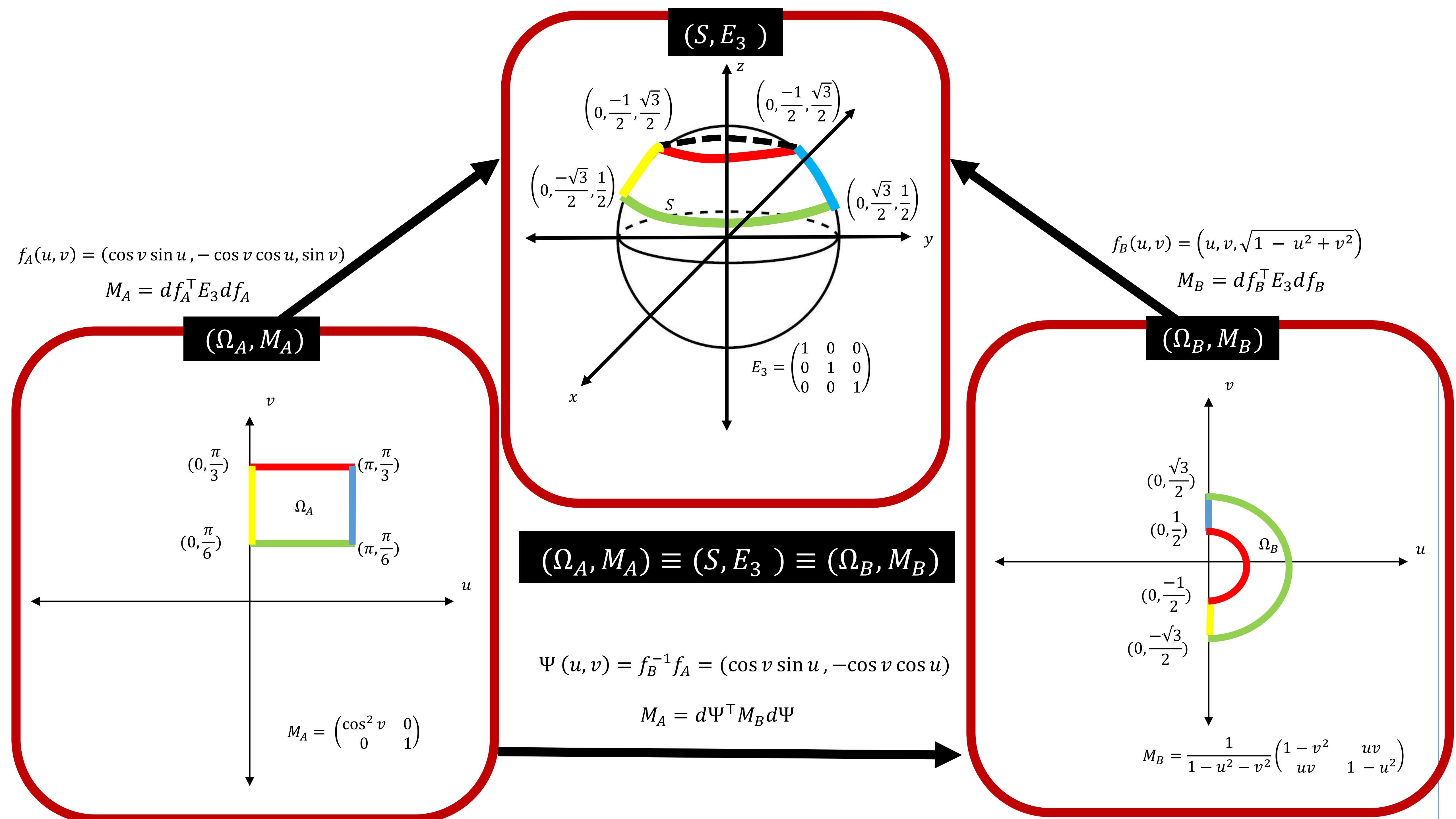
Lecture 3

Introduction to Geometry Processing

Spring 2017

Johns Hopkins University

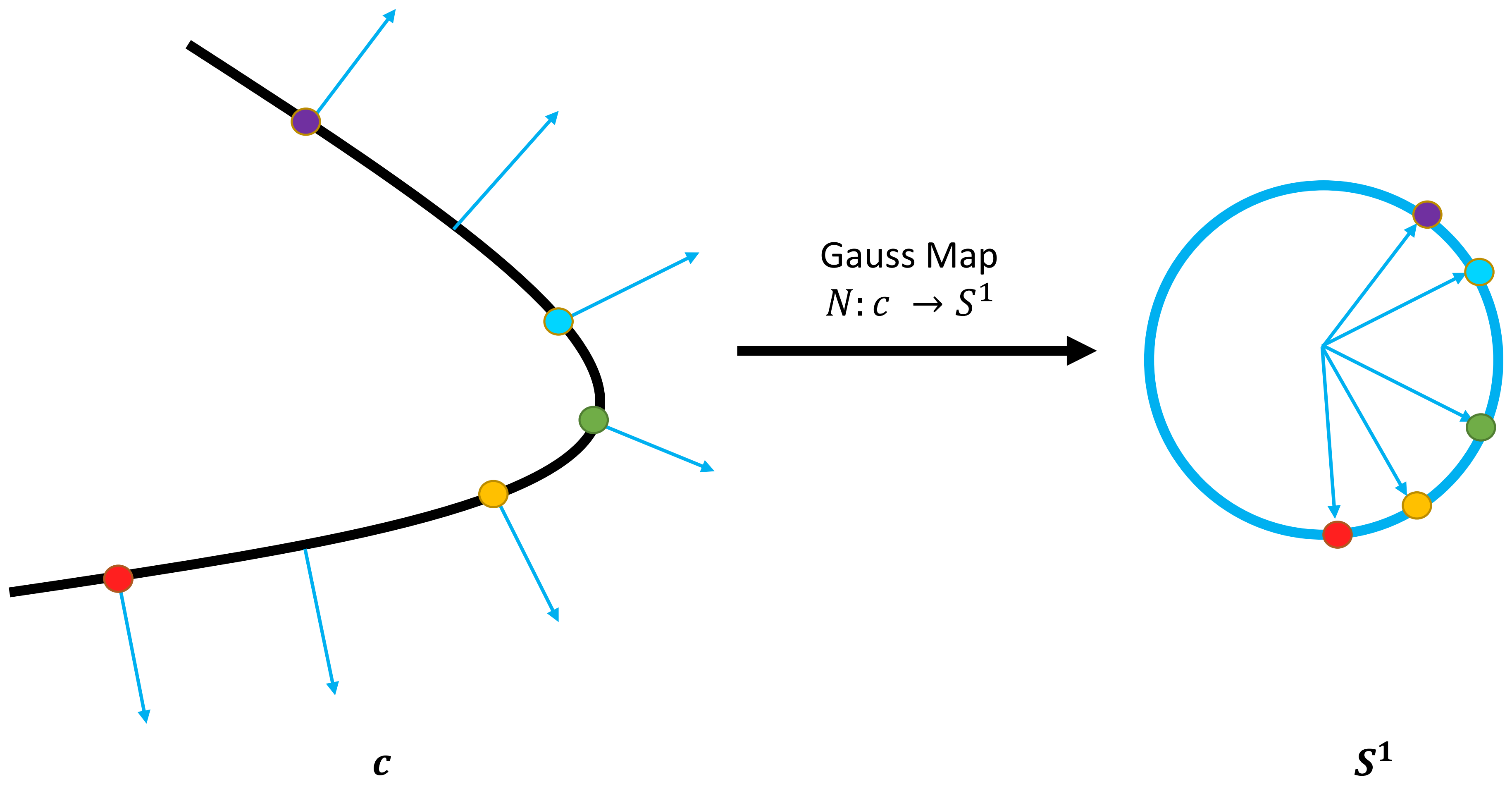


(S, E_3)
z
$\left(0, \frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(0, \frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(0, \frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$
$\left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
S
y
x
$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$f_A(u, v) = (\cos v \sin u, -\cos v \cos u, \sin v)$
$M_A = df_A^\top E_3 df_A$
$f_B(u, v) = \left(u, v, \sqrt{1 - u^2 + v^2}\right)$
$M_B = df_B^\top E_3 df_B$
(Ω_A, M_A)
v
$(0, \frac{\pi}{3})$
$(\pi, \frac{\pi}{3})$
Ω_A
$(0, \frac{\pi}{6})$
$(\pi, \frac{\pi}{6})$
u
$M_A = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$
$(\Omega_A, M_A) \equiv (S, E_3) \equiv (\Omega_B, M_B)$
$\Psi(u, v) = f_B^{-1} f_A = (\cos v \sin u, -\cos v \cos u)$
$M_A = d\Psi^\top M_B d\Psi$
(Ω_B, M_B)
v
$(0, \frac{\sqrt{3}}{2})$
$(0, \frac{1}{2})$
Ω_B
u
$(0, \frac{-1}{2})$
$(0, \frac{-\sqrt{3}}{2})$
$M_B = \frac{1}{1 - u^2 - v^2} \begin{pmatrix} 1 - v^2 & uv \\ uv & 1 - u^2 \end{pmatrix}$

