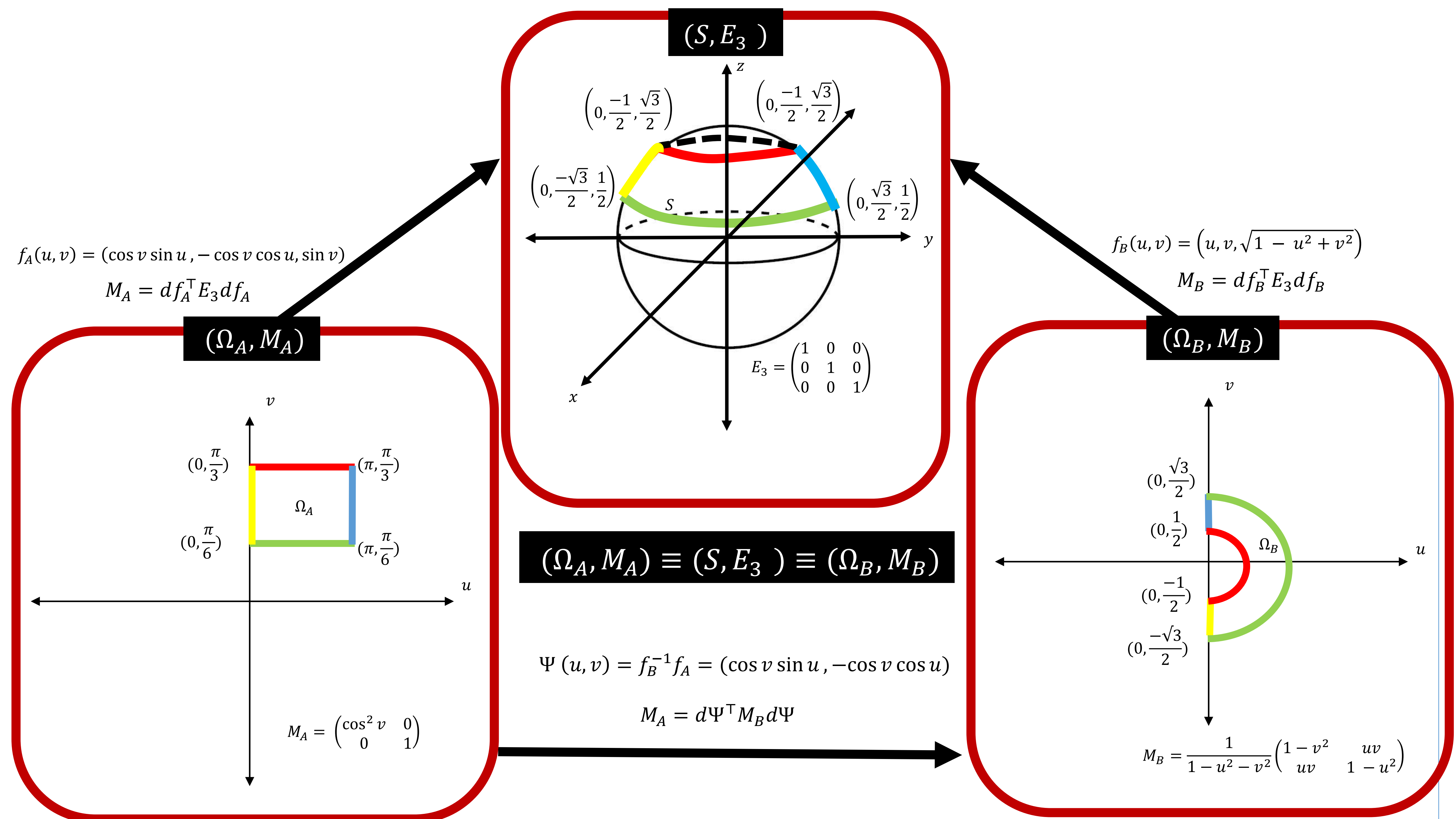
Lecture 4

Introduction to Geometry Processing

Spring 2017

Johns Hopkins University

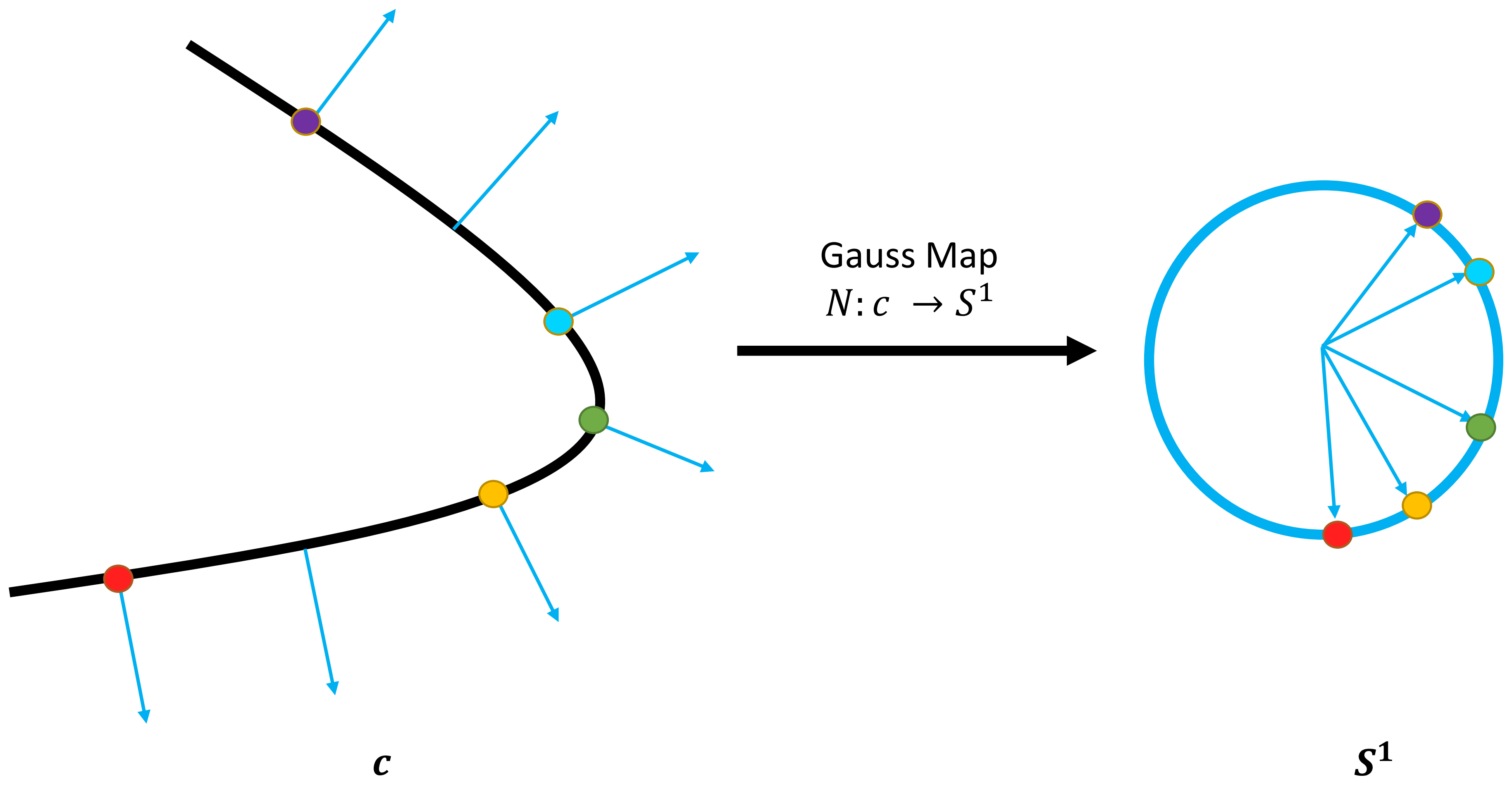


(S, E_3)
z
y
x
S
$\left(0, \frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(0, \frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(0, \frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$
$\left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$f_A(u, v) = (\cos v \sin u, -\cos v \cos u, \sin v)$
$M_A = df_A^\top E_3 df_A$
$f_B(u, v) = \left(u, v, \sqrt{1 - u^2 + v^2}\right)$
$M_B = df_B^\top E_3 df_B$
(Ω_A, M_A)
v
u
$(0, \frac{\pi}{3})$
$(\pi, \frac{\pi}{3})$
Ω_A
$(0, \frac{\pi}{6})$
$(\pi, \frac{\pi}{6})$
$M_A = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$
$(\Omega_A, M_A) \equiv (S, E_3) \equiv (\Omega_B, M_B)$
$\Psi(u, v) = f_B^{-1} f_A = (\cos v \sin u, -\cos v \cos u)$
$M_A = d\Psi^\top M_B d\Psi$
(Ω_B, M_B)
v
u
$(0, \frac{\sqrt{3}}{2})$
$(0, \frac{1}{2})$
Ω_B
$(0, \frac{-1}{2})$
$(0, \frac{-\sqrt{3}}{2})$
$M_B = \frac{1}{1 - u^2 - v^2} \begin{pmatrix} 1 - v^2 & uv \\ uv & 1 - u^2 \end{pmatrix}$

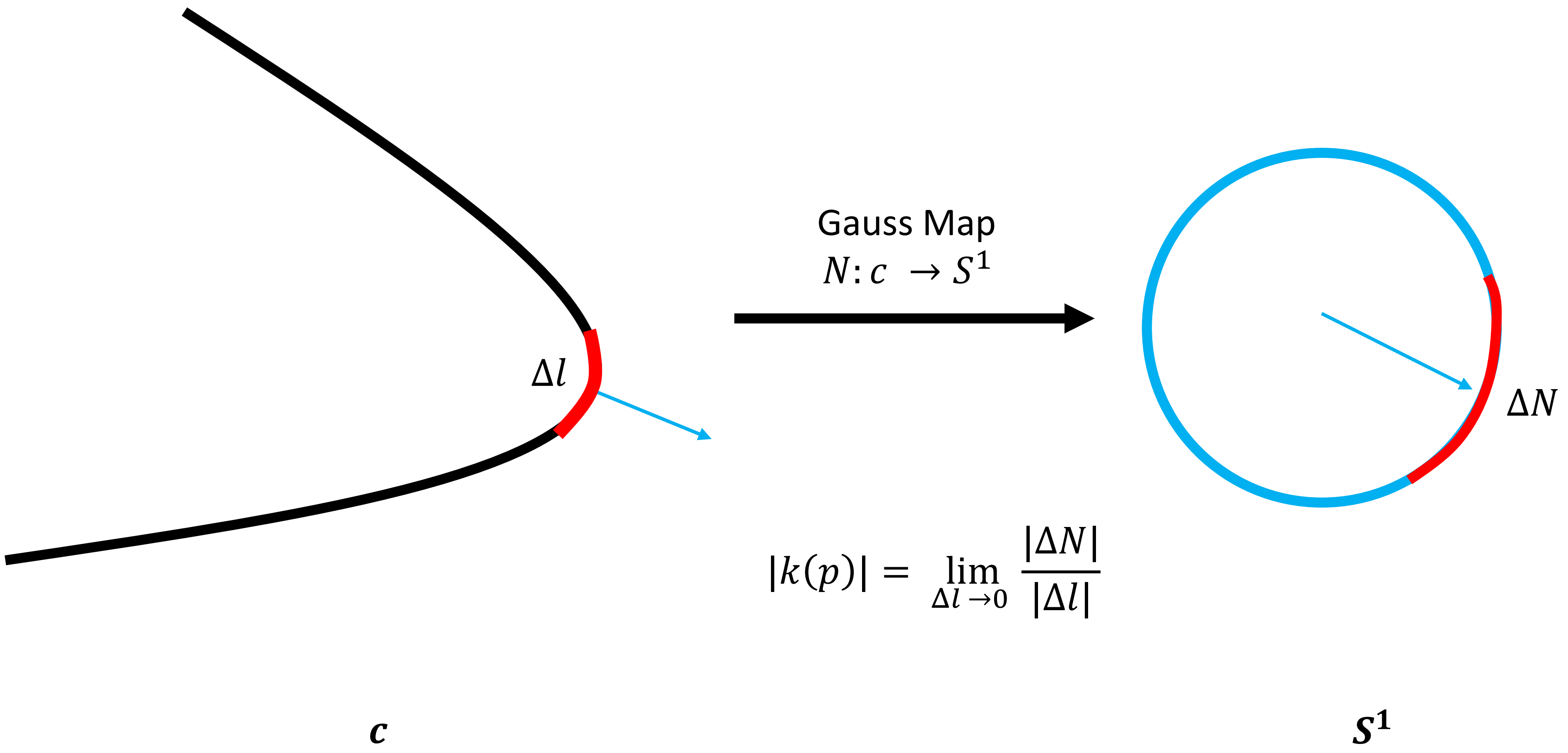
Johann Carl Friedrich Gauss
(1777- 1855)

- Number theory
- Algebra
- Statistics
- Analysis
- **Differential geometry**
- Mechanics
- Electrostatics
- Astronomy
- Matrix theory
- Optics

