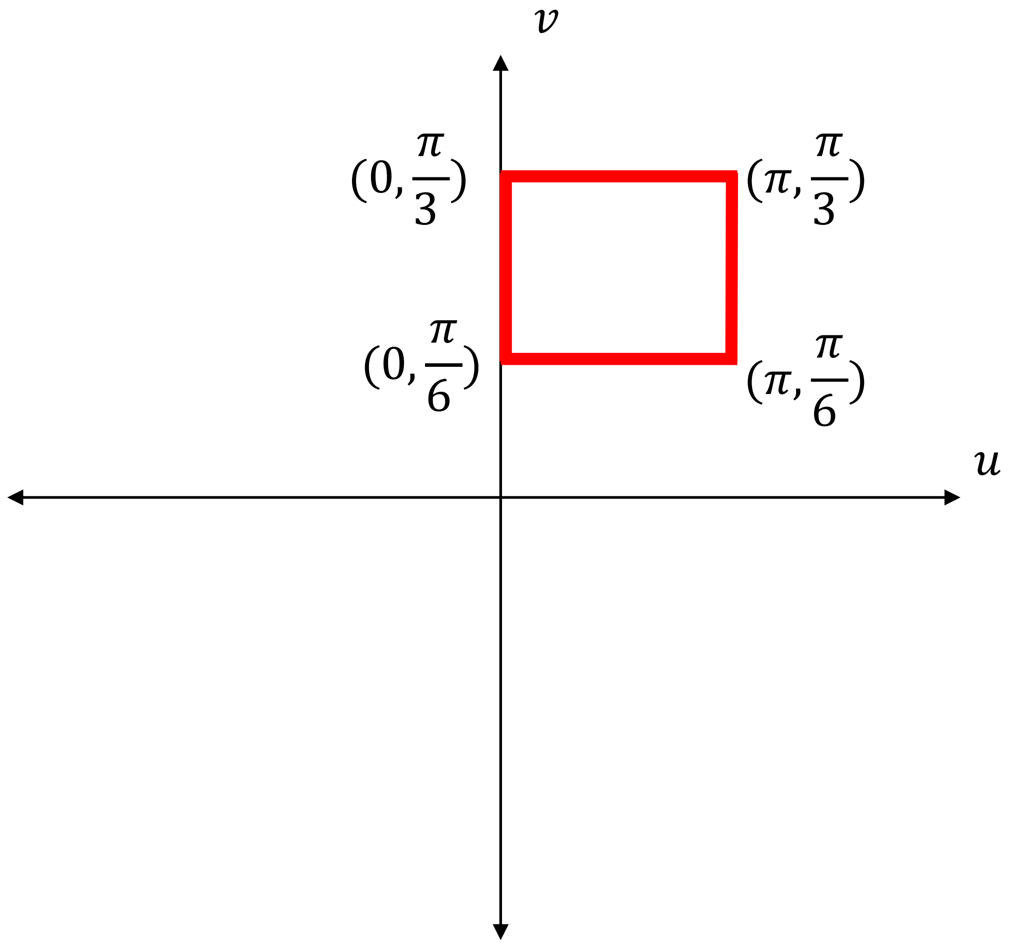
Lecture 2

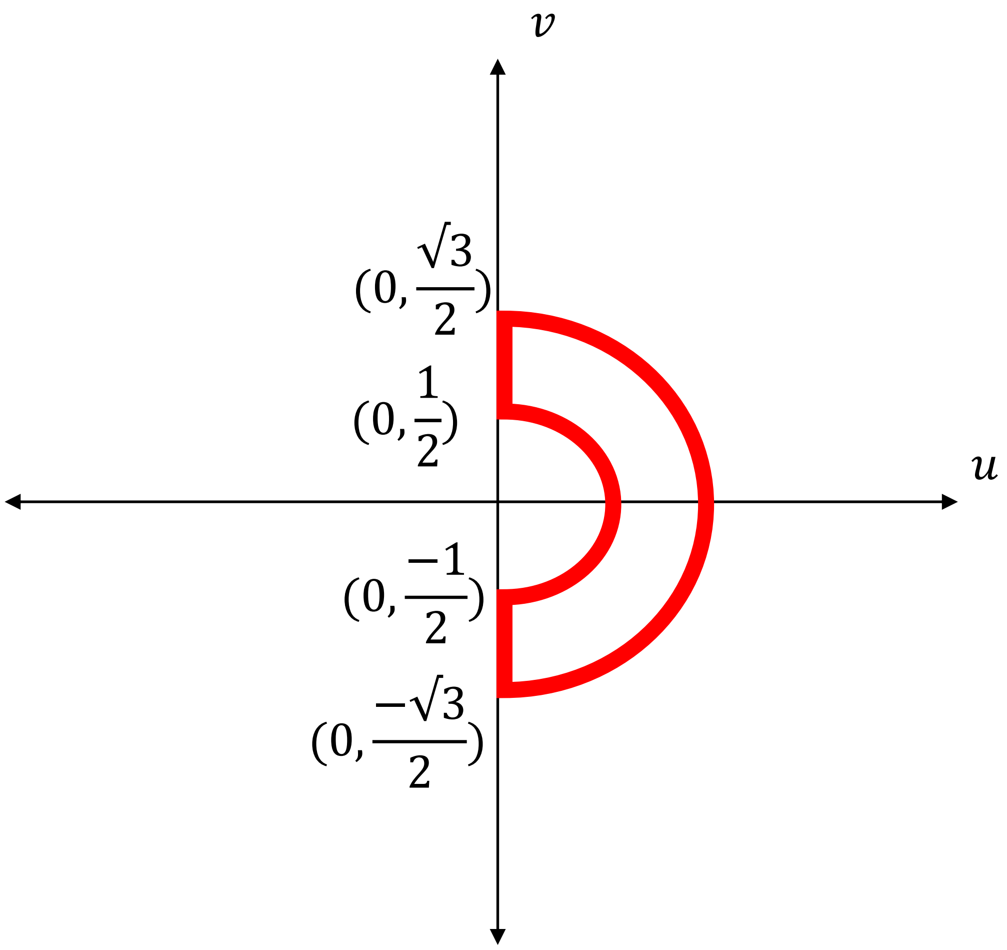
Introduction to Geometry Processing

Spring 2017