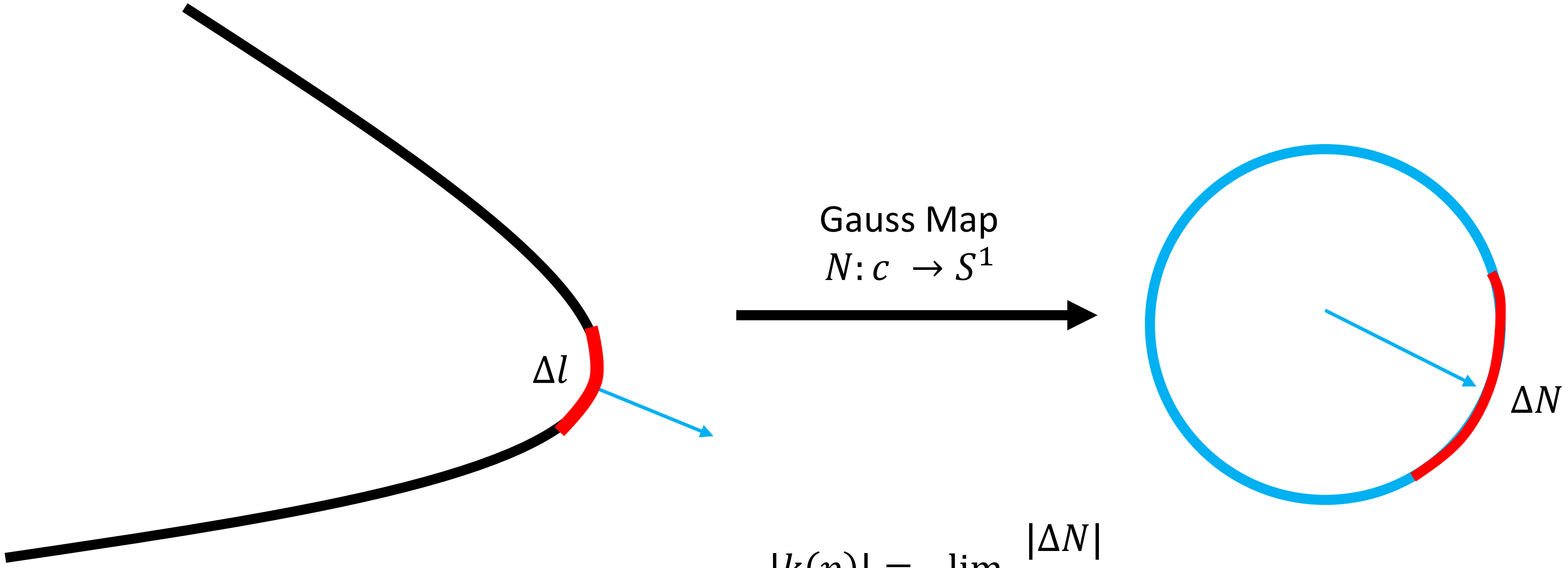
Johann Carl Friedrich Gauss
(1777- 1855)

- Number theory
- Algebra
- Statistics
- Analysis
- **Differential geometry**
- Mechanics
- Electrostatics
- Astronomy
- Matrix theory
- Optics

Planar Curves

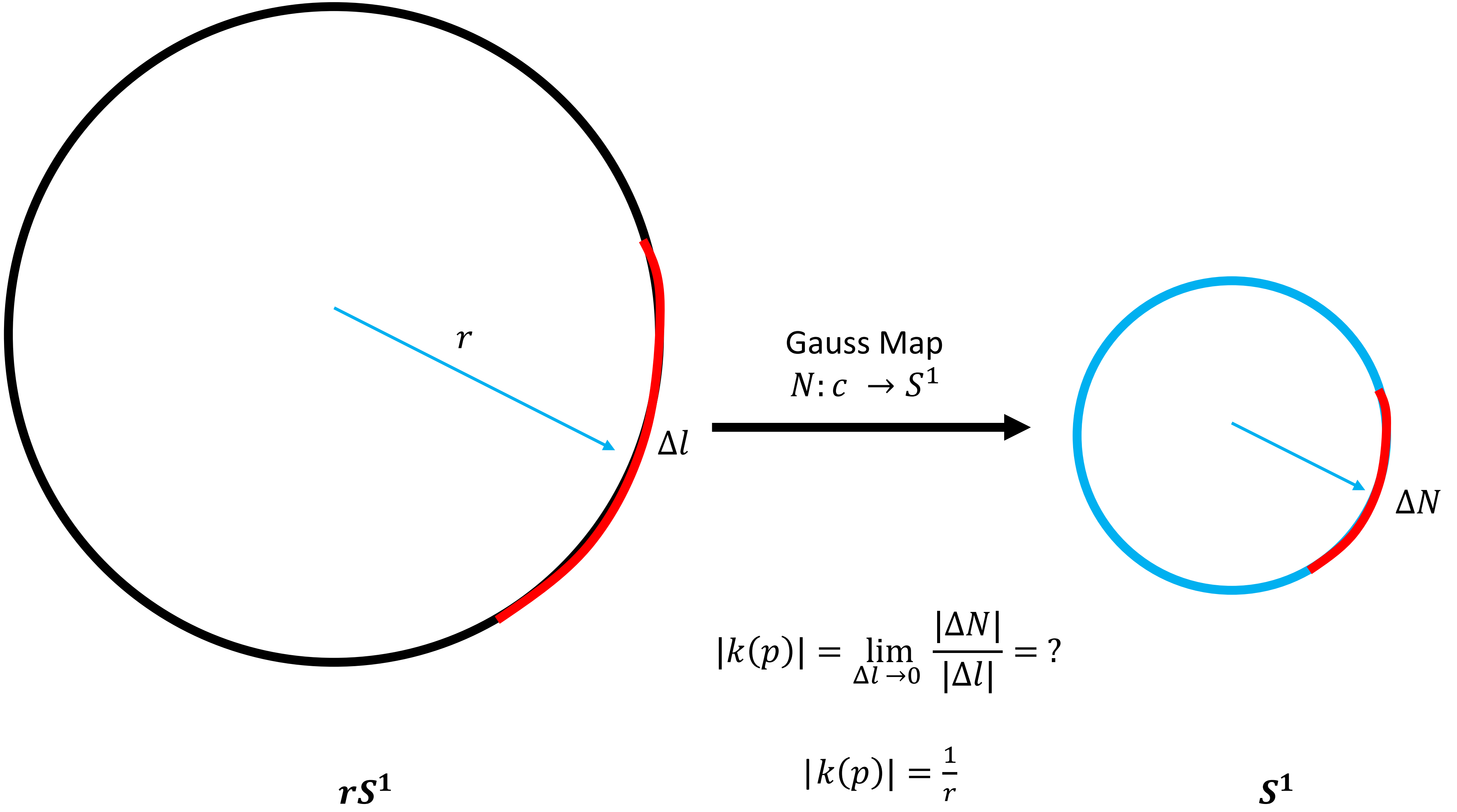


Curvature



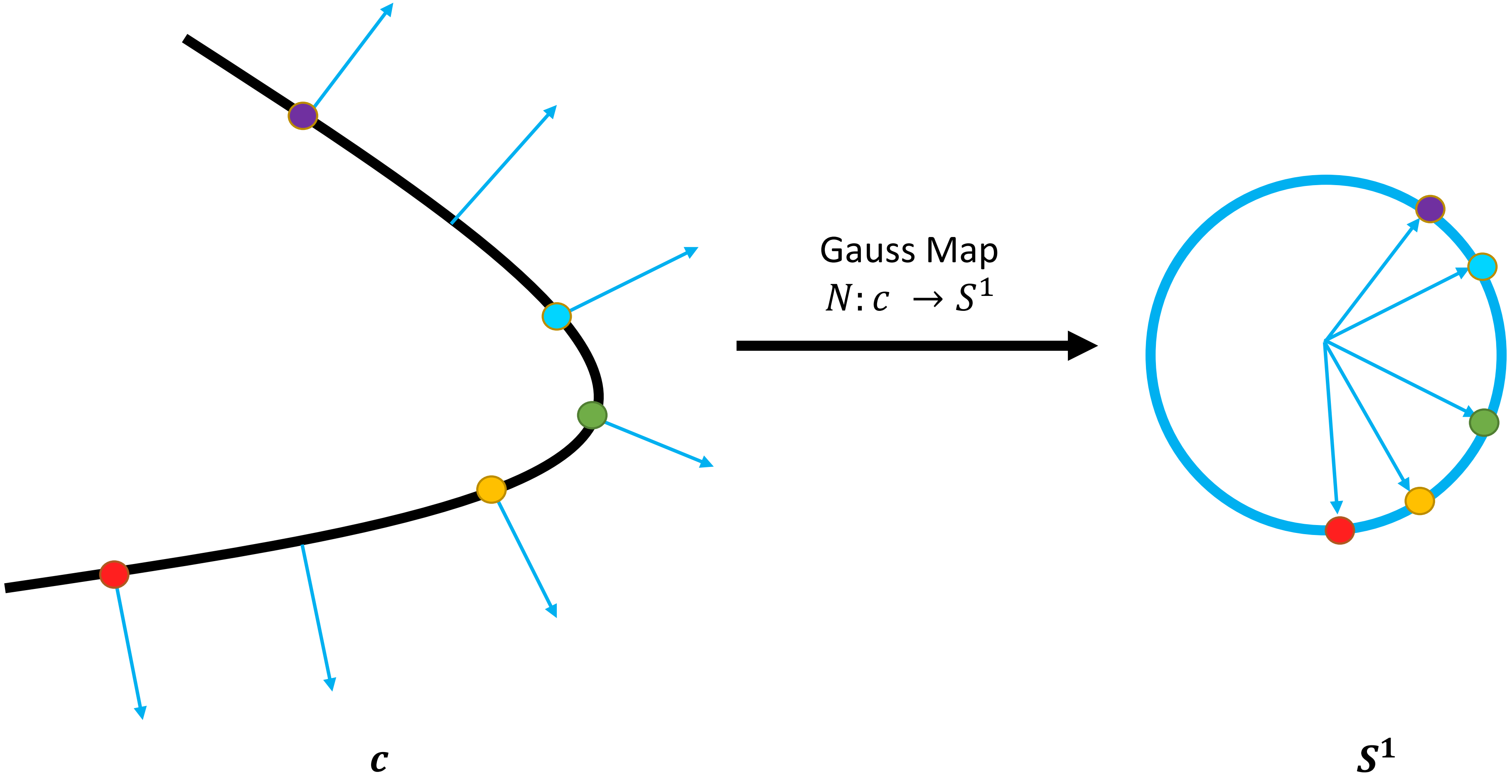
$$|k(p)| = \lim_{\Delta l \to 0} \frac{|\Delta N|}{|\Delta l|}$$