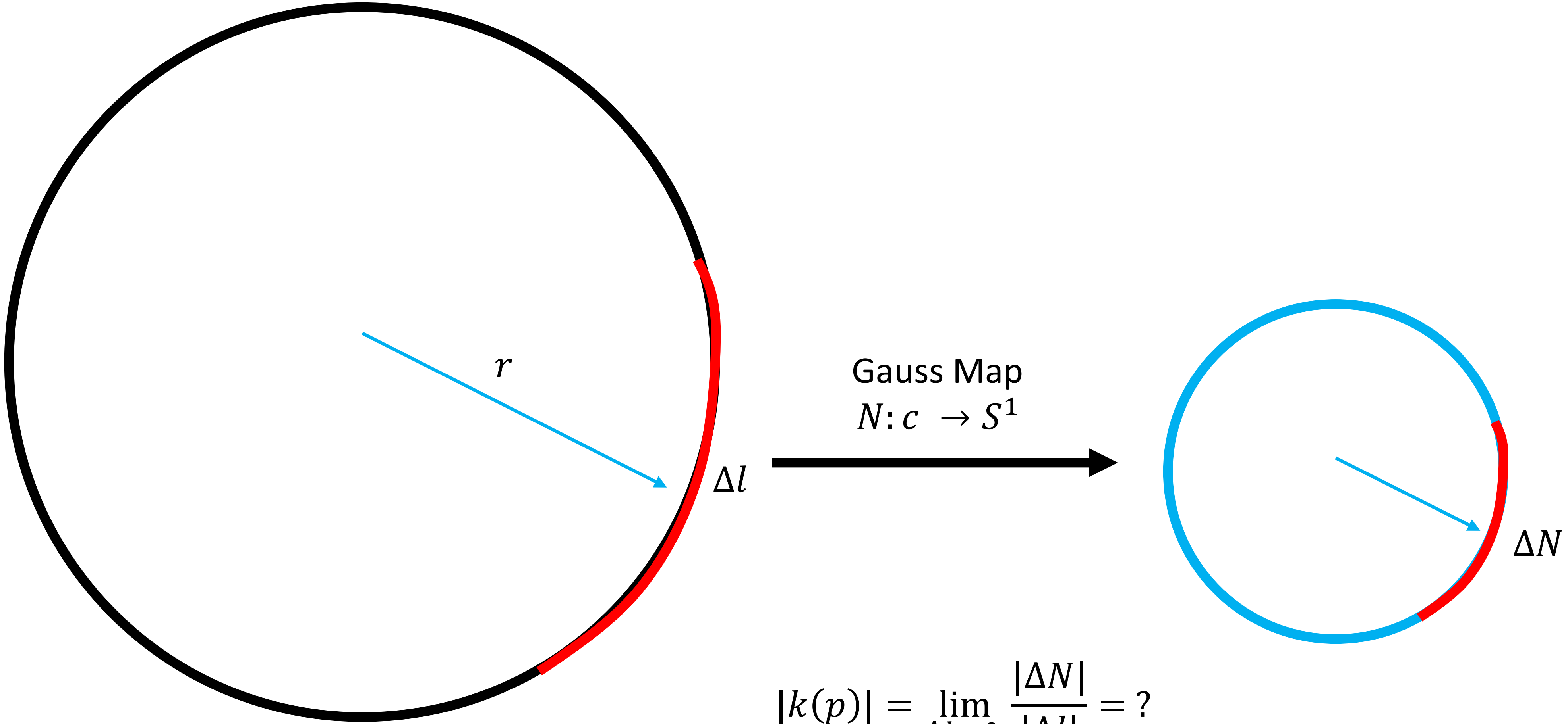
Planar Curves



Curvature



$$|k(p)| = \lim_{\Delta l \rightarrow 0} \frac{|\Delta N|}{|\Delta l|}$$

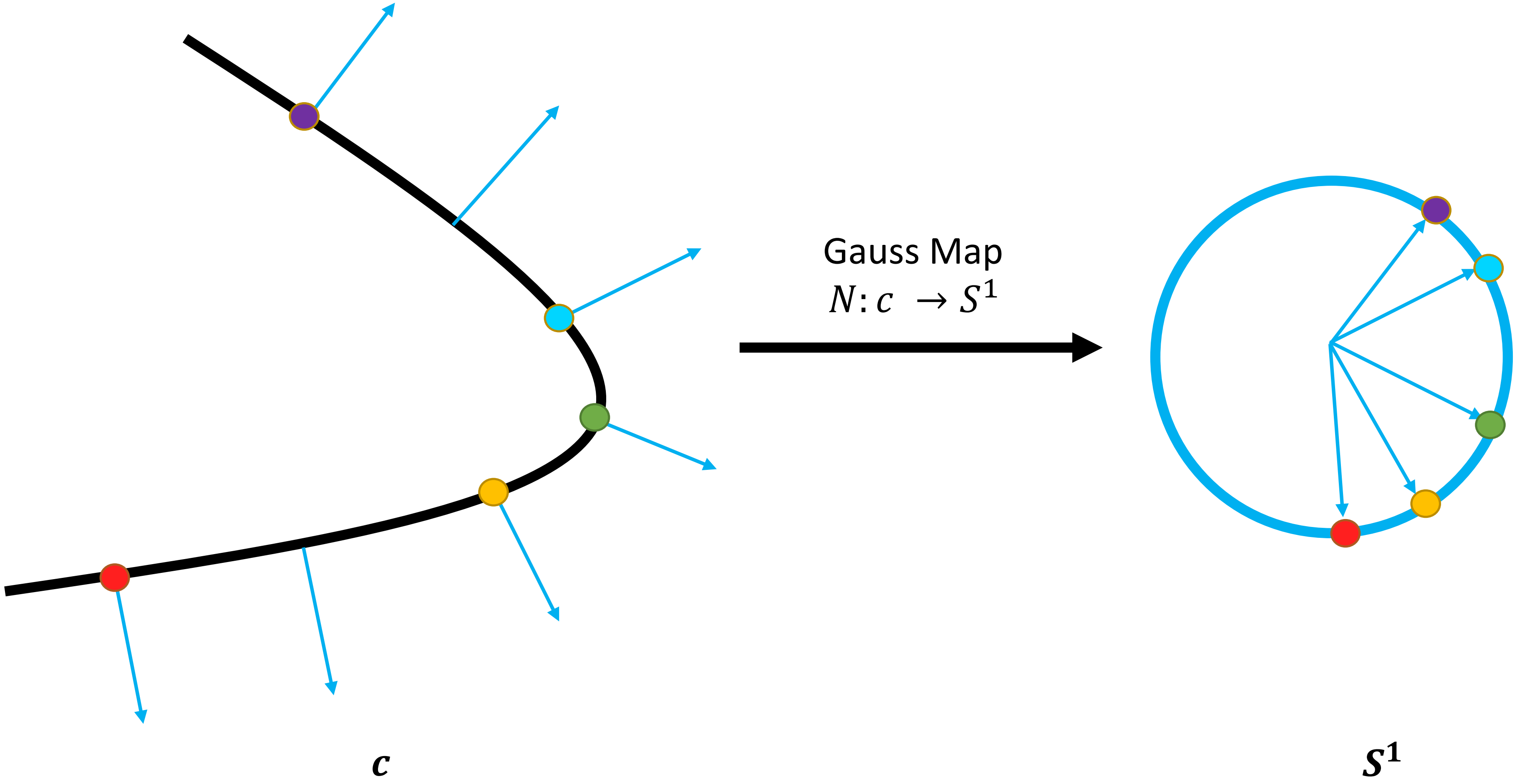


$$|k(p)| = \lim_{\Delta l \to 0} \frac{|\Delta N|}{|\Delta l|} = ?$$

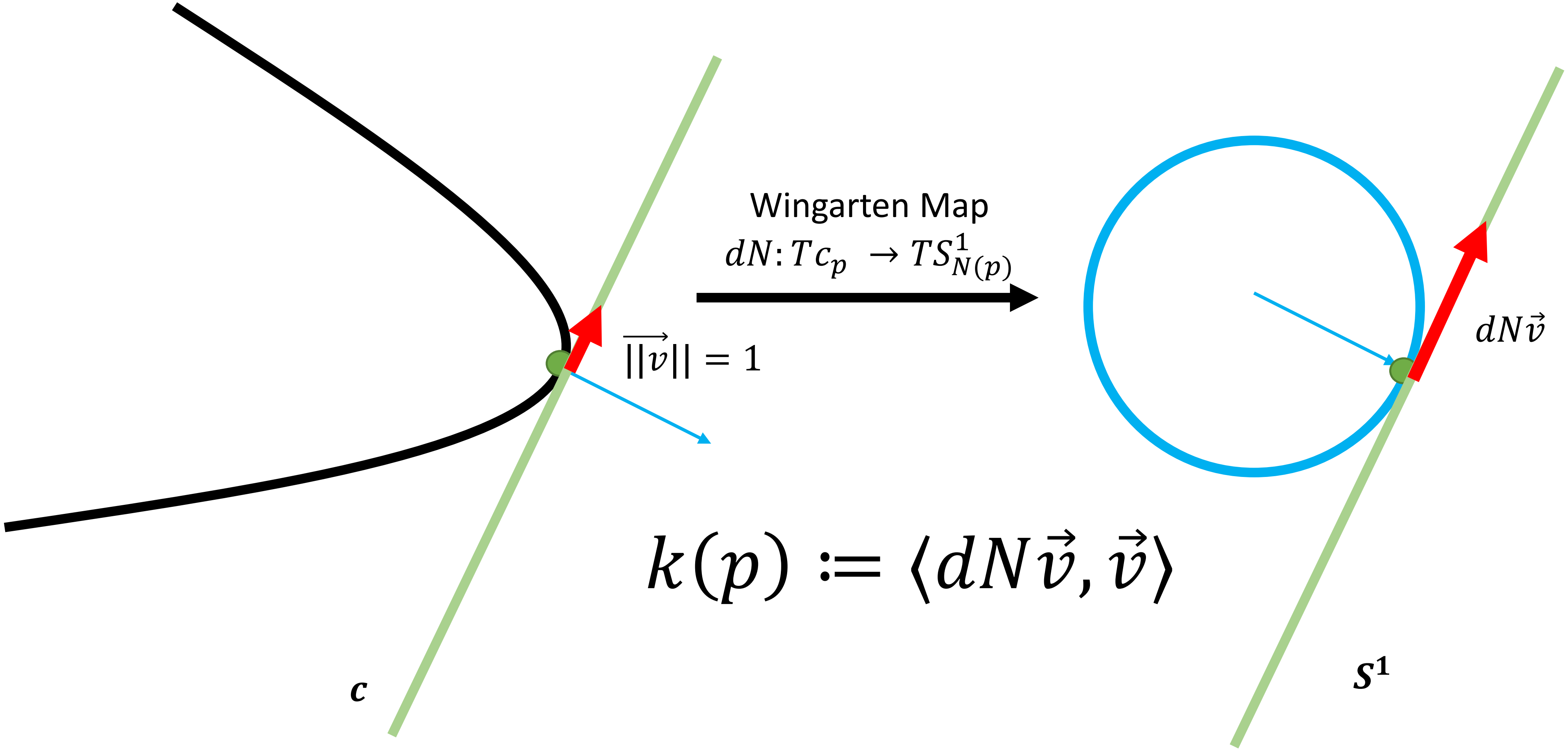
$$|k(p)| = \frac{1}{r}$$

$\boldsymbol{rS^1}$ $\boldsymbol{S^1}$

Weingarten Map



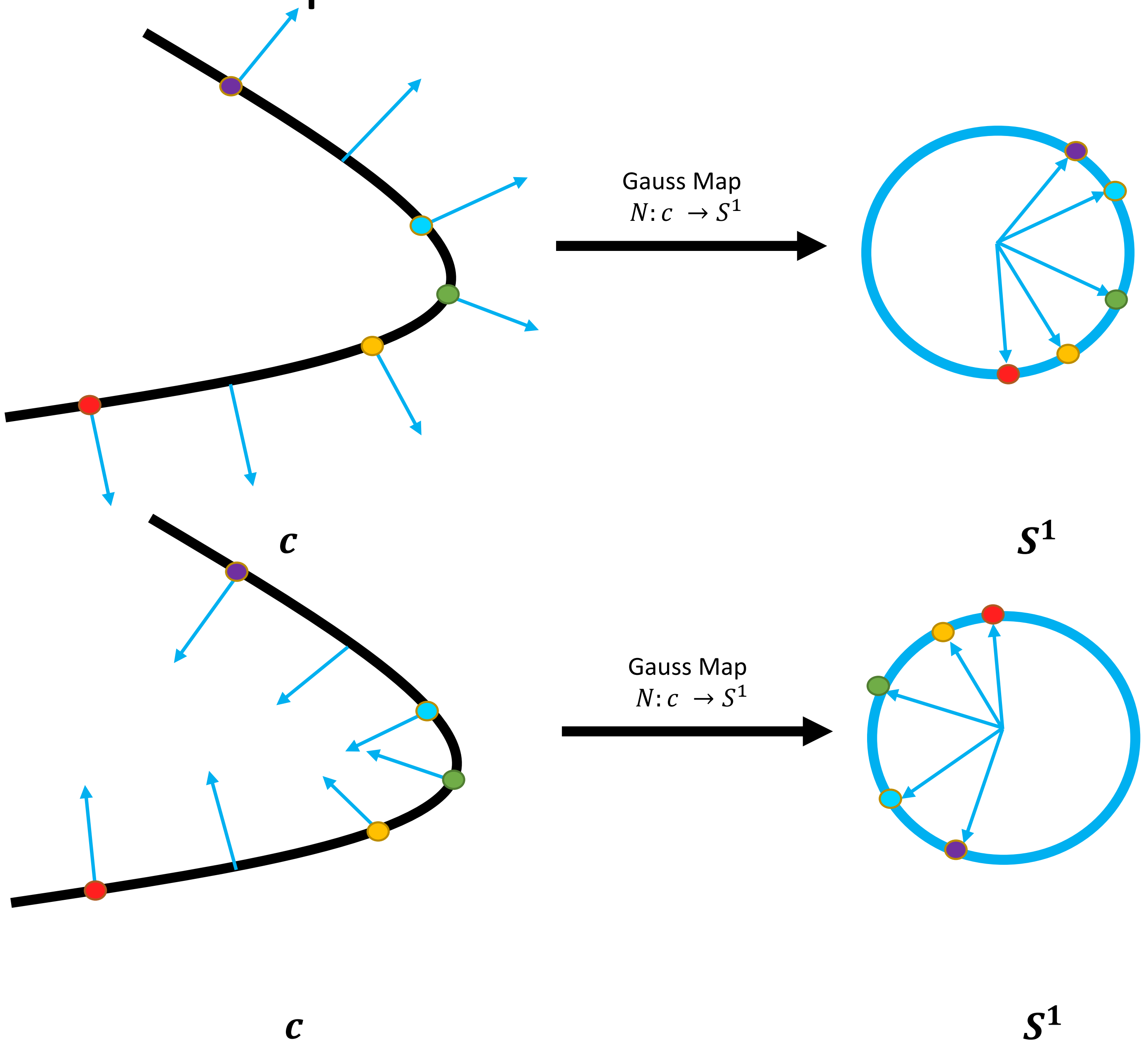
Weingarten Map



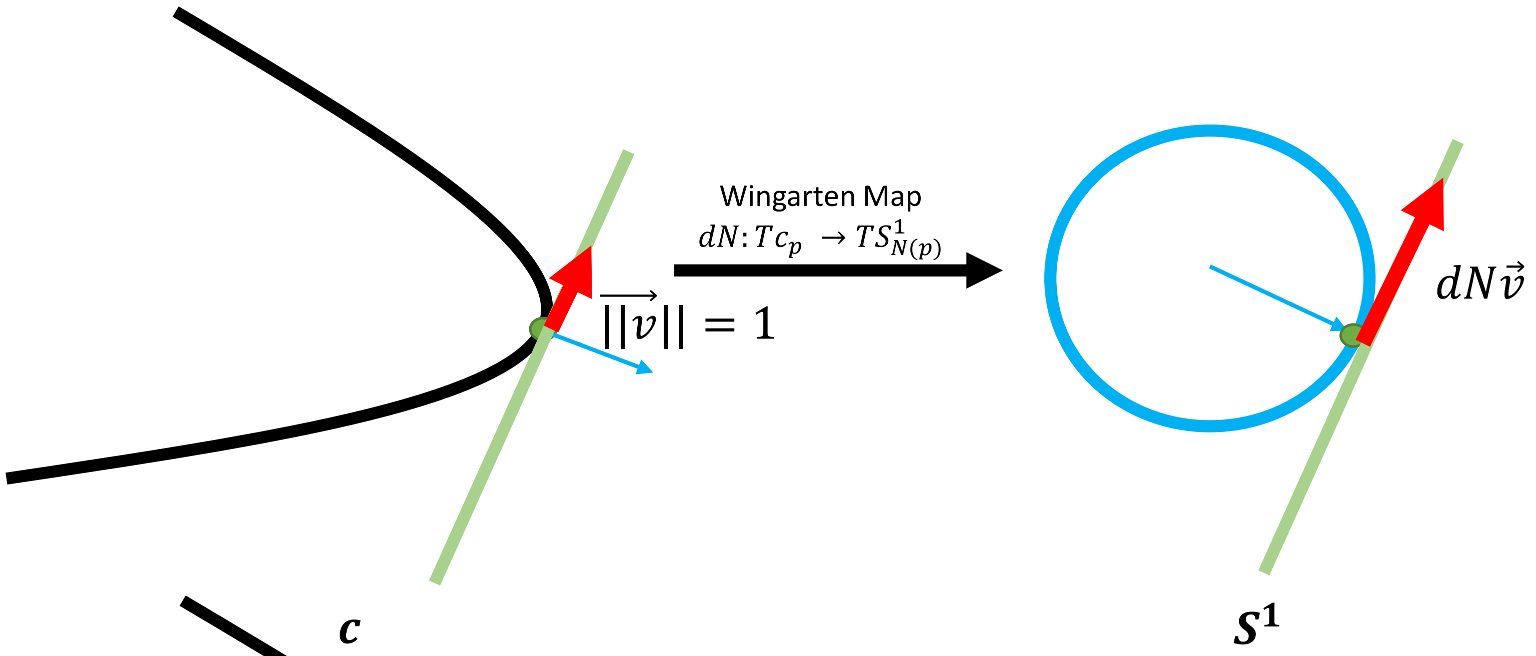
$$k(p) \coloneqq \langle dN\vec{v}, \vec{v} \rangle$$