Johns Hopkins University



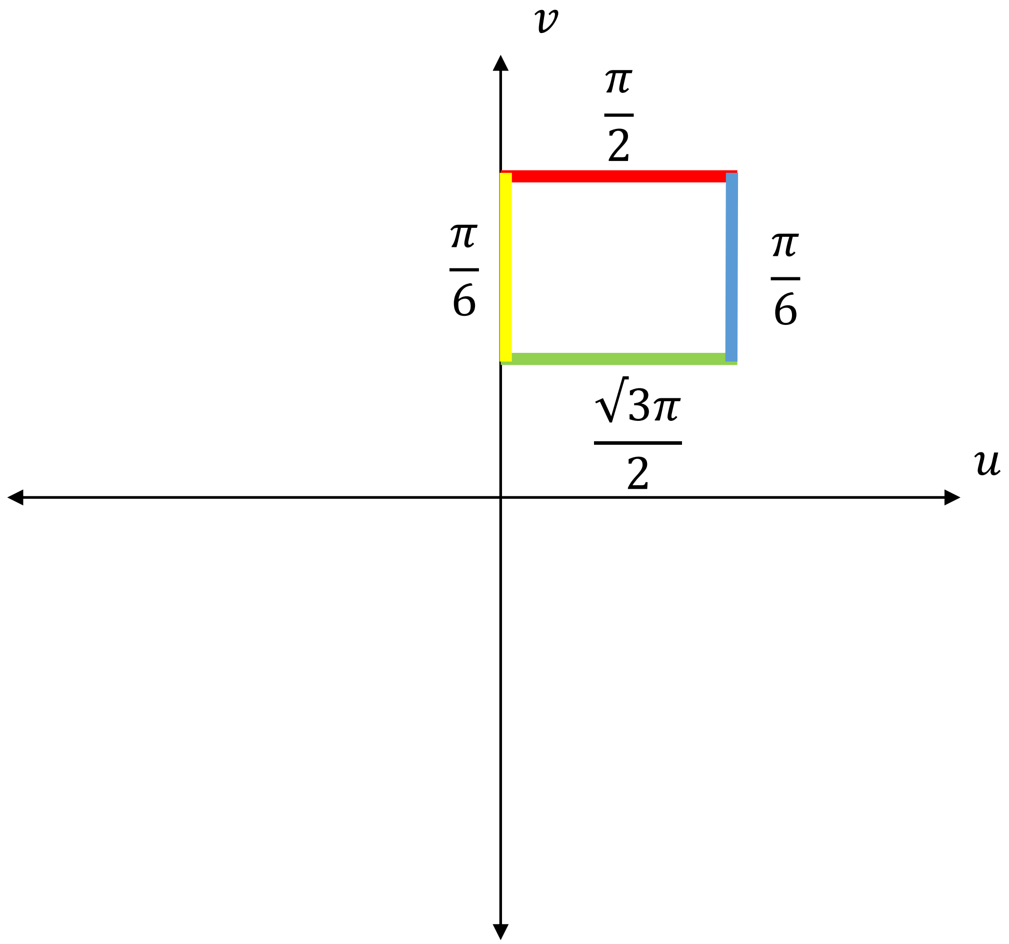
$$M_A = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$