$\boldsymbol{c}$ $\boldsymbol{S^1}$

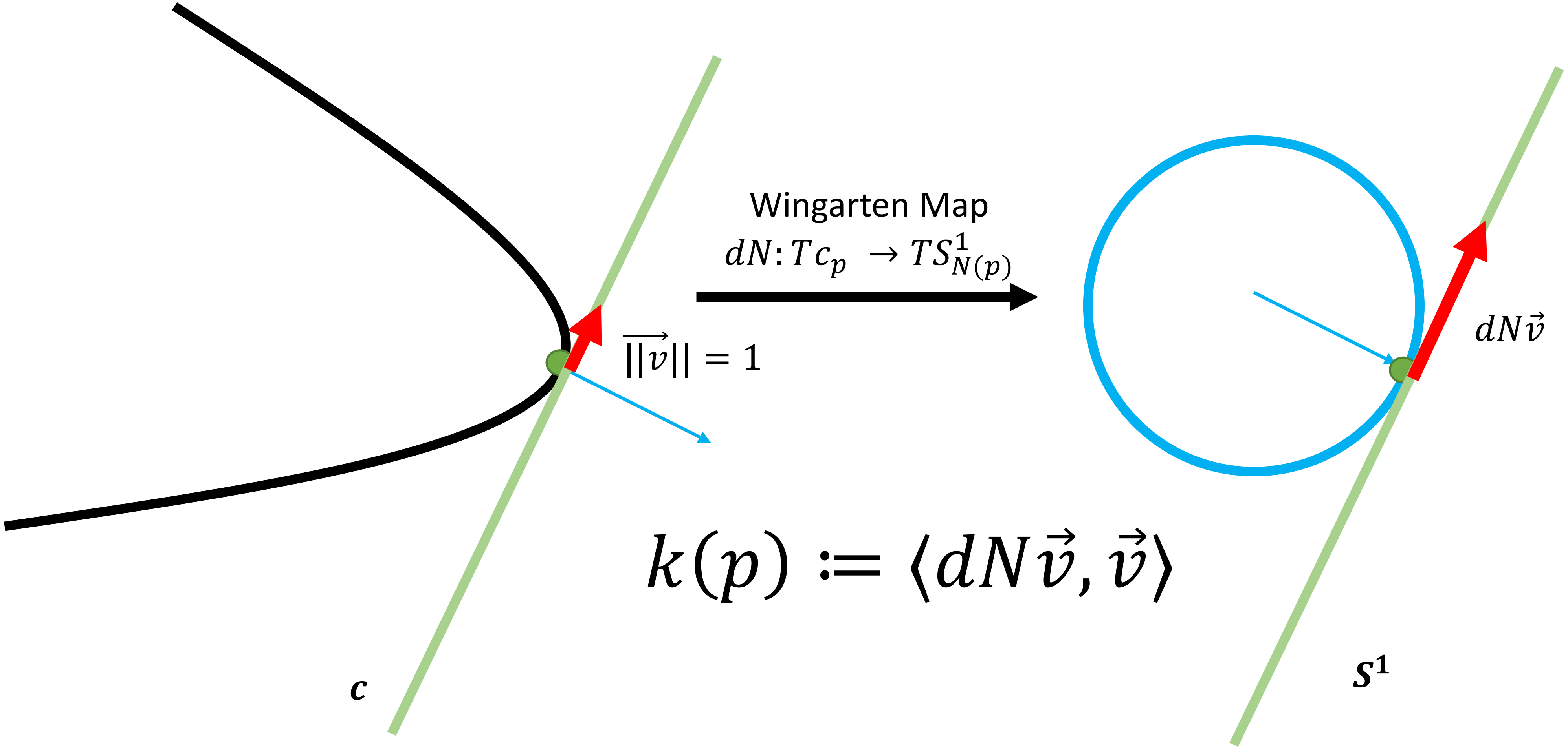


r
Δl
Gauss Map
$N: c \to S^1$
ΔN
$|k(p)| = \lim_{\Delta l \to 0} \frac{|\Delta N|}{|\Delta l|} = ?$
$|k(p)| = \frac{1}{r}$
$\boldsymbol{rS^1}$
$\boldsymbol{S^1}$