Remark $|k(p)| = |\langle dN\vec{v}, \vec{v}\rangle| = |dN\vec{v}| = \lim_{\Delta l \to 0} \frac{|\Delta N|}{|\Delta l|}$

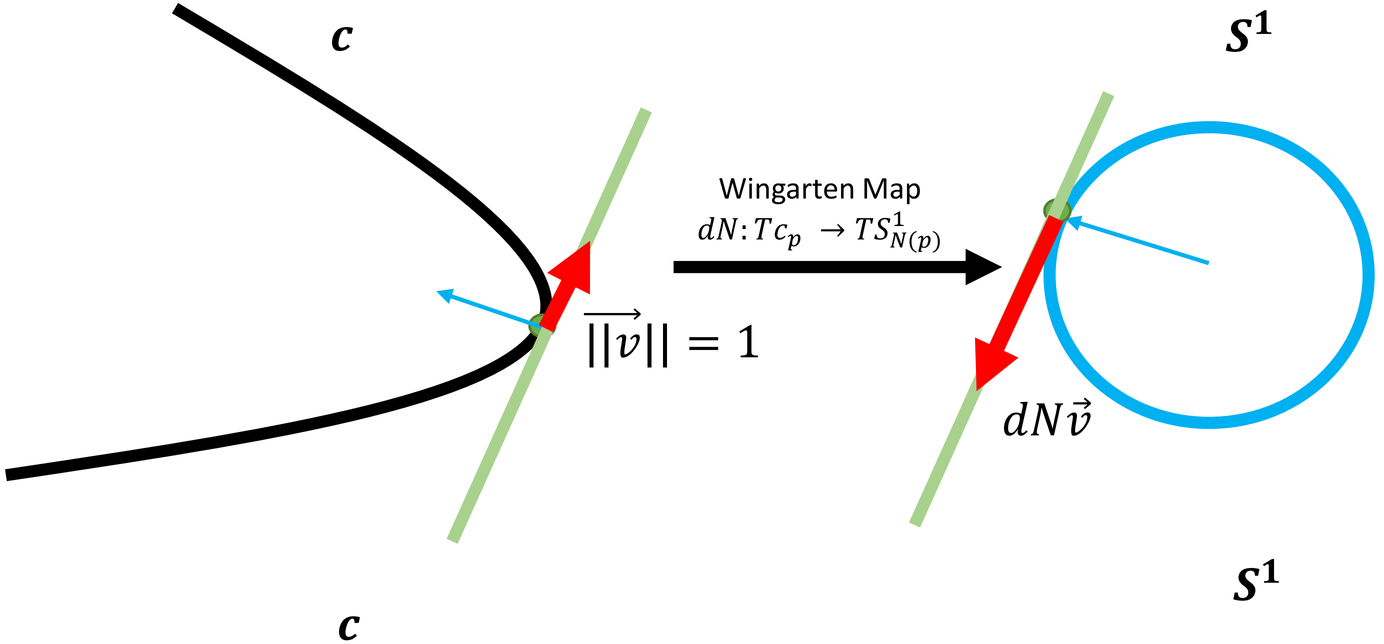
Orientation is important!



Orientation is important!



$k(p) = \langle dN\vec{v}, \vec{v} \rangle > 0$

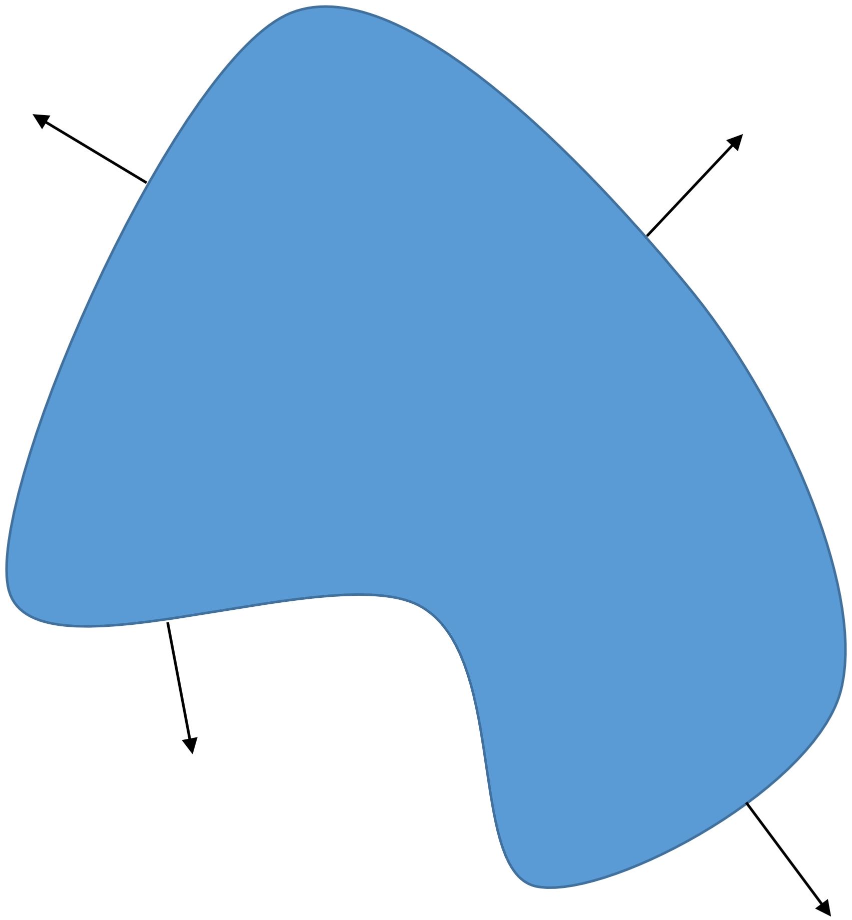


$k(p) = \langle dN\vec{v}, \vec{v} \rangle < 0$

Where the orientation come from?

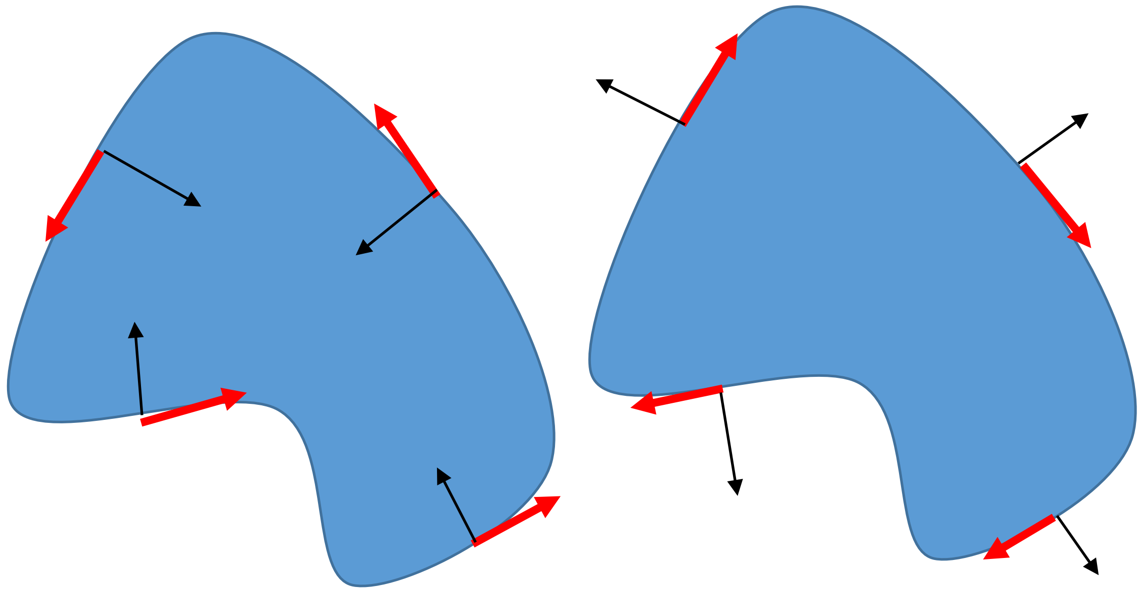
(1) Prescribed

e.g., to point to the exterior of a region.



