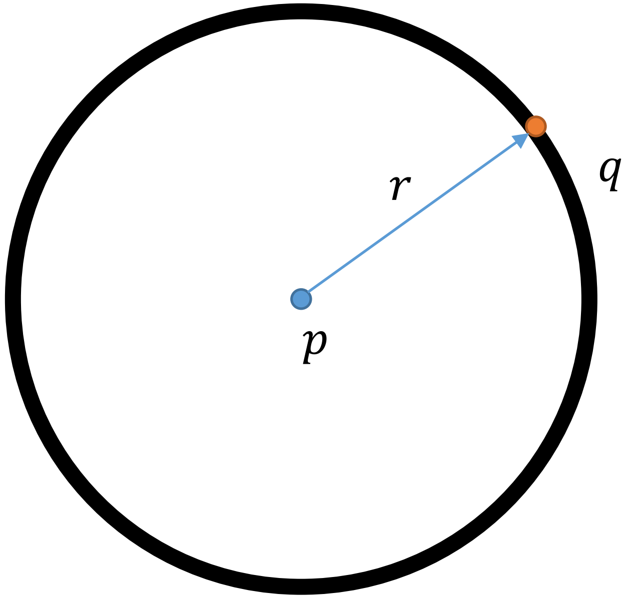
$$\pi r^2 \left(\bar{f}_{B_r(p)} - f(p) \right) = \int_0^s 2\pi s (\bar{f}_{\partial B_s(p)} - f(p)) ds$$

$$\pi r^2 \left(\bar{f}_{B_r(p)} - f(p) \right) = \int_0^s \pi s^3 \Delta f(p) + O(s^4)\, ds = \frac{\pi r^4}{4} \Delta f(p) + O(r^5)$$

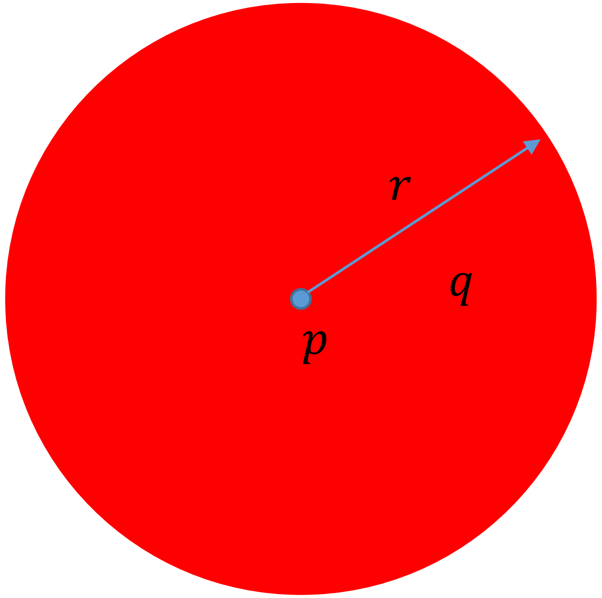
$$\Delta f(p) = \lim_{r \to 0} \frac{4 \left(\bar{f}_{B_r(p)} - f(p) \right)}{r^2}$$

Harmonic Functions

A function $f: \Omega \subset R^2 \rightarrow R^2$ is said harmonic, if $\Delta f(x, y) = 0 \; \forall (x, y) \in \Omega$



If f is harmonic, then $\bar{f}_{\partial B_r(p)} = f(p) \; \forall p \in \Omega$, and $\partial B_r \subset \Omega$



If f is harmonic, then $\bar{f}_{B_r(p)} = f(p) \; \forall p \in \Omega$, and $B_r \subset \Omega$

Harmonic Functions

If f is harmonic, then $\exists\ x, y \in \partial\Omega$ such that $f(x) \leq f(q) \leq f(y)$, for all $q \in \Omega$