Weingarten Map



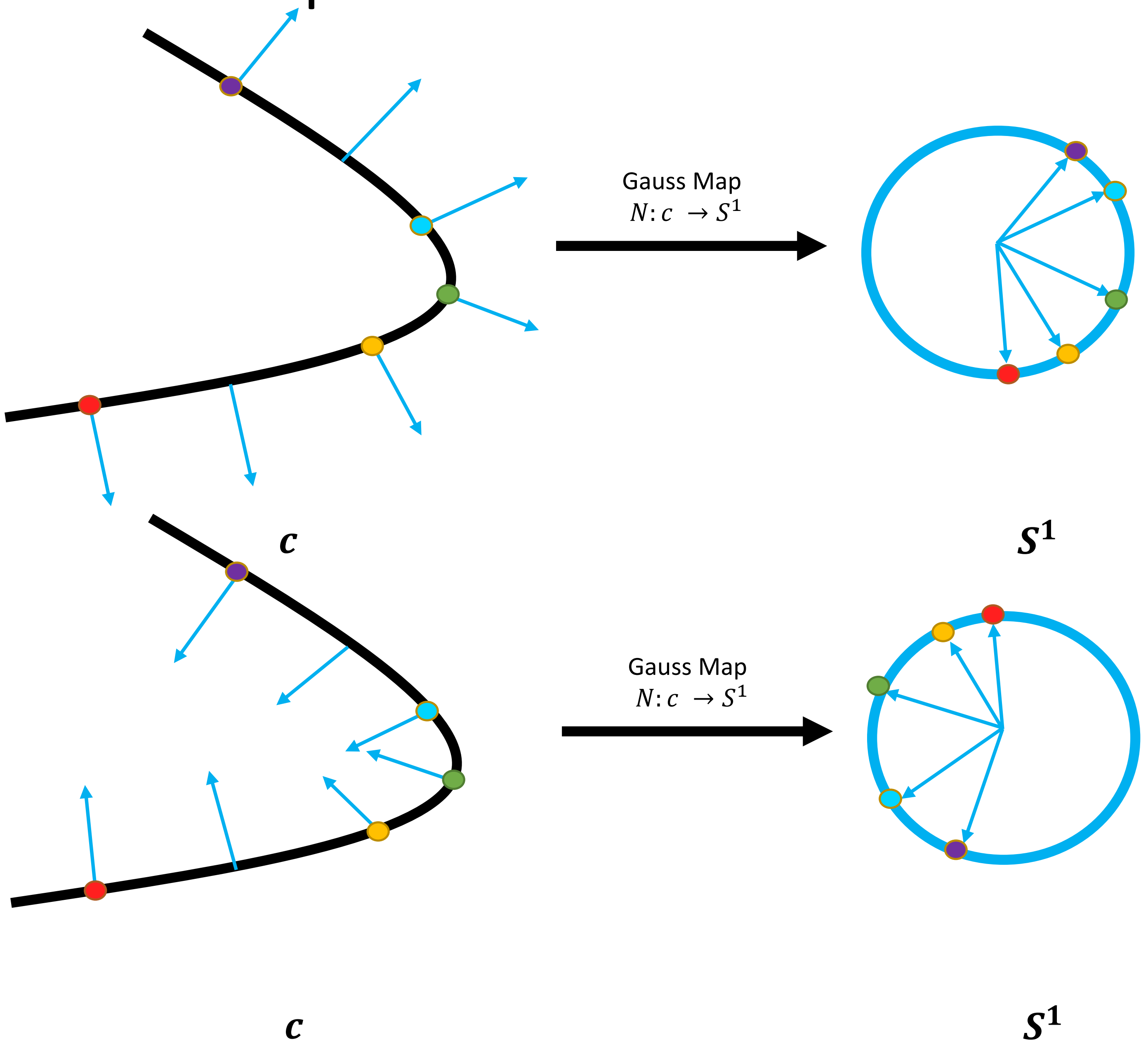
Weingarten Map



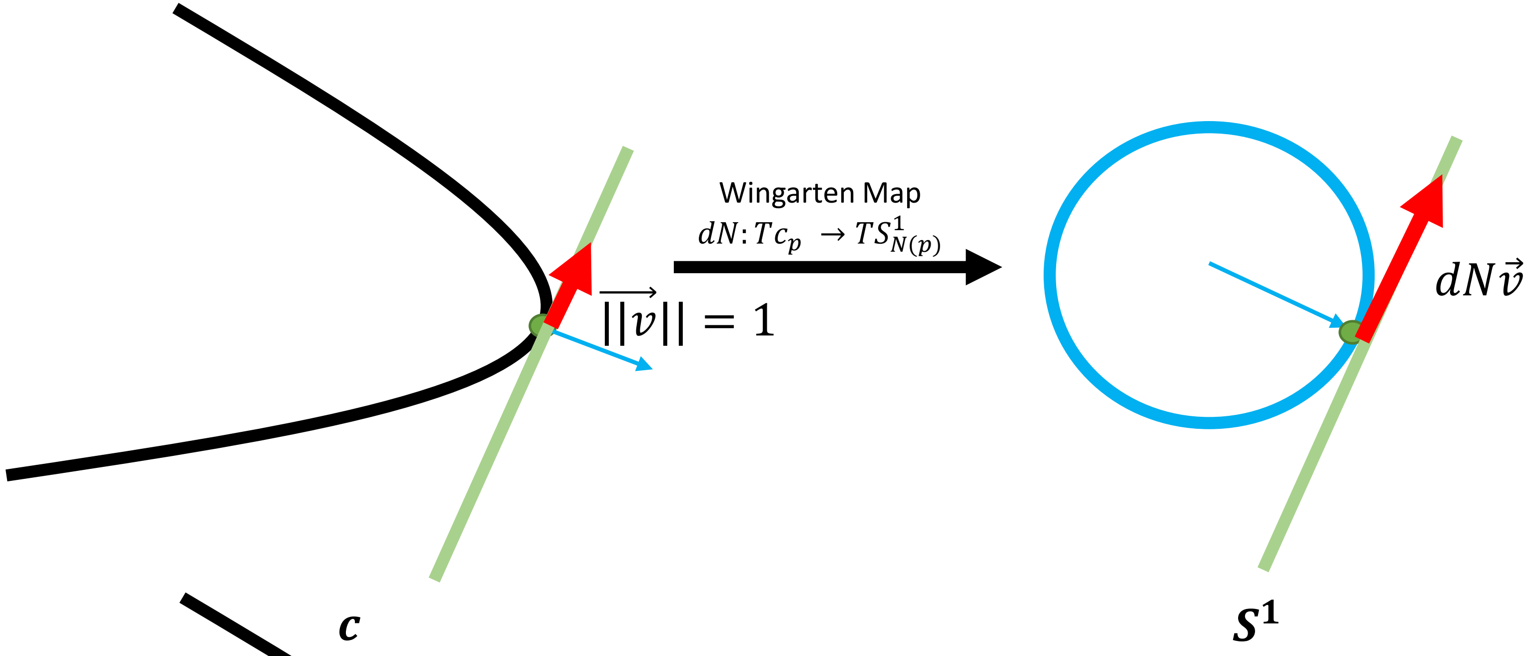
$$k(p) \coloneqq \langle dN\vec{v}, \vec{v}\rangle$$

Remark $|k(p)| = |\langle dN\vec{v}, \vec{v}\rangle| = |dN\vec{v}| = \lim_{\Delta l \to 0} \frac{|\Delta N|}{|\Delta l|}$

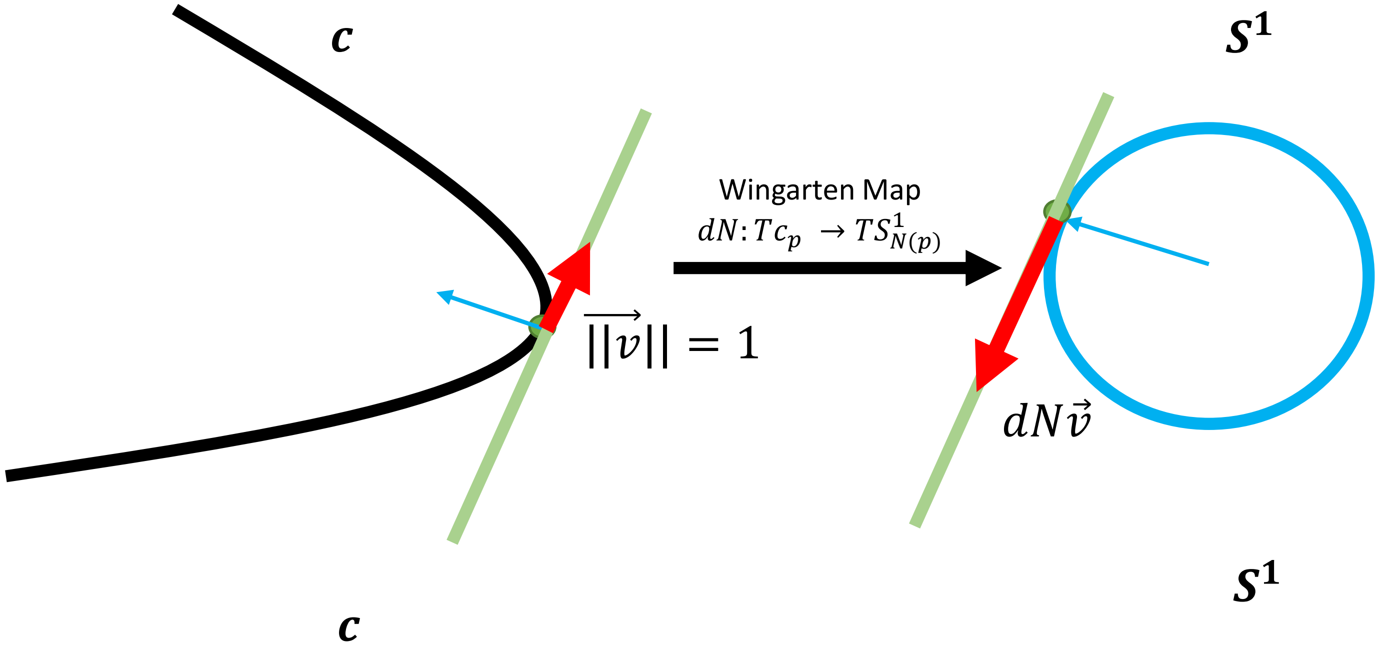
Orientation is important!



Orientation is important!



$k(p) = \langle dN\vec{v}, \vec{v} \rangle > 0$

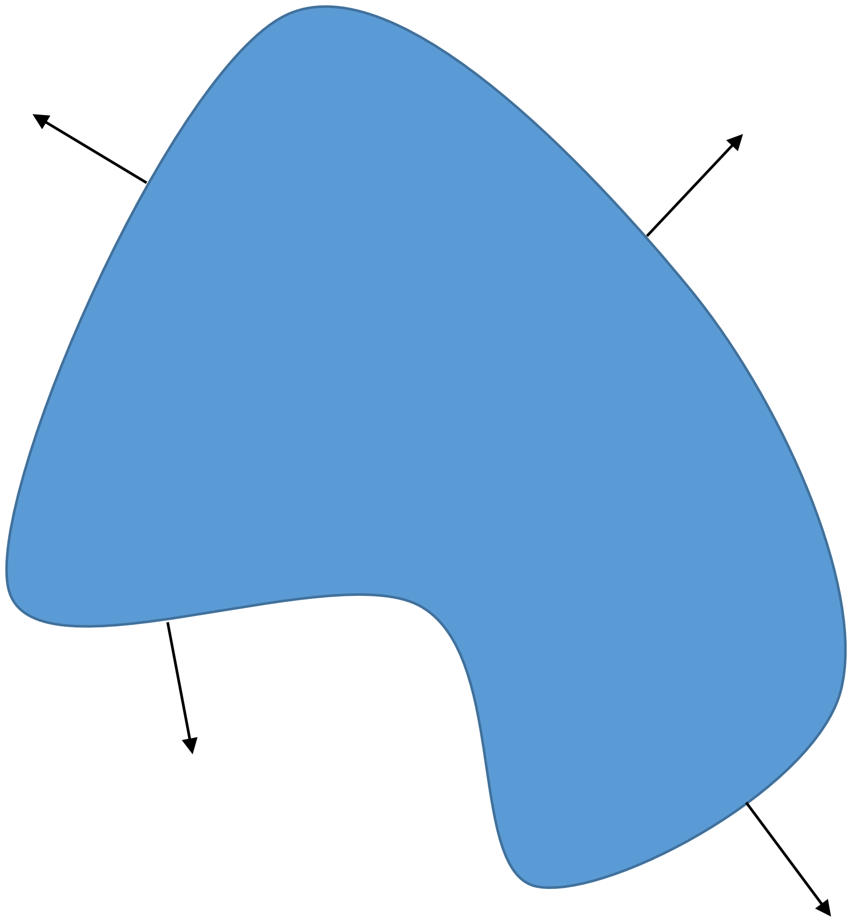


$k(p) = \langle dN\vec{v}, \vec{v} \rangle < 0$

Where the orientation come from?