(2) Induced by a parametrization

e.g., the 90 degree rotation of the tangent unit vector.



Exercise

$\gamma : [0, l] \to c$ is said to be an arc-length parametrization of the curve c if

$$\mathrm{Length}(\gamma[0,t]) = t, \quad \forall t \in [0, l].$$

- $\left||\gamma'(t)|\right| = 1$
- $\langle \gamma'(t), \gamma''(t) \rangle = 0.$
- $\langle (N \circ \gamma)'(t), \gamma'(t) \rangle = -\langle (N \circ \gamma)(t), \gamma''(t) \rangle$

Conclude

$$k\big(\gamma(t)\big) = -\langle (N \circ \gamma)(t), \gamma''(t) \rangle$$

Computing curvatures

Exercise:

Given **parametrizations** $\gamma_1 = I_1 \to c$, $\gamma_2 : I_2 \to c$, the gauss map $N: c \to S^1$, and $p = \gamma_1(t_1) = \gamma_2(t_2)$, prove:

$$\frac{\langle (N \circ \gamma_1)'(t_1), {\gamma_1}'(t_1) \rangle}{\langle {\gamma_1}'(t_1), {\gamma_1}'(t_1) \rangle} = \frac{\langle (N \circ \gamma_2)'(t_2), {\gamma_2}'(t_2) \rangle}{\langle {\gamma_2}'(t_2), {\gamma_2}'(t_2) \rangle}$$

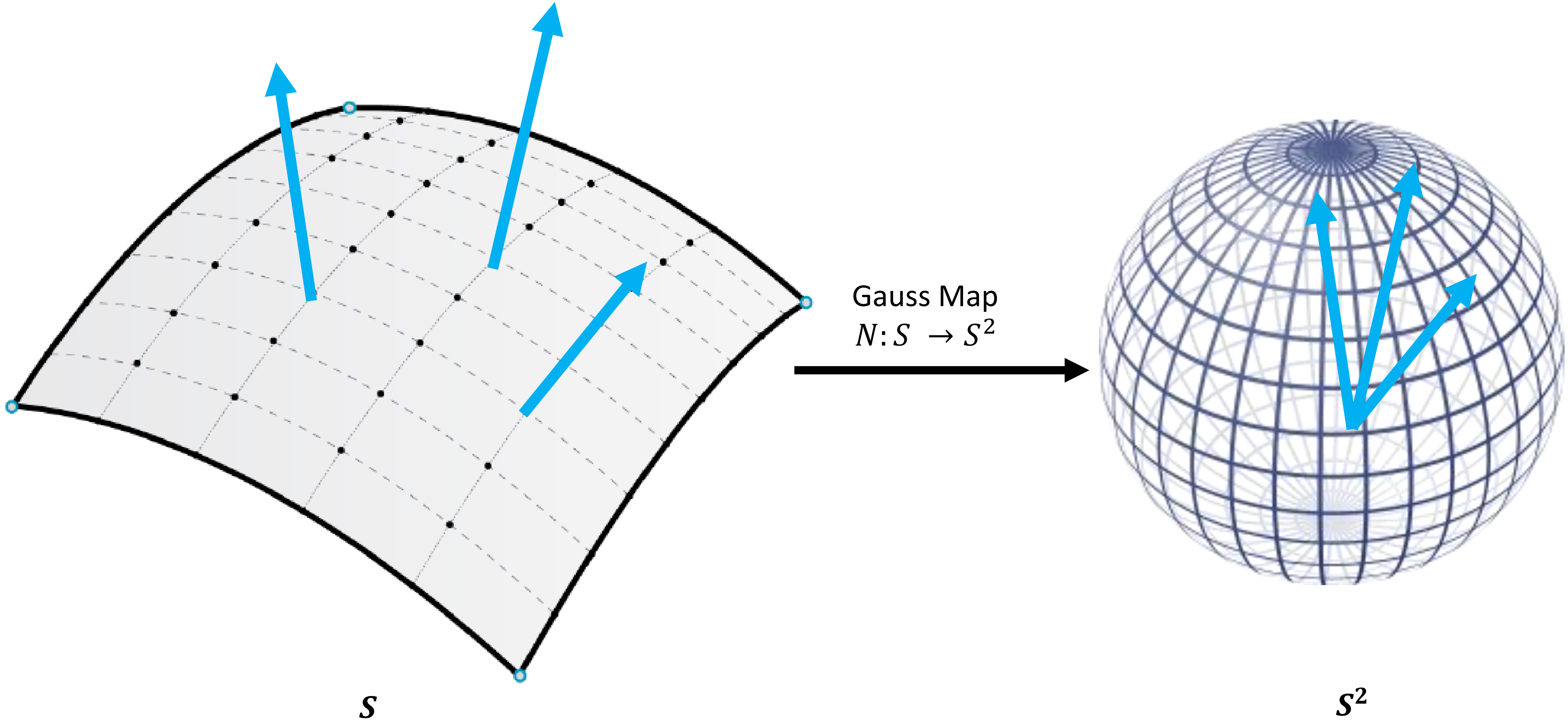
Conclusion: For $p \in c$ we have $k(p) = \dfrac{\langle (N \circ \gamma)'(t), \gamma'(t) \rangle}{\langle \gamma'(t), \gamma'(t) \rangle}$ for any $\gamma = I \to c$, such that $p = \gamma(t)$.

Exercise:

Given $\gamma = I \to c$, prove: $\langle (N \circ \gamma)'(t), \gamma'(t) \rangle = -\langle (N \circ \gamma)(t), \gamma''(t) \rangle$

Conclusion: For $p \in c$ we have $k(p) = \dfrac{-\langle (N \circ \gamma)(t), \gamma''(t) \rangle}{\langle \gamma'(t), \gamma'(t) \rangle}$ for any $\gamma = I \to c$, such that $p = \gamma(t)$.