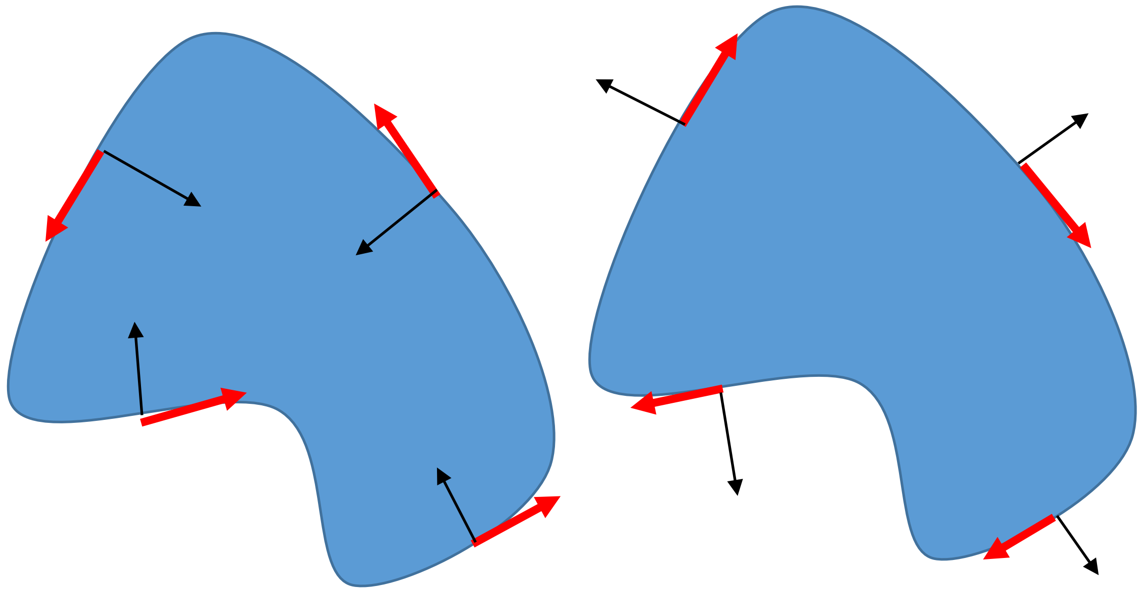
(1) Prescribed

e.g., to point to the exterior of a region.



(2) Induced by a parametrization

e.g., the 90 degree rotation of the tangent unit vector.



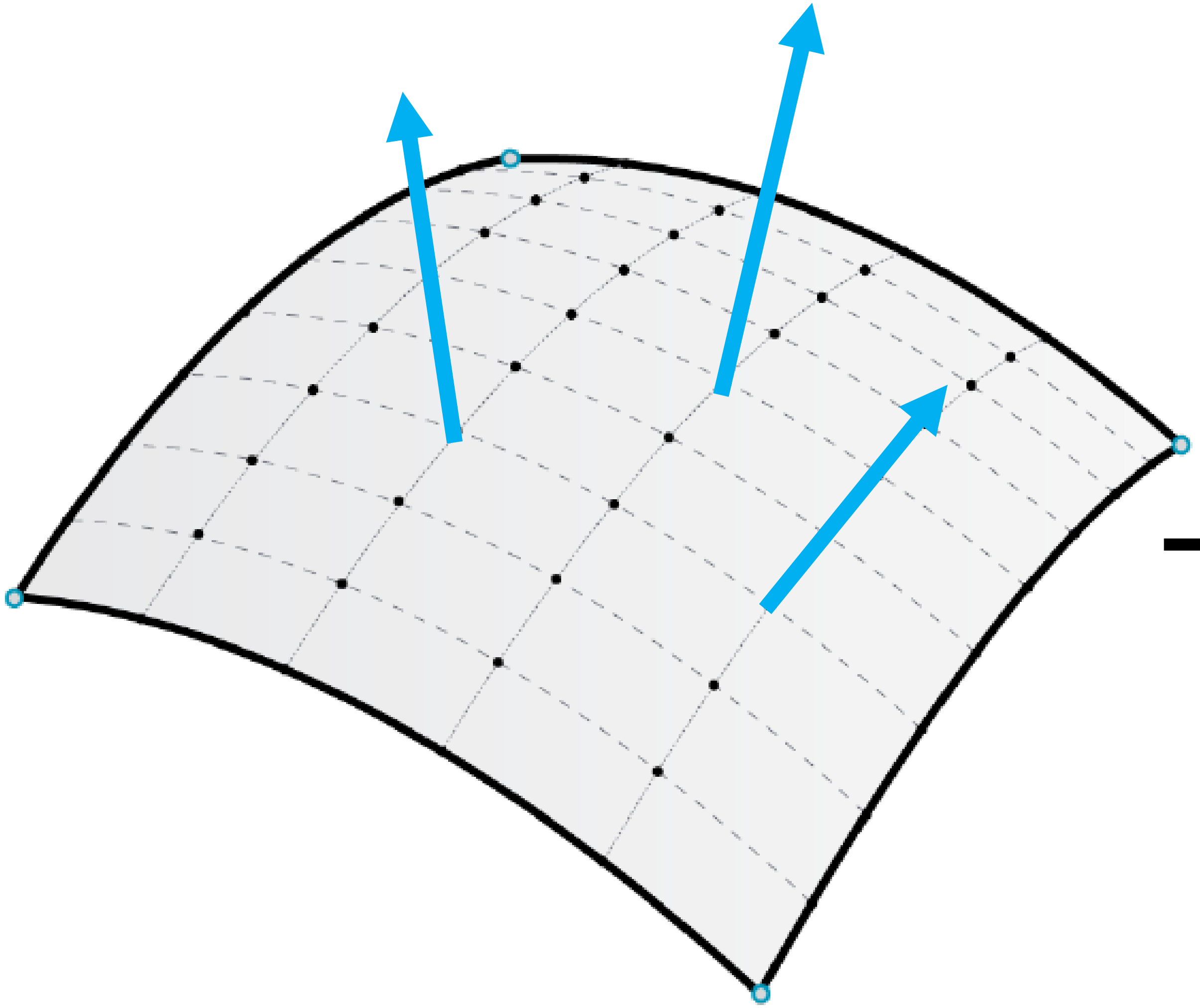
Exercise

$\gamma: [0, l] \to c$ is said to be an arc-length parametrization of the curve c if

$$\text{Length}(\gamma[0, t]) = t, \quad \forall t \in [0, l].$$

- $\left||\gamma'(t)|\right| = 1$
- $\langle \gamma'(t), \gamma''(t) \rangle = 0.$
- $|k(\gamma(t))| = \left||\gamma''(t)|\right|$

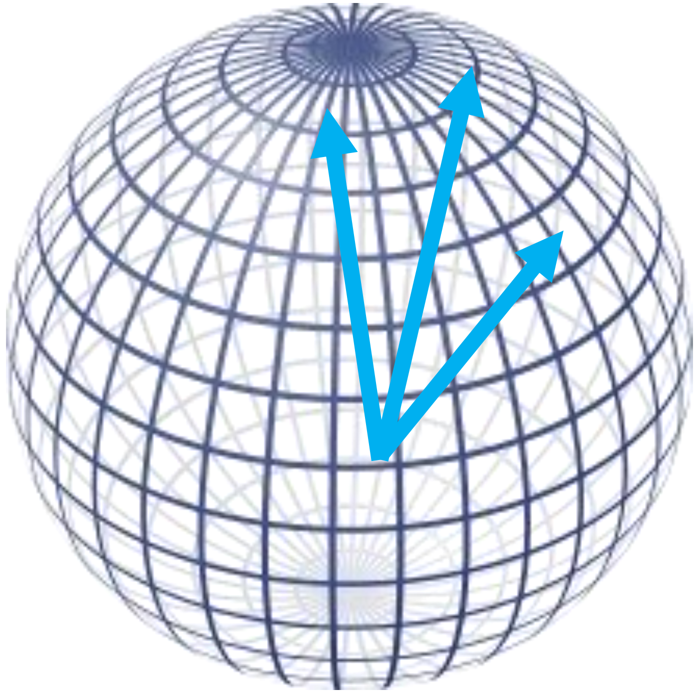
Surface



S

Gauss Map

$N: S \rightarrow S^2$

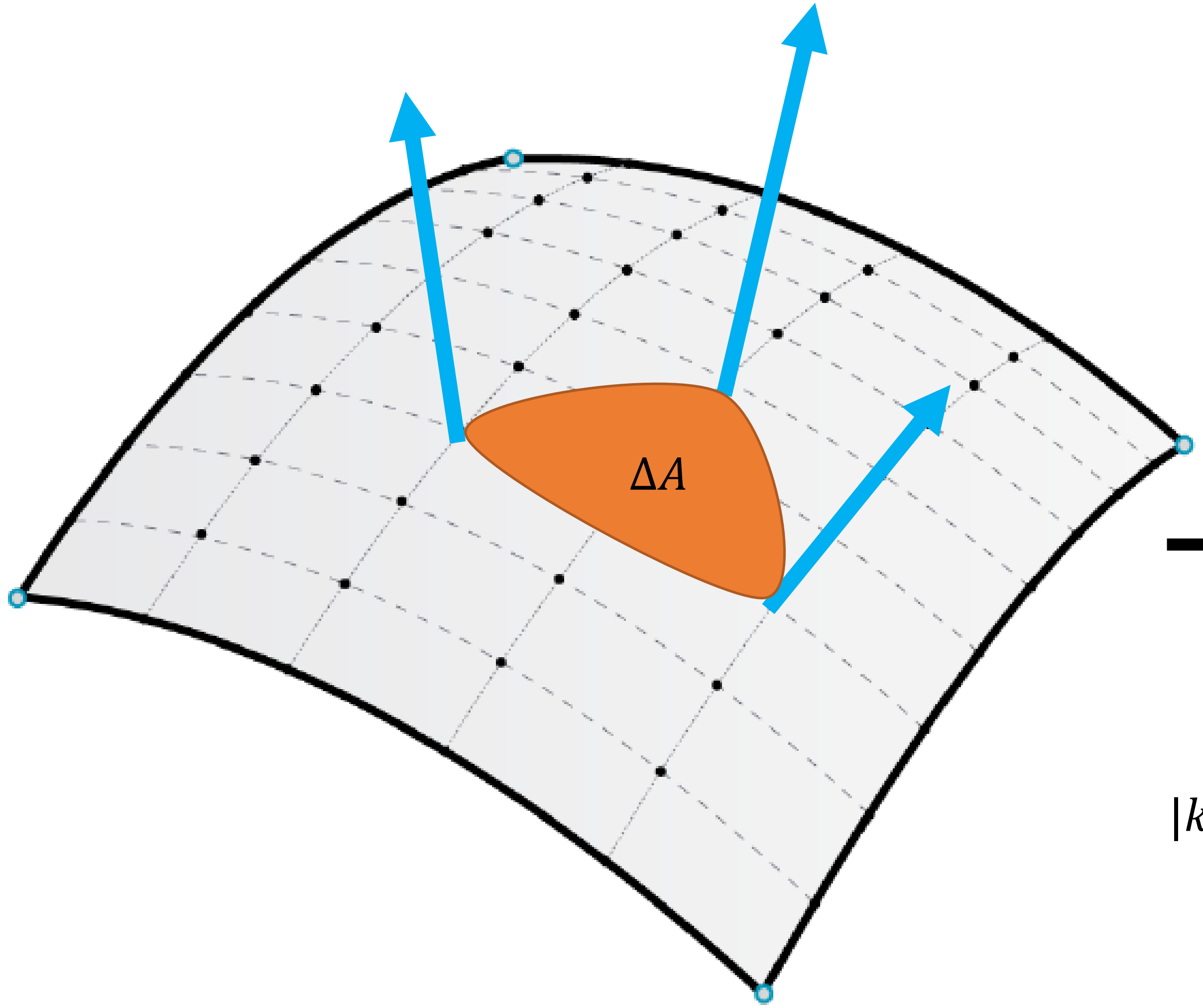


S^2

http://grasshopperprimer.com/en/1-foundations/1-5/1_surface-geometry.html

https://en.wikipedia.org/wiki/Sphere

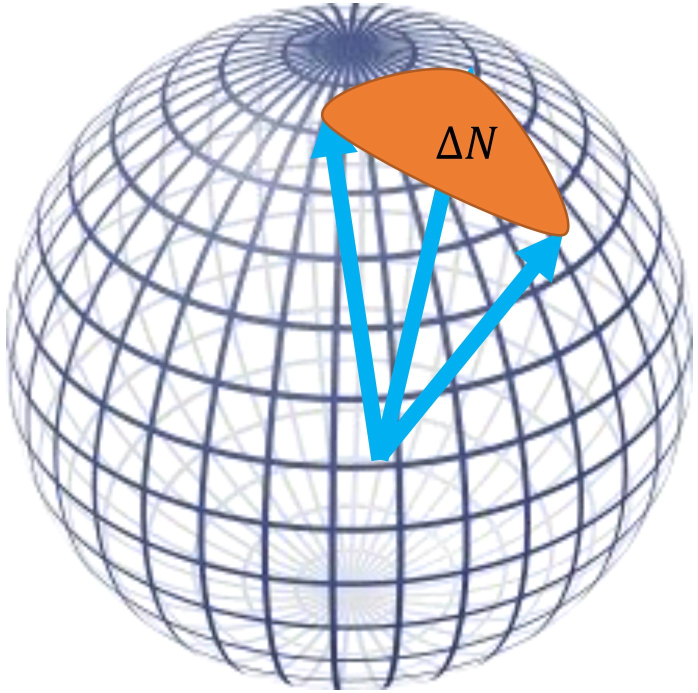
Curvature



S

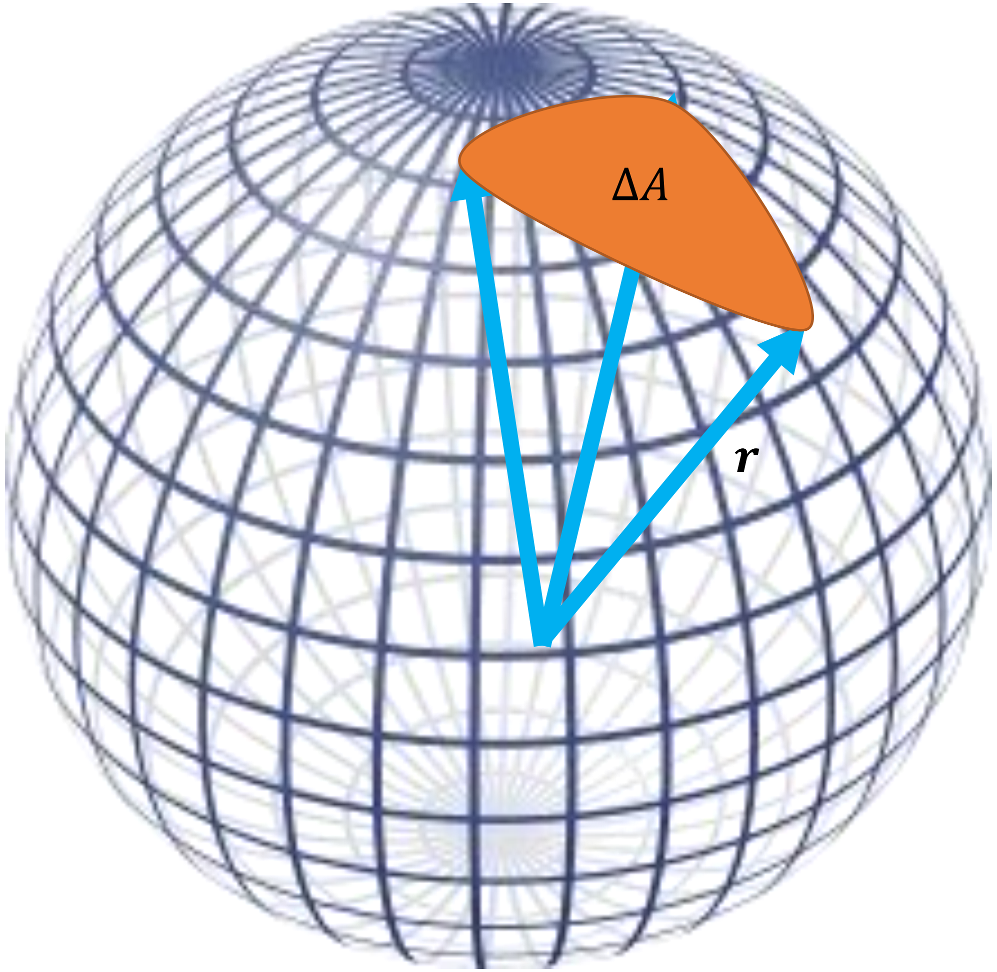
Gauss Map
$N: S \to S^2$

$$|k_G(p)| = \lim_{\Delta A \to 0} \frac{|\Delta N|}{|\Delta A|}$$