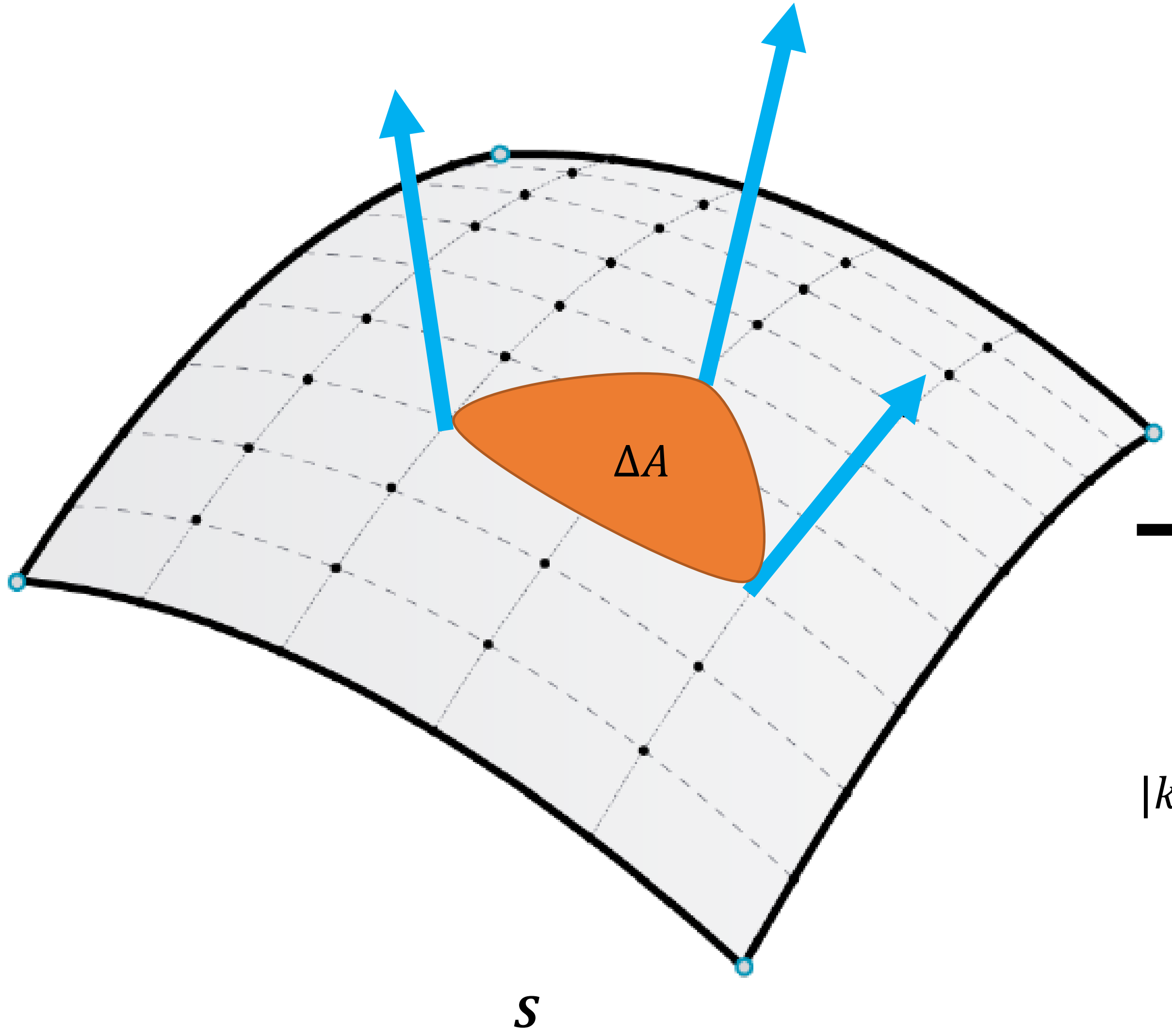
Surface



http://grasshopperprimer.com/en/1-foundations/1-5/1_surface-geometry.html

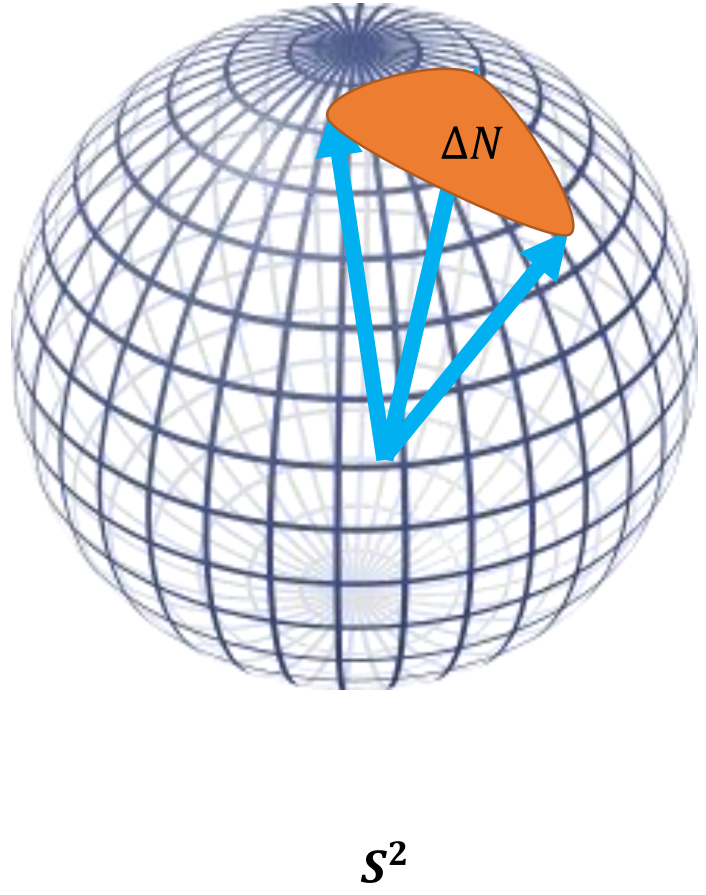
https://en.wikipedia.org/wiki/Sphere

Curvature

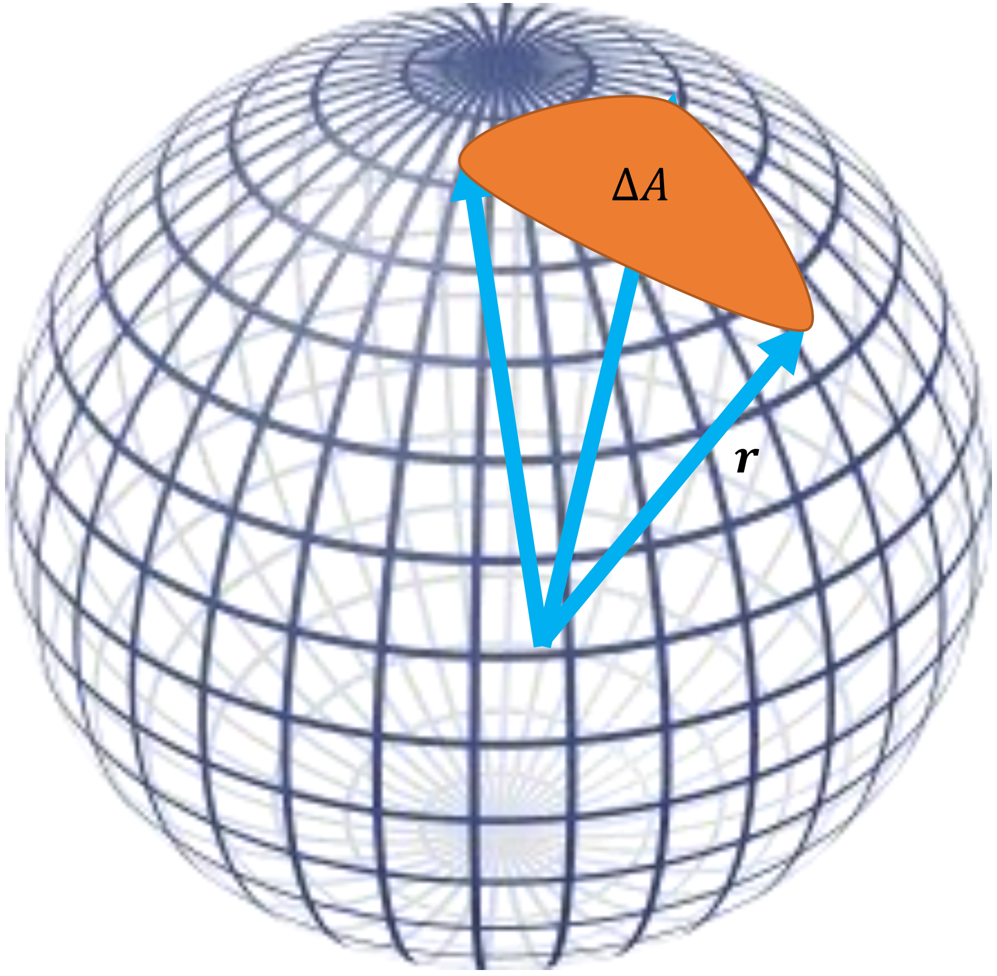


Gauss Map
$N: S \to S^2$

$$|k_G(p)| = \lim_{\Delta A \to 0} \frac{|\Delta N|}{|\Delta A|}$$



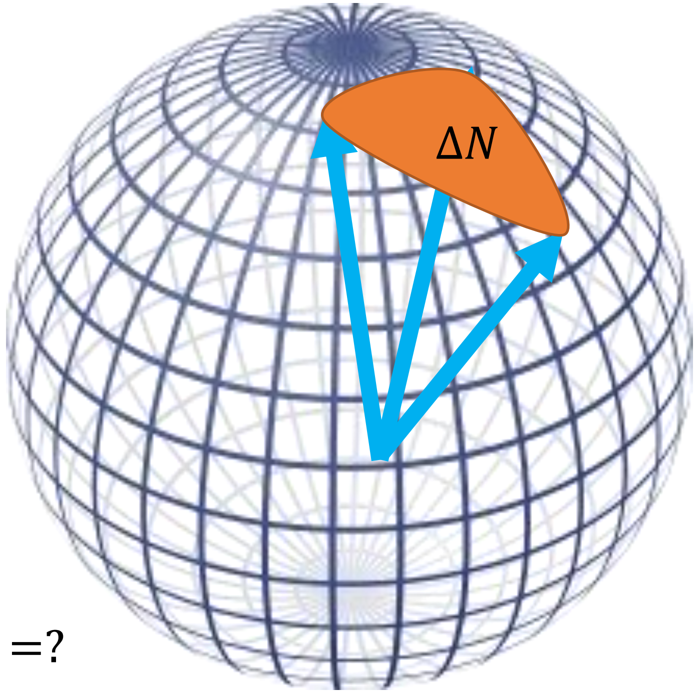
Curvature



rS^2

Gauss Map

$N: S \to S^2$

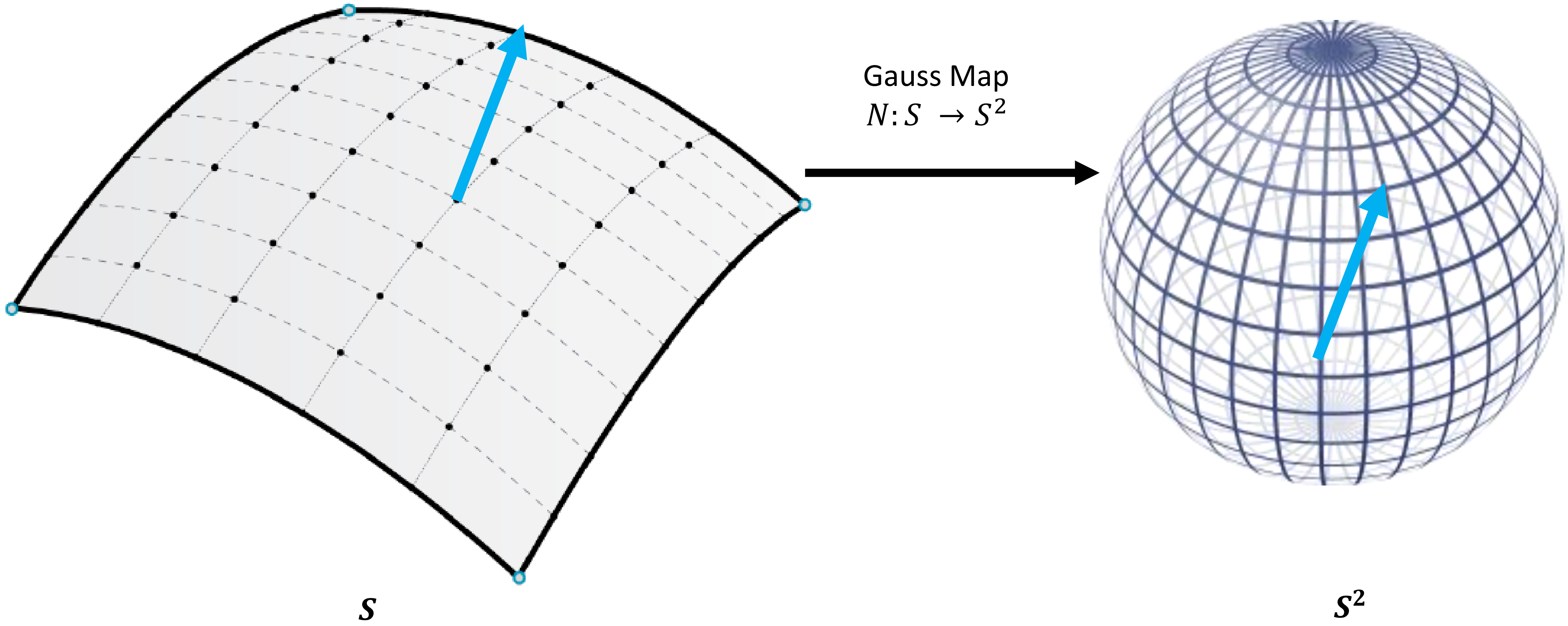


$$|k_G(p)| = \lim_{\Delta A \to 0} \frac{|\Delta N|}{|\Delta A|} =?$$

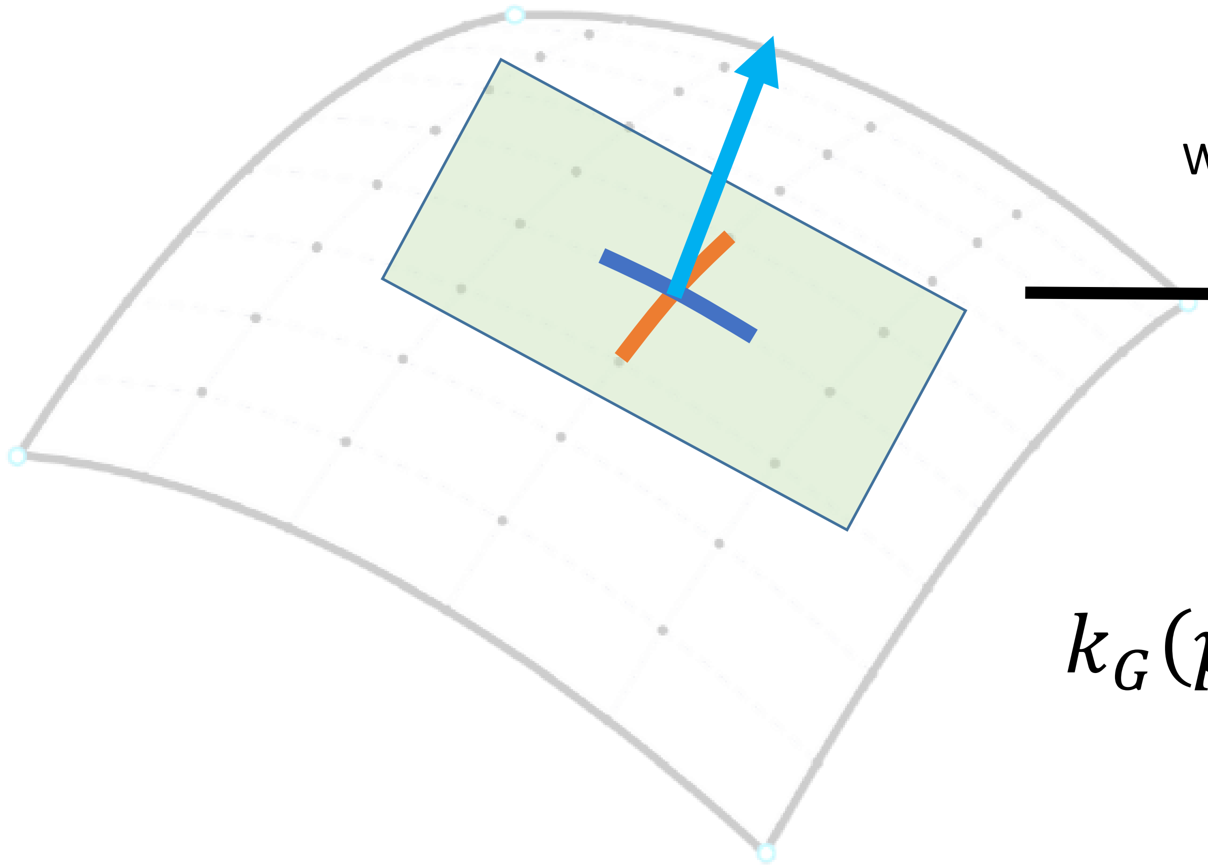
$$|k_G(p)| = \frac{1}{r^2}$$

S^2

Weingarten Map

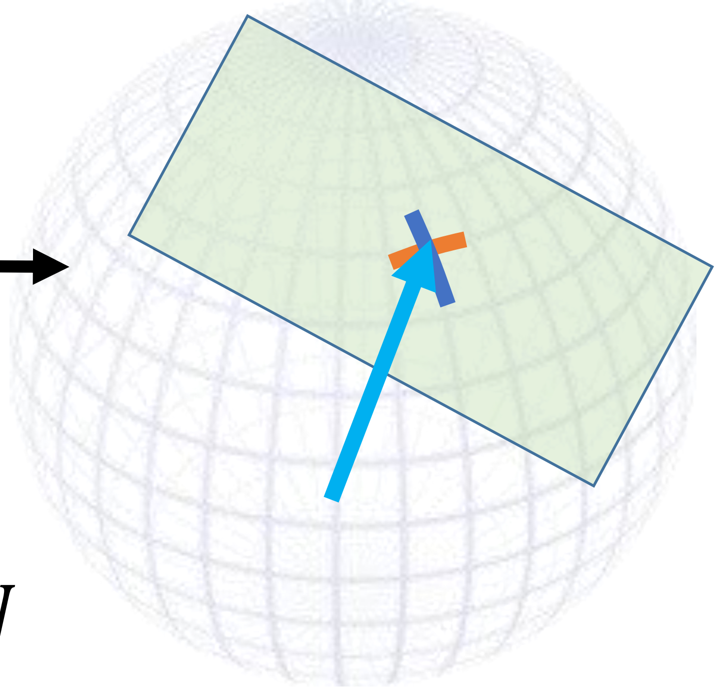


Weingarten Map



Weingarten Map

$dN : TS_p \to TS^2_{N(p)}$

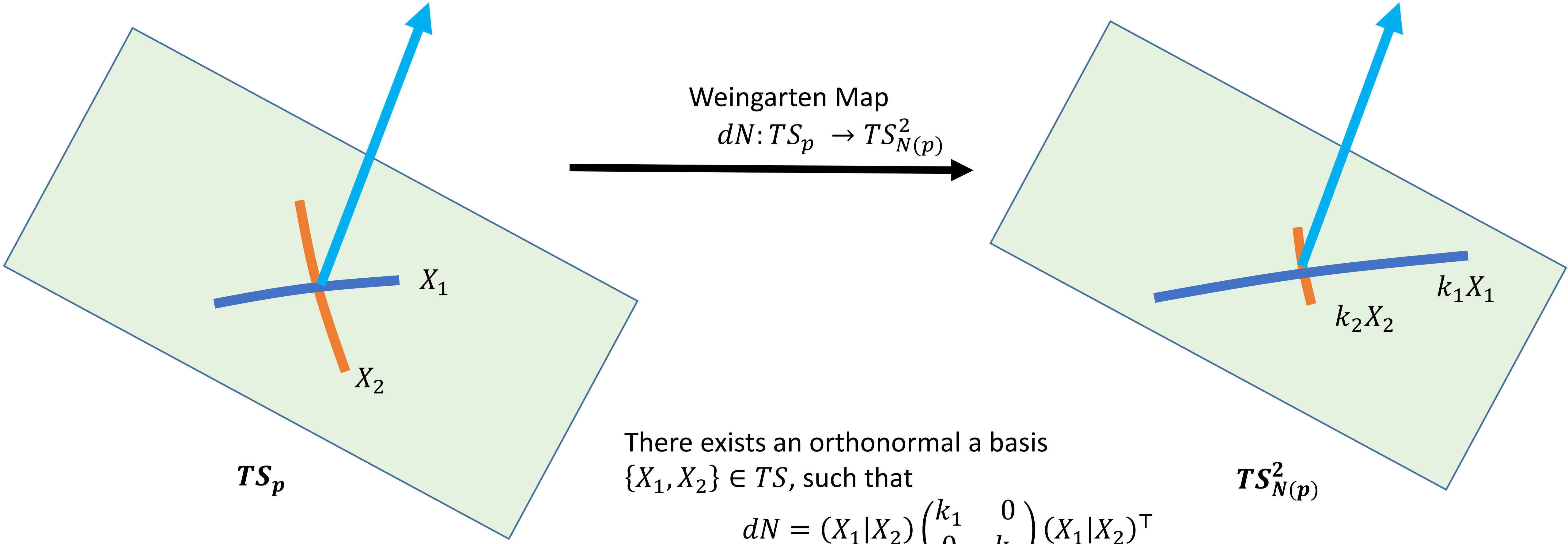


$$k_G(p) = \det dN$$

S $\qquad$ S^2

Remark: $|k_G(p)| = |\det(dN)| = \lim_{\Delta A \to 0} \frac{\Delta N}{\Delta A}$

Principal Curvatures



There exists an orthonormal a basis $\{X_1, X_2\} \in TS$, such that

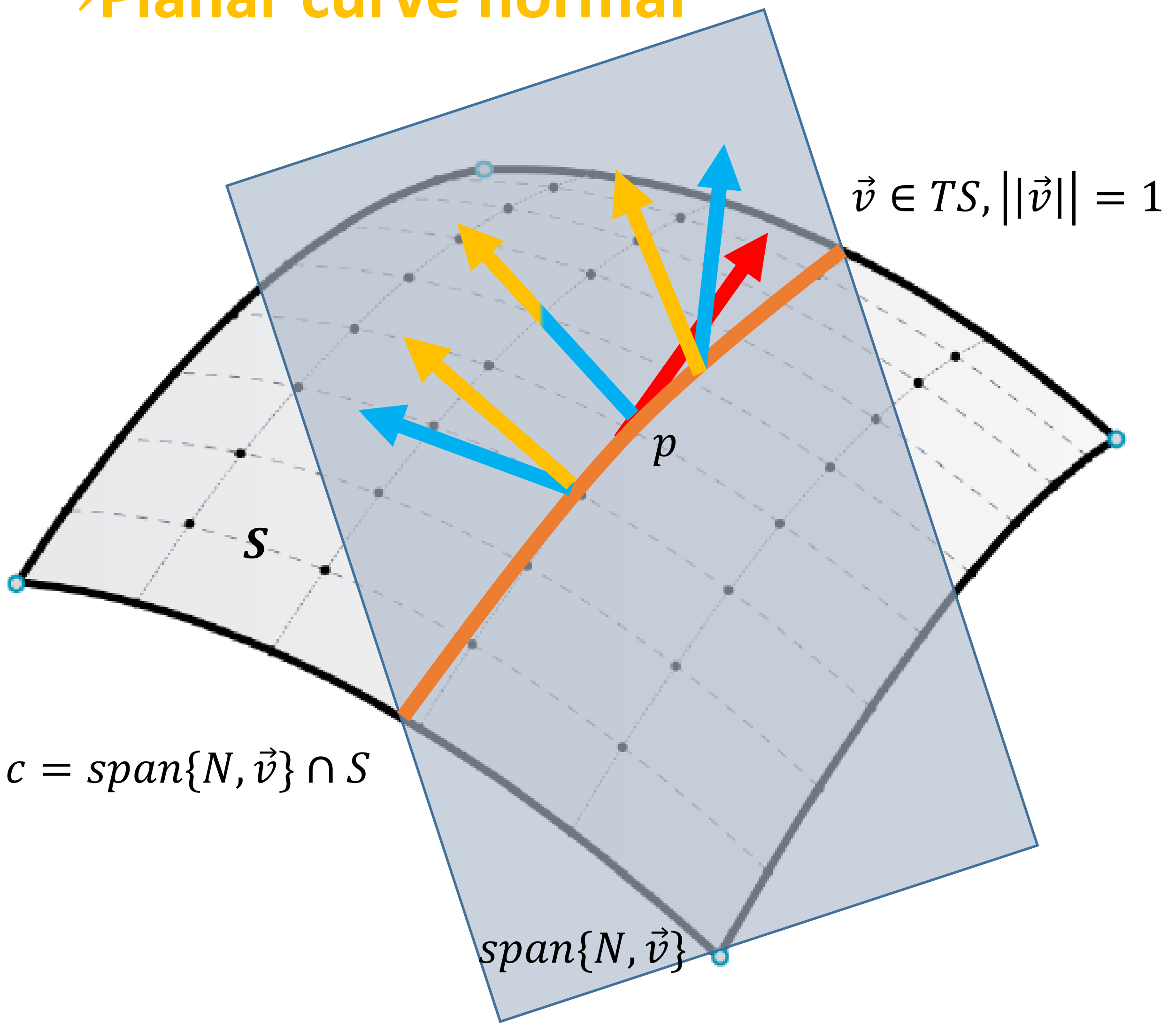
$$dN = (X_1|X_2)\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}(X_1|X_2)^\top$$

- X_1, X_2 are called principal directions
- k_1, k_2 are called principal curvatures

Normal Curvature

N → Surface normal

$\widetilde{N}$ → Planar curve normal



- Prove that $k(p; c) = \langle dN(\vec{v}), \vec{v} \rangle$

$$k(p;c) = \langle d\widetilde{N}(\vec{v}), \vec{v} \rangle = \frac{\langle (\widetilde{N} \circ \gamma)'(t), \gamma'(t) \rangle}{\langle \gamma'(t), \gamma'(t) \rangle}$$
$$= \frac{-\langle (\widetilde{N} \circ \gamma)(t), \gamma''(t) \rangle}{\langle \gamma'(t), \gamma'(t) \rangle} = \frac{-\langle (N \circ \gamma)(t), \gamma''(t) \rangle}{\langle \gamma'(t), \gamma'(t) \rangle} =$$
$$= \frac{\langle (N \circ \gamma)'(t), \gamma'(t) \rangle}{\langle \gamma'(t), \gamma'(t) \rangle} = \langle dN(\vec{v}), \vec{v} \rangle$$

- Let $\vec{v} = \cos(\theta) X_1 + \sin(\theta) X_2$. Prove,

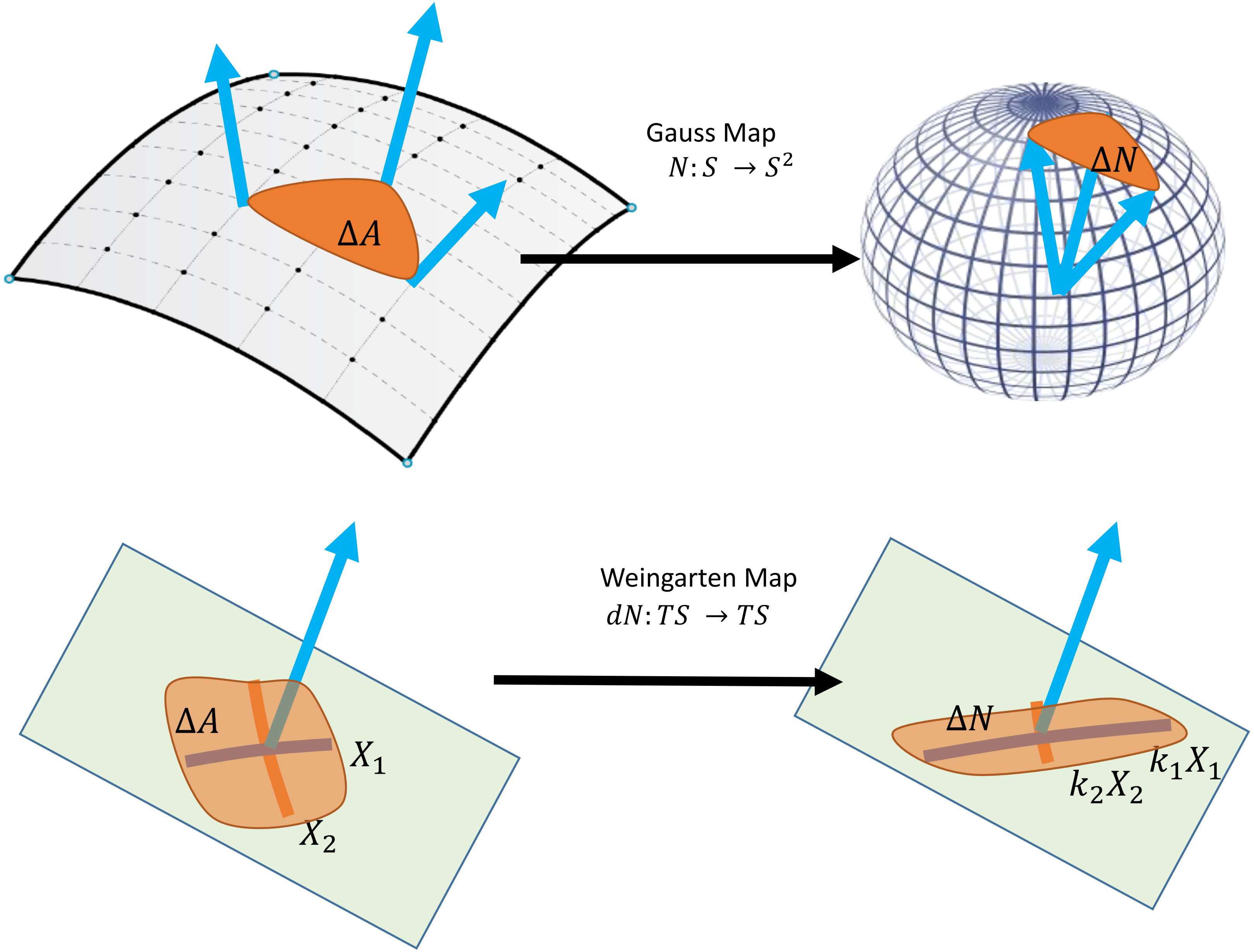
$$k(p;c) = \cos(\theta)^2\, k_1 + \sin(\theta)^2\, k_2$$

- Conclude,

$$\min(k_1, k_2) \leq k(p;c) \leq \max(k_1, k_2)$$

for all normal curves c.