S^2

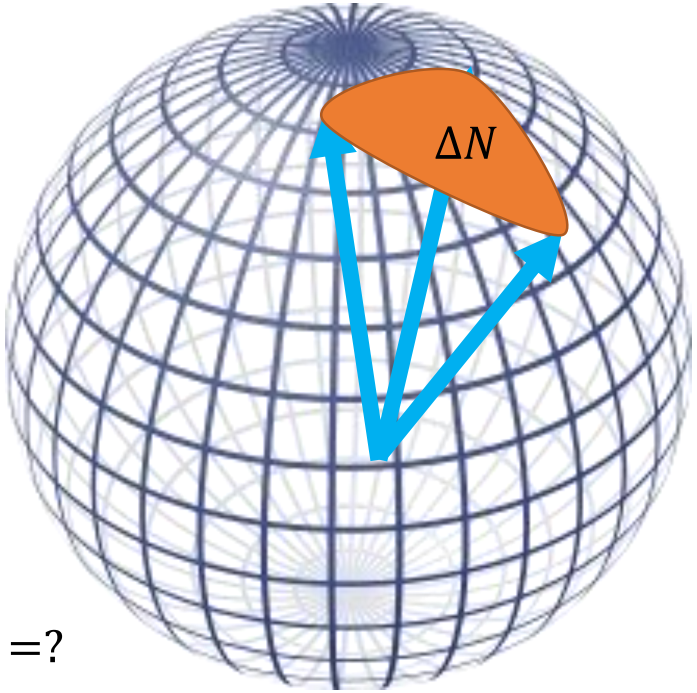
Curvature



rS^2

Gauss Map

$N: S \to S^2$

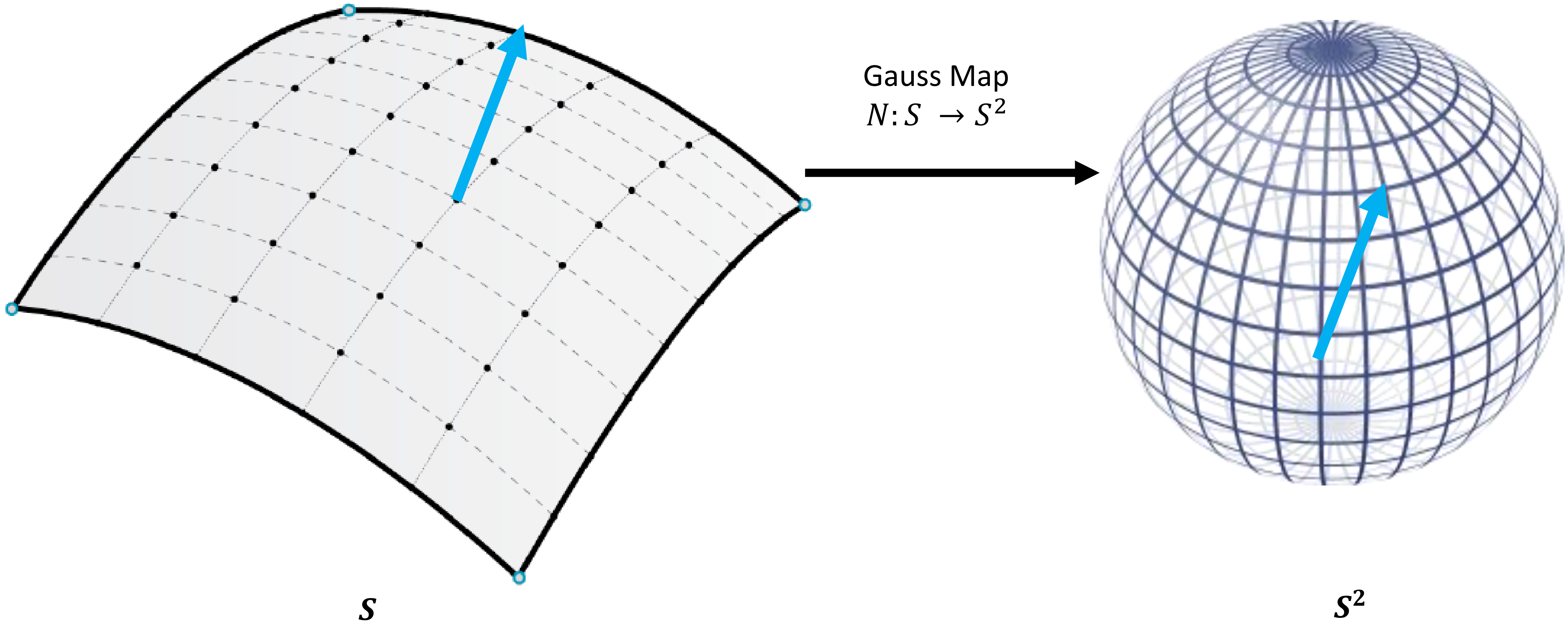


S^2

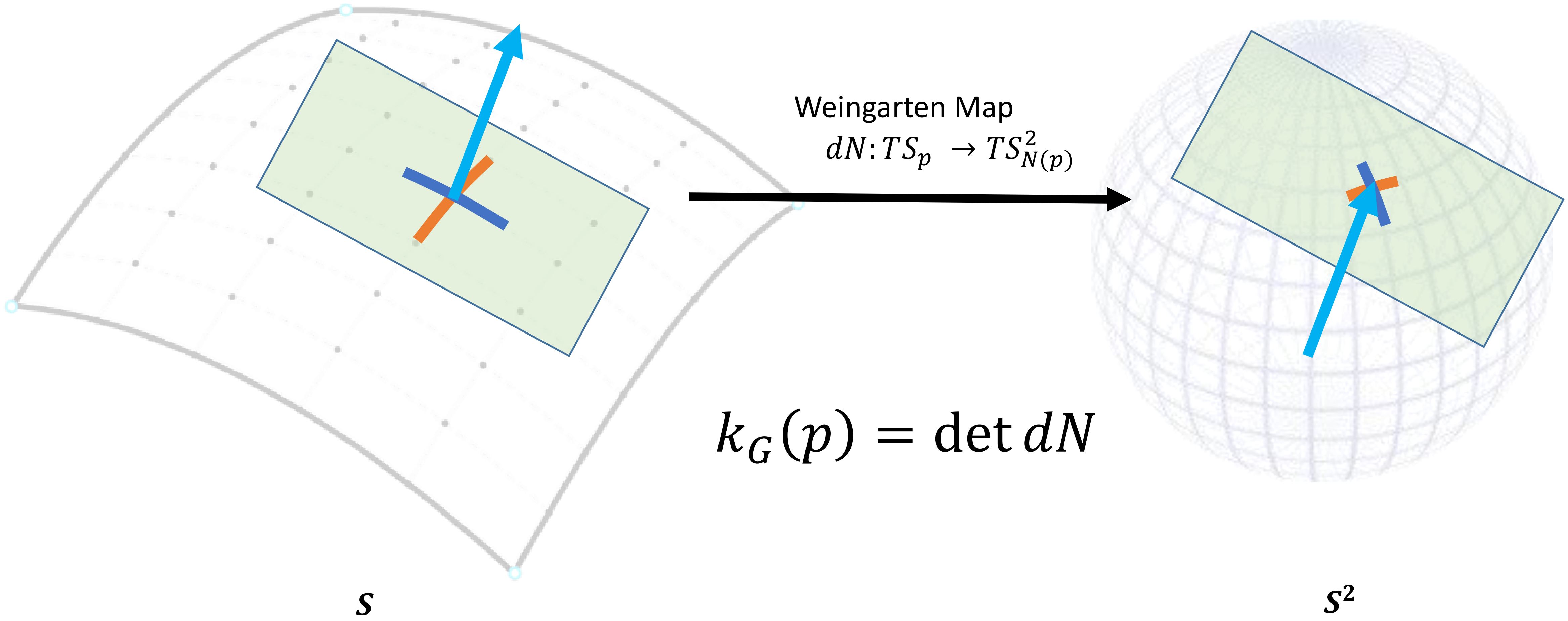
$$|k_G(p)| = \lim_{\Delta A \to 0} \frac{|\Delta N|}{|\Delta A|} = ?$$

$$|k_G(p)| = \frac{1}{r^2}$$

Weingarten Map

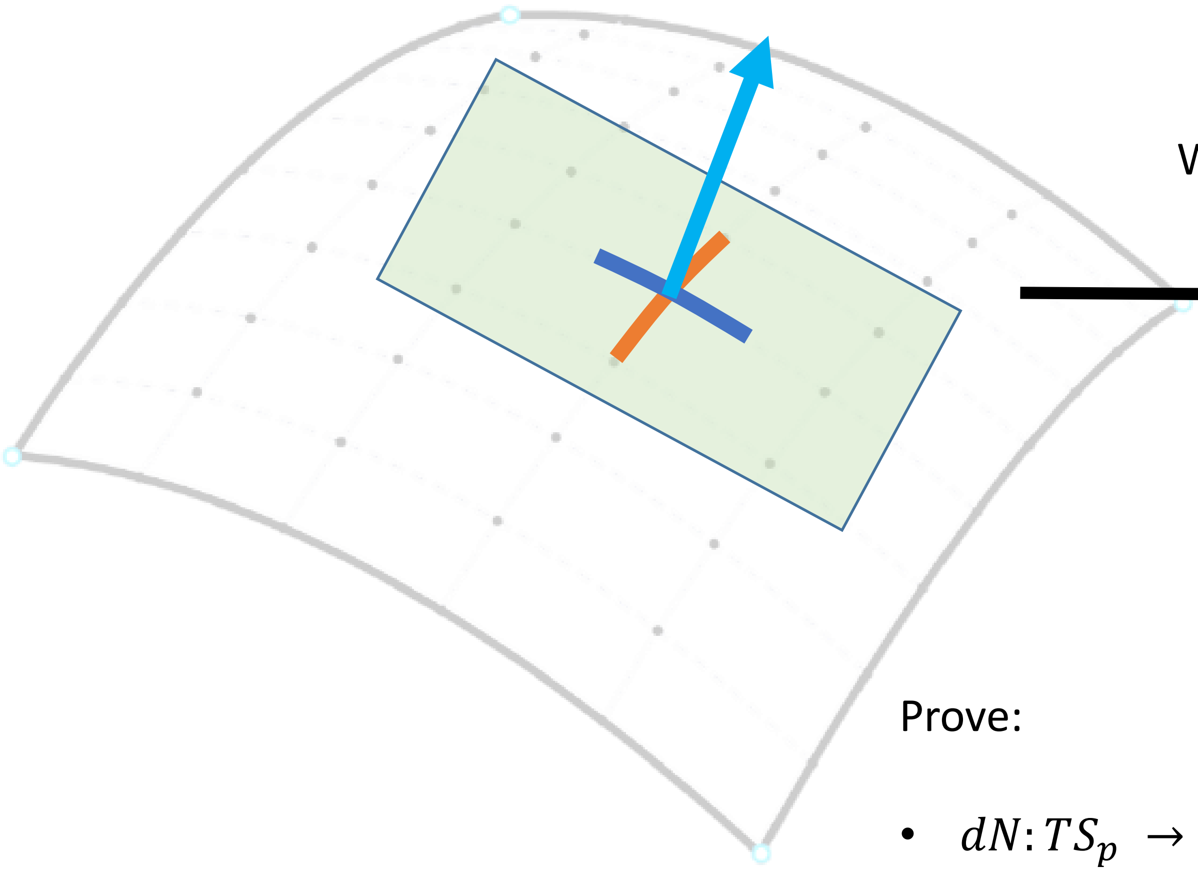


Weingarten Map



Remark: $|k_G(p)| = |\det(dN)| = \lim_{\Delta A \to 0} \frac{|\Delta N|}{|\Delta A|}$

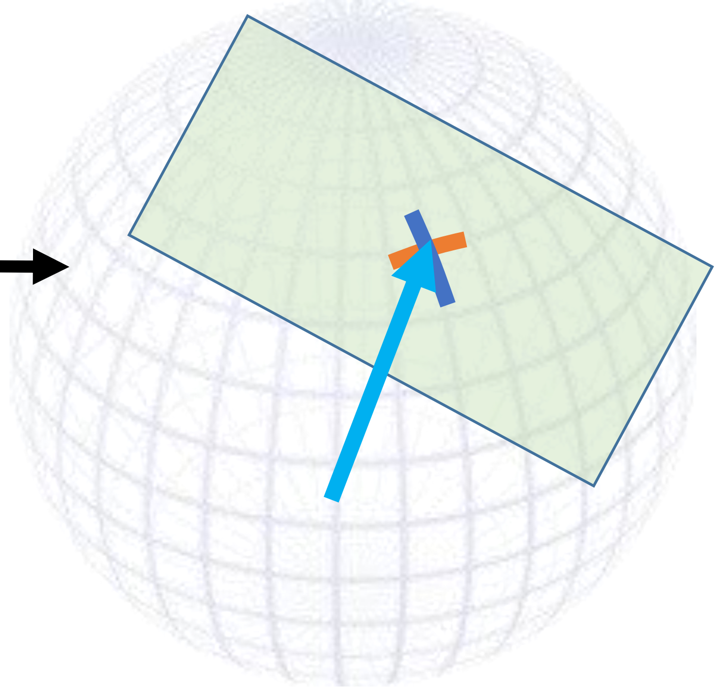
Weingarten Map



S

Weingarten Map

$dN: TS_p \to TS^2_{N(p)}$



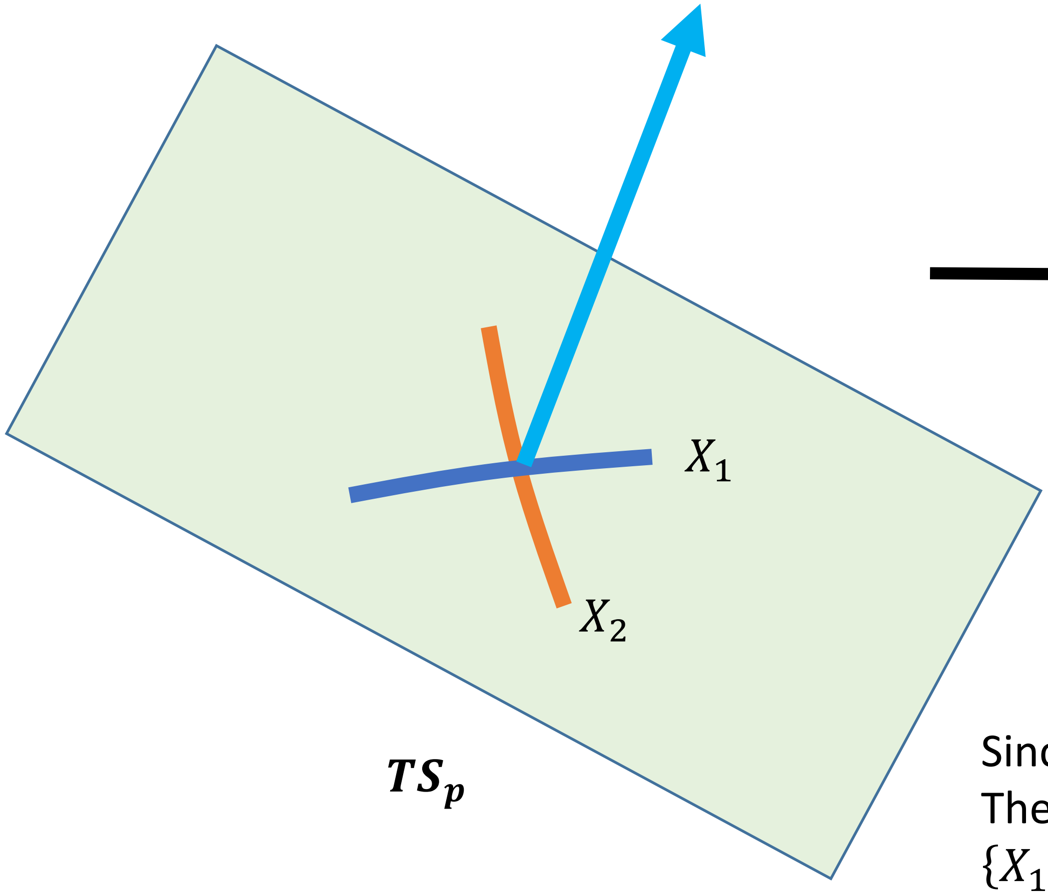
S^2

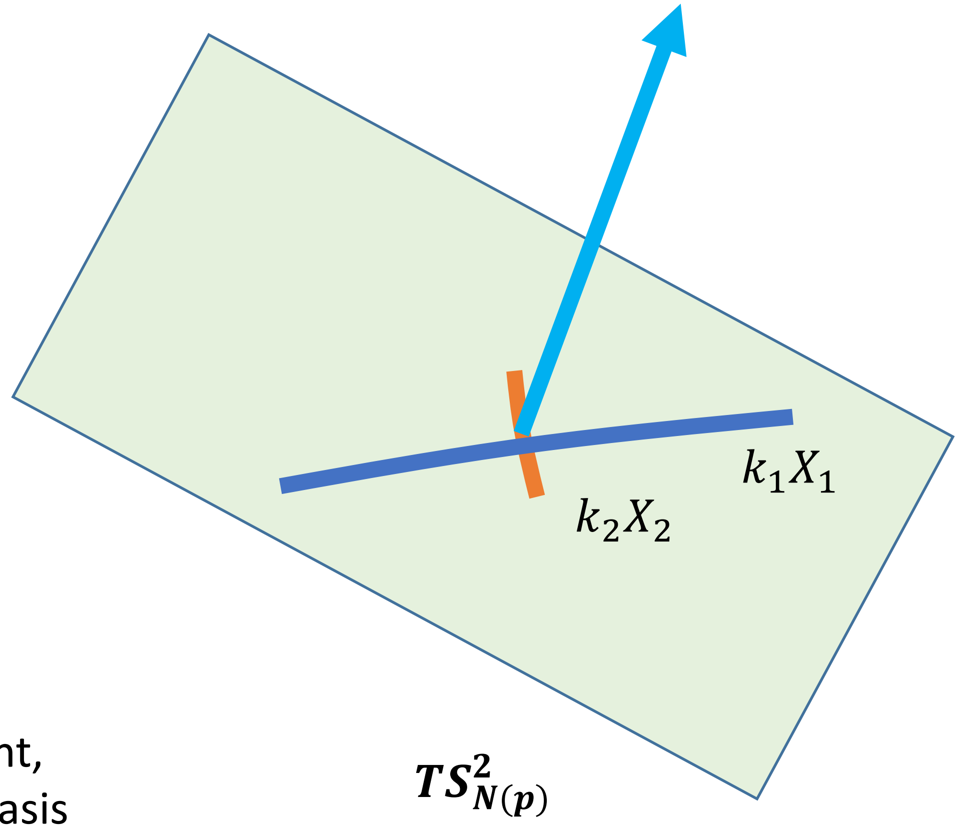
Prove:

- $dN: TS_p \to TS^2_{N(p)}$ is selfadjoint

$$\langle dN(X), Y \rangle = \langle X, dN(Y) \rangle$$

Principal Curvature





Since $dN: TS \ \rightarrow TS^2$ is selfadjoint,
There exists an orthonormal a basis
$\{X_1, X_2\} \in TS$, such that

$$dN = (X_1|X_2)\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}(X_1|X_2)^\top$$

- X_1, X_2 are called principal directions
- k_1, k_2 are called principal curvatures

http://grasshopperprimer.com/en/1-foundations/1-5/1_surface-geometry.html

https://en.wikipedia.org/wiki/Sphere