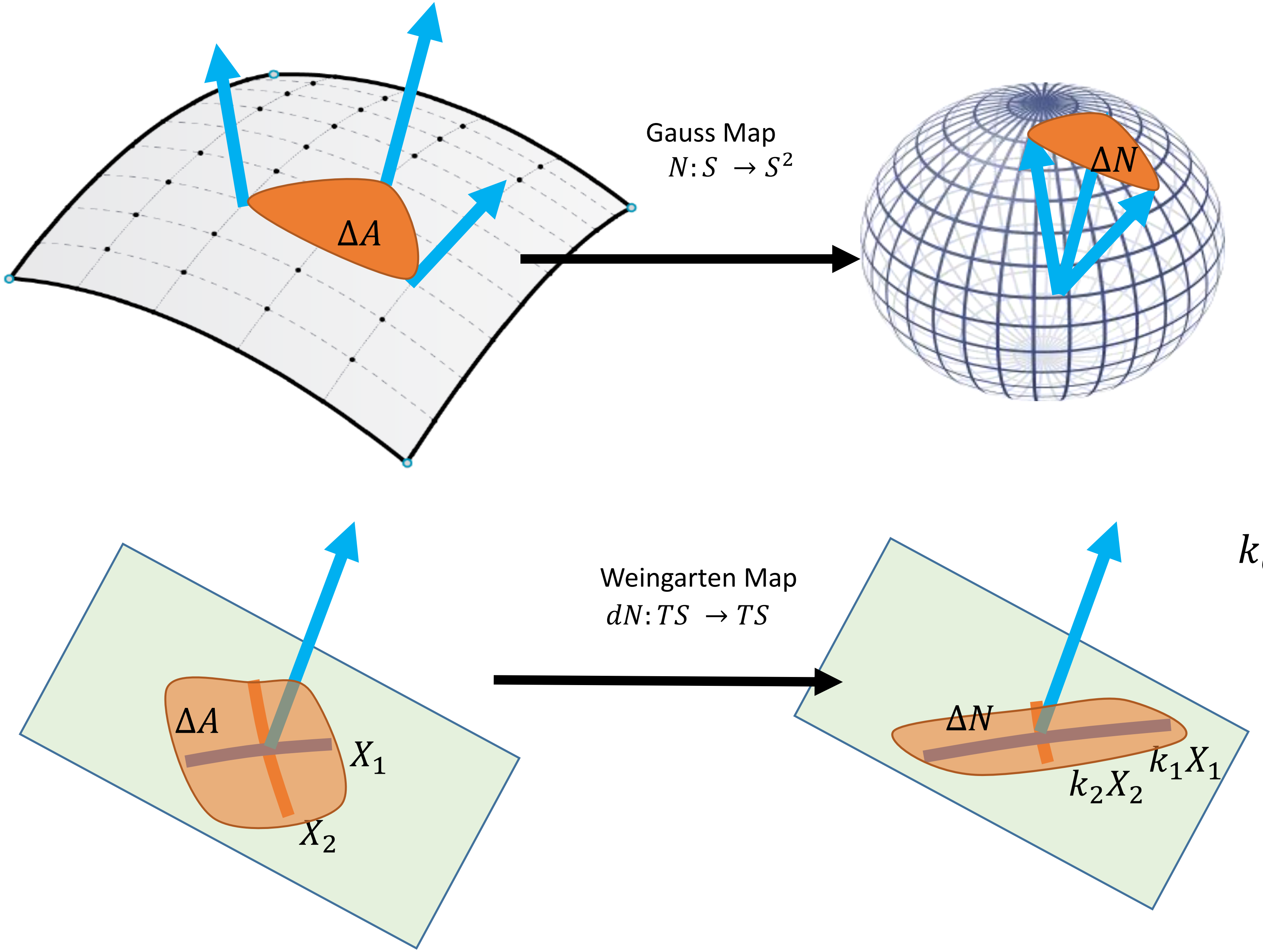
Gauss Curvature and the Weingarten Map



$$|k_G(p)| = \lim_{\Delta A \to 0} \frac{|\Delta N|}{|\Delta A|}$$

$$|k_G(p)| = |k_1 k_2|$$

Gauss Curvature and the Weingarten Map

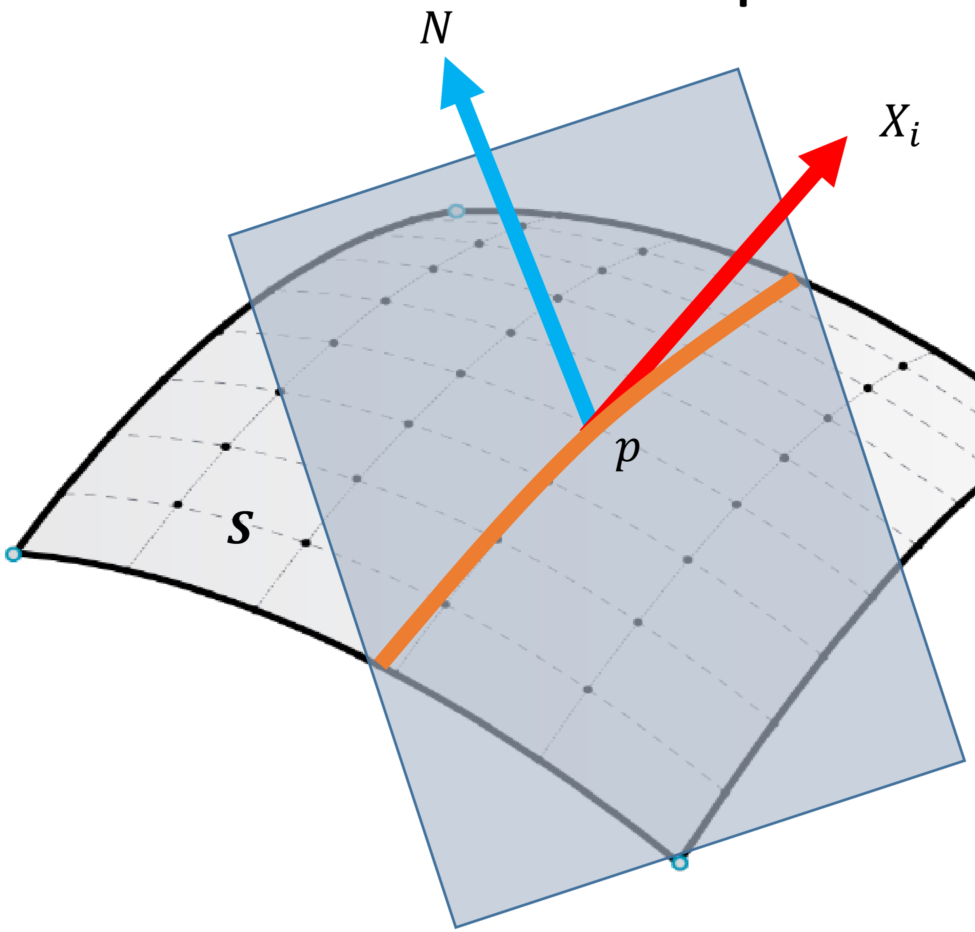


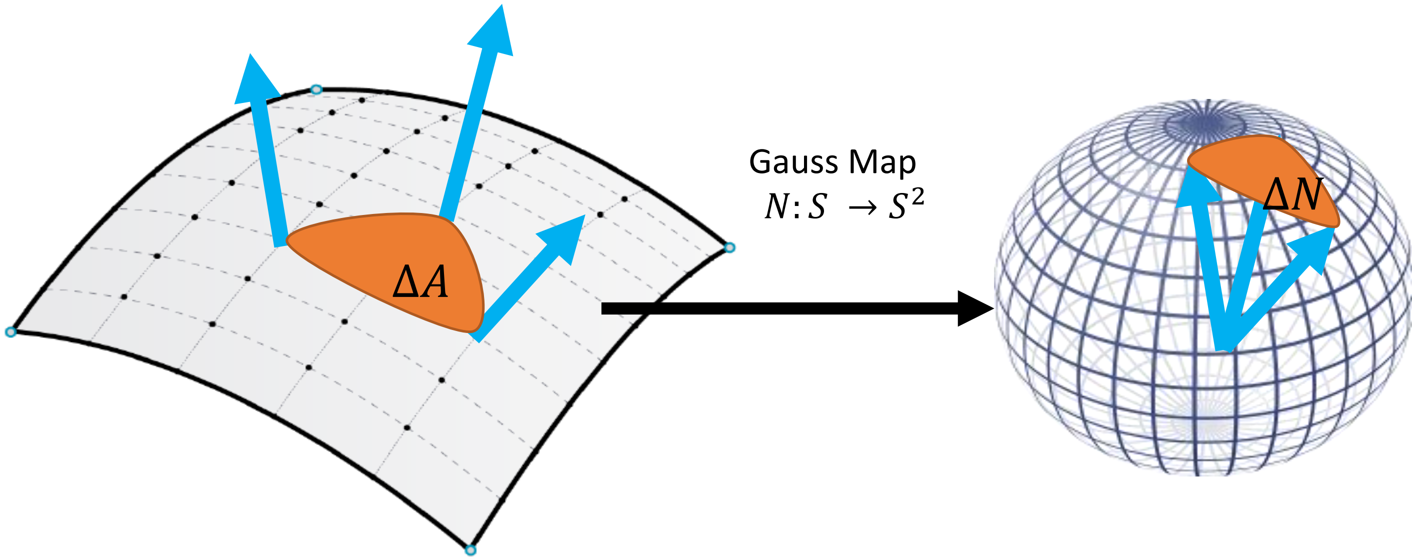
$$k_G(p) = \det(dN)$$

$$k_G(p) = \det[\,(X_1|X_2)\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}(X_1|X_2)^{\top}]$$

$$k_G(p) = k_1 k_2$$

Is orientation important?





For principal curvatures it is!

$$k_i = \langle dN(X_i), X_i \rangle \neq \langle d(-N)X_i, X_i \rangle = -k_i$$

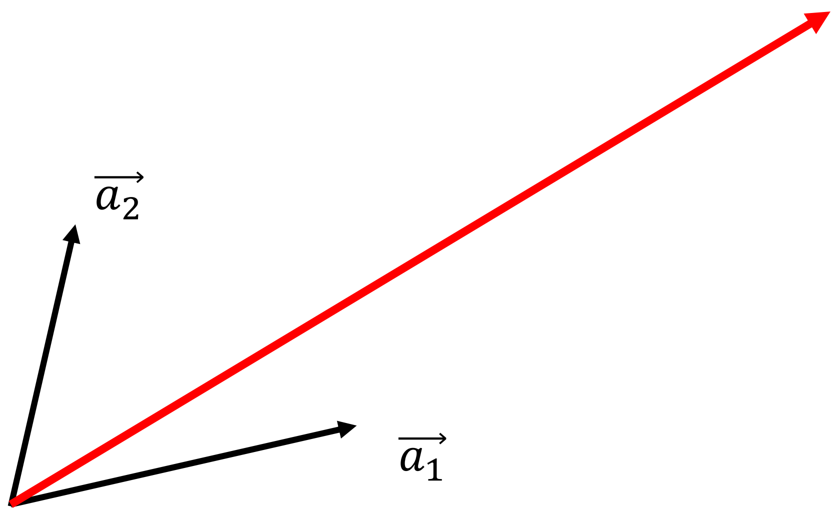
For gauss curvature is not!

$$k_G(p) = k_1 k_2 = \det(dN) = \det\big(d(-N)\big) = (-k_1)(-k_2) = k_G(p)$$

Computing surface curvatures

Exercise:

$\vec{b} = x_1\vec{a}_1 + x_2\vec{a}_2$

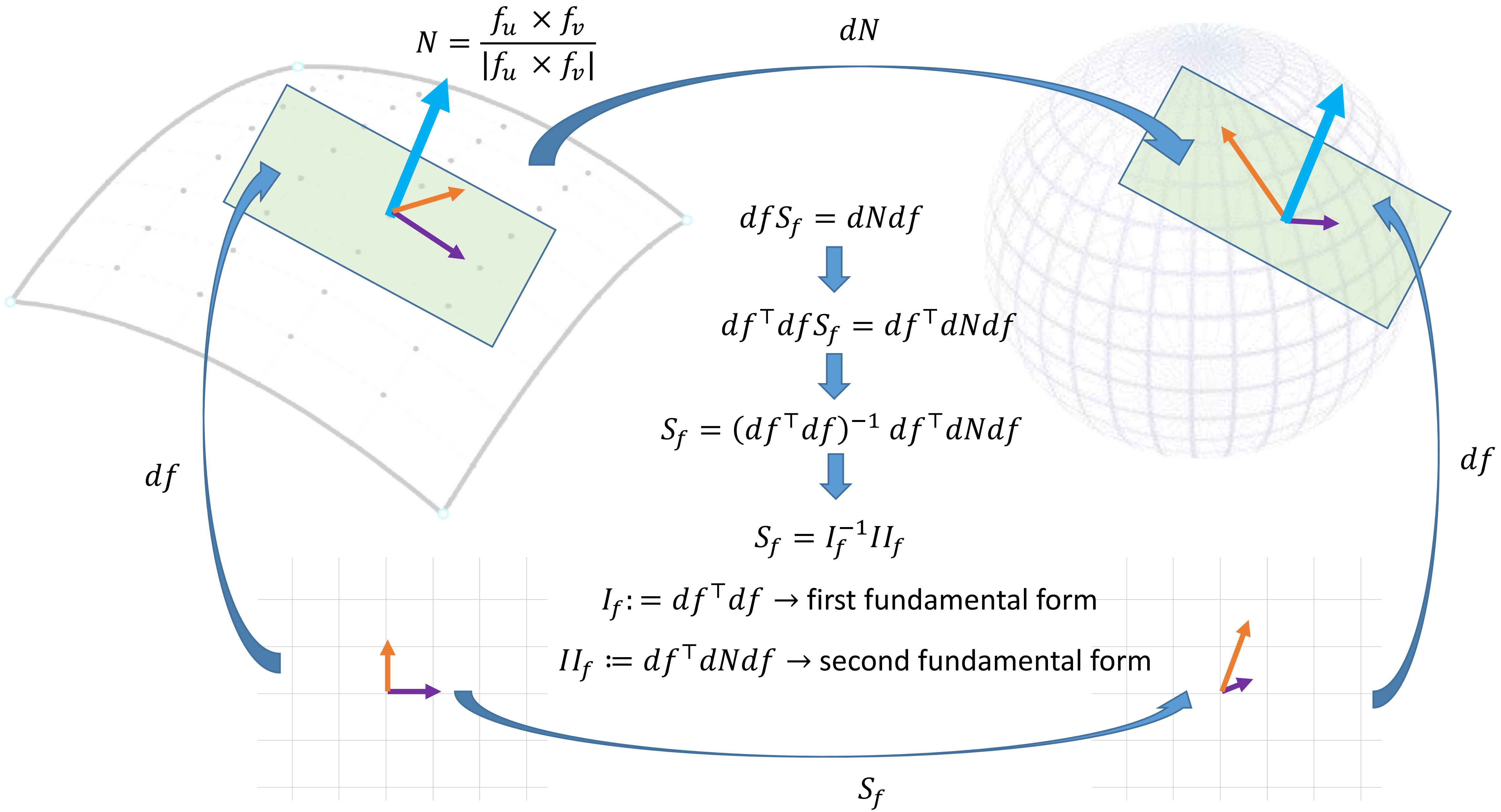


$x_1, x_2 = ?$

$$\begin{aligned}\langle b, \vec{a}_1\rangle &= x_1\langle \vec{a}_1, a_1\rangle + x_2\langle \vec{a}_2, \vec{a}_1\rangle \\ \langle \vec{b}, \vec{a}_2\rangle &= x_1\langle \vec{a}_1, \vec{a}_2\rangle + x_2\langle \vec{a}_2, \vec{a}_2\rangle\end{aligned} \Rightarrow \begin{pmatrix}\langle \vec{b}, \vec{x}_1\rangle \\ \langle \vec{b}, \vec{x}_2\rangle\end{pmatrix} = \begin{pmatrix}\langle \vec{a}_1, a_1\rangle & \langle \vec{a}_2, \vec{a}_1\rangle \\ \langle a_1, a_2\rangle & \langle \vec{a}_2, \vec{a}_2\rangle\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix} \Rightarrow \begin{pmatrix}x_1 \\ x_2\end{pmatrix} = \begin{pmatrix}\langle \vec{a}_1, a_1\rangle & \langle \vec{a}_2, \vec{a}_1\rangle \\ \langle a_1, a_2\rangle & \langle \vec{a}_2, \vec{a}_2\rangle\end{pmatrix}^{-1}\begin{pmatrix}\langle \vec{b}, \vec{a}_1\rangle \\ \langle \vec{b}, \vec{a}_2\rangle\end{pmatrix}$$

Let $A = (\vec{a}_1|\vec{a}_2)$ $\Rightarrow$ $\min_x ||Ax - b||^2$ $\Rightarrow$ $x = (A^\top A)^{-1}A^\top b$

Computing surface curvatures



$$df S_f = dN df$$

$$df^\top df S_f = df^\top dN df$$

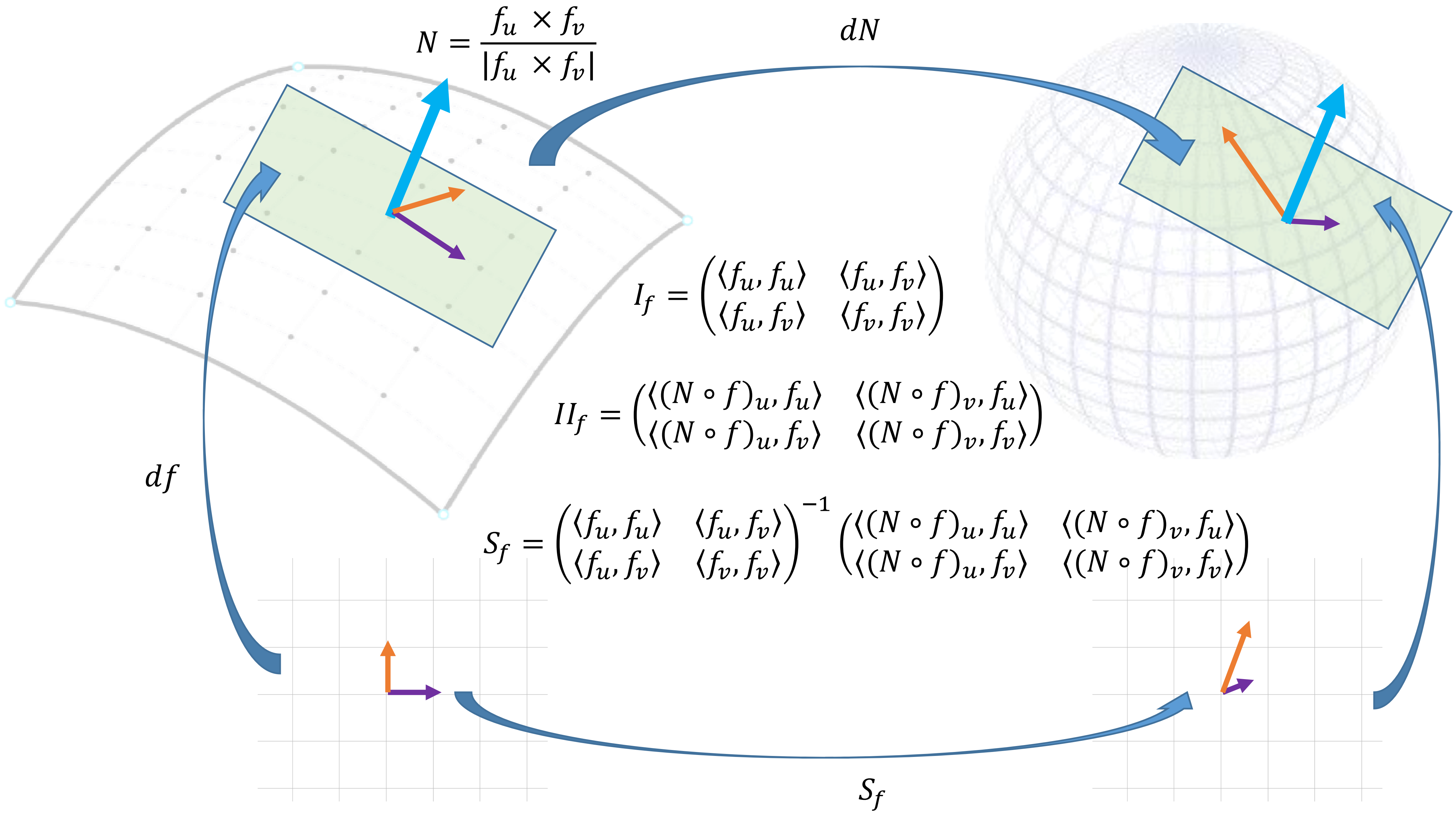
$$S_f = (df^\top df)^{-1}\, df^\top dN df$$

$$S_f = I_f^{-1} II_f$$

$I_f := df^\top df \rightarrow$ first fundamental form

$II_f \coloneqq df^\top dN df \rightarrow$ second fundamental form

Computing surface curvatures



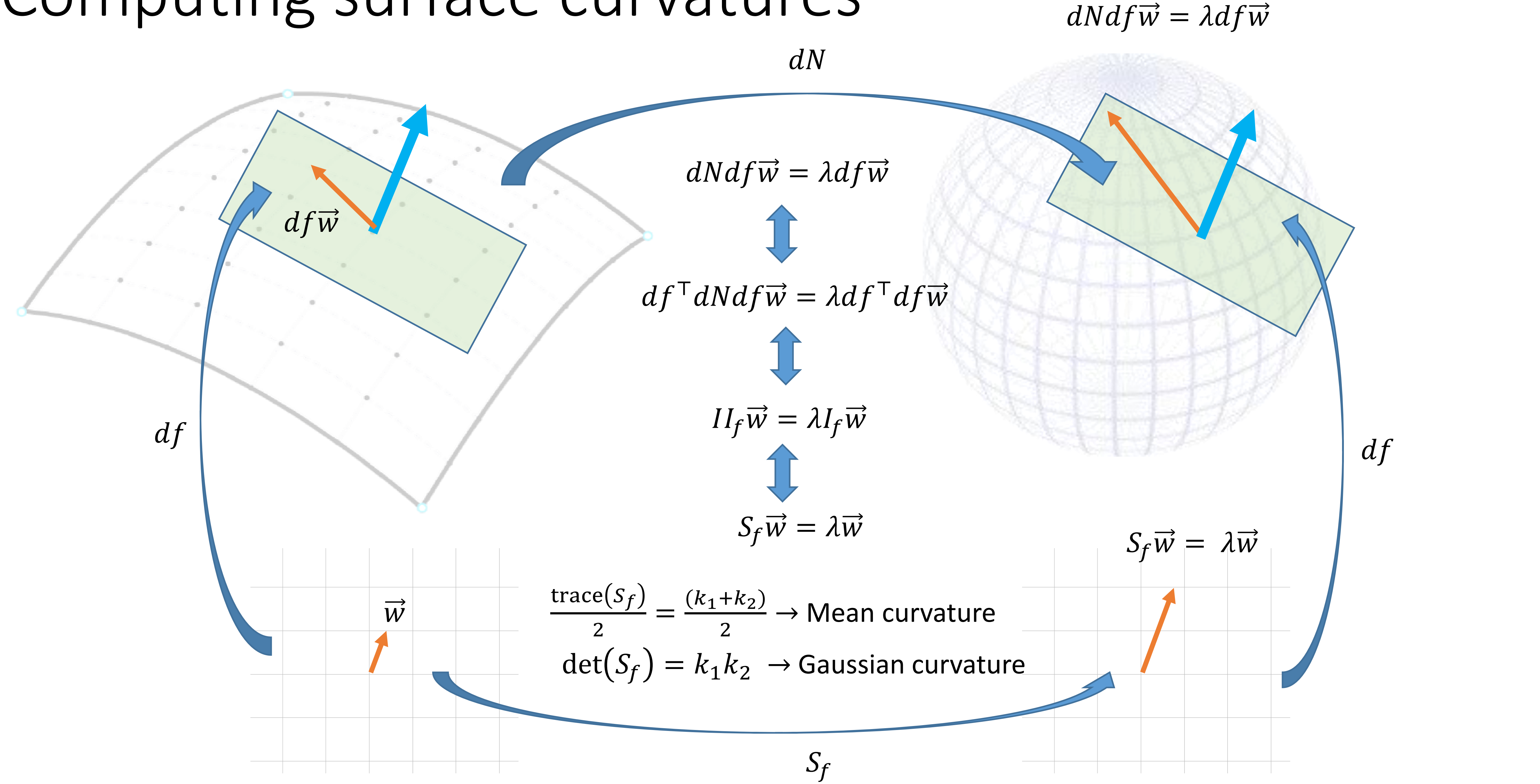
$$N = \frac{f_u \times f_v}{|f_u \times f_v|}$$

$$I_f = \begin{pmatrix} \langle f_u, f_u \rangle & \langle f_u, f_v \rangle \\ \langle f_u, f_v \rangle & \langle f_v, f_v \rangle \end{pmatrix}$$

$$II_f = \begin{pmatrix} \langle (N \circ f)_u, f_u \rangle & \langle (N \circ f)_v, f_u \rangle \\ \langle (N \circ f)_u, f_v \rangle & \langle (N \circ f)_v, f_v \rangle \end{pmatrix}$$

$$S_f = \begin{pmatrix} \langle f_u, f_u \rangle & \langle f_u, f_v \rangle \\ \langle f_u, f_v \rangle & \langle f_v, f_v \rangle \end{pmatrix}^{-1} \begin{pmatrix} \langle (N \circ f)_u, f_u \rangle & \langle (N \circ f)_v, f_u \rangle \\ \langle (N \circ f)_u, f_v \rangle & \langle (N \circ f)_v, f_v \rangle \end{pmatrix}$$

Computing surface curvatures



Computing surface curvatures

Exercise:

Given a parametrization $f\colon \Omega \subset R^2 \to S$, prove that

$$\langle (N \circ f)_u, f_u \rangle = -\langle N, f_{uu} \rangle$$
$$\langle (N \circ f)_v, f_v \rangle = -\langle N, f_{vv} \rangle$$
$$\langle (N \circ f)_u, f_v \rangle = -\langle N, f_{vu} \rangle$$
$$\langle (N \circ f)_v, f_u \rangle = -\langle N, f_{uv} \rangle$$

Conclude, the second fundamental form is symmetric and equivalent to,

$$II_f = -\begin{pmatrix} \langle N, f_{uu} \rangle & \langle N, f_{uv} \rangle \\ \langle N, f_{uv} \rangle & \langle N, f_{uv} \rangle \end{pmatrix}$$

Then,

$$S_f = -\begin{pmatrix} \langle f_u, f_u \rangle & \langle f_u, f_v \rangle \\ \langle f_u, f_v \rangle & \langle f_v, f_v \rangle \end{pmatrix}^{-1} \begin{pmatrix} \langle N, f_{uu} \rangle & \langle N, f_{uv} \rangle \\ \langle N, f_{uv} \rangle & \langle N, f_{uv} \rangle \end{pmatrix}$$

Principal Curvatures

Exercise:

Prove that $\langle dNX, Y\rangle = \langle X, dNY\rangle$ for all $X, Y \in T_pS$.

Hint: Write $X = df\omega_x,\ Y = df\omega_y$.

Exercise:

If $dNX_1 = k_1X_1$ and $dNX_2 = k_2X_2$ with $k_1 \neq k_2$, then

$$\langle X_1, X_2\rangle = 0$$