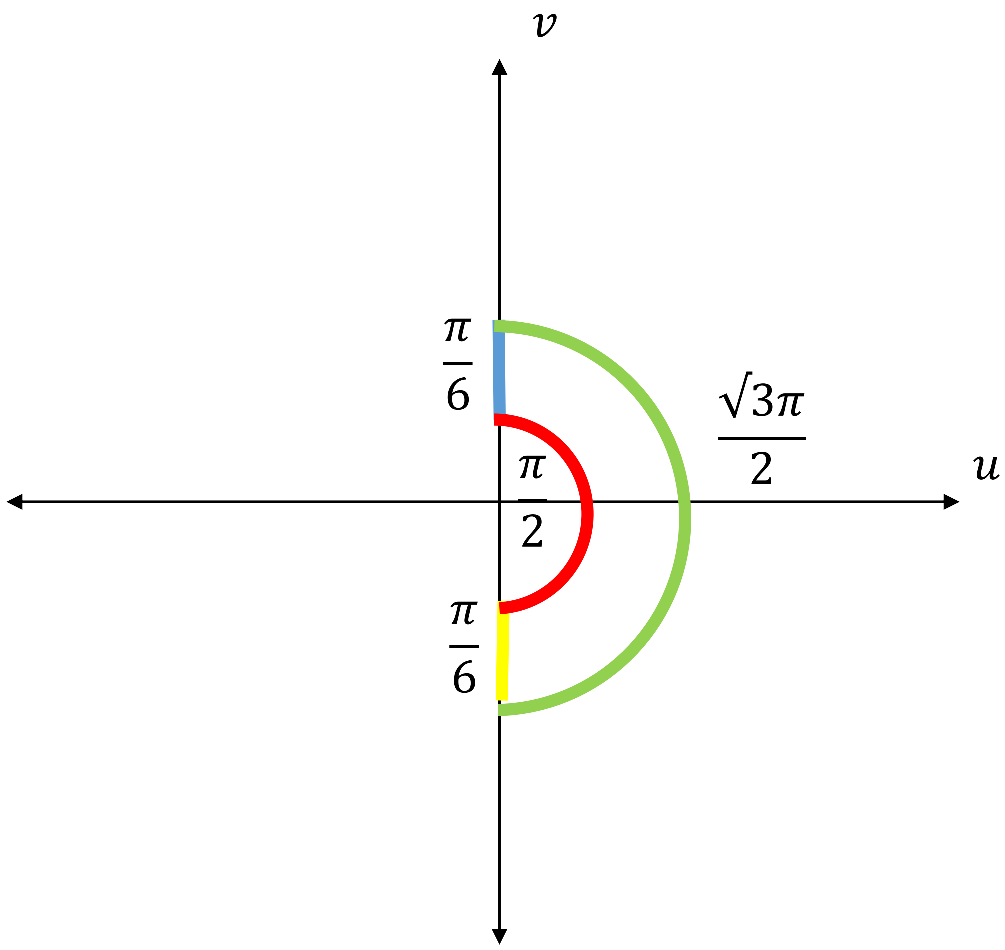
$$M_B = \frac{1}{1 - u^2 - v^2} \begin{pmatrix} 1 - v^2 & uv \\ uv & 1 - u^2 \end{pmatrix}$$

$$Length_M(c) = \int_I \sqrt{\left(\frac{d\gamma}{dt}\right)^{\top} M \left(\frac{d\gamma}{dt}\right)} dt$$

$$\mathrm{Area}_{\mathrm{M}}(\Omega) = \int_\Omega \sqrt{\det(M)} du dv$$



$$M_A = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$



$$M_B = \frac{1}{1 - u^2 - v^2} \begin{pmatrix} 1 - v^2 & uv \\ uv & 1 - u^2 \end{pmatrix}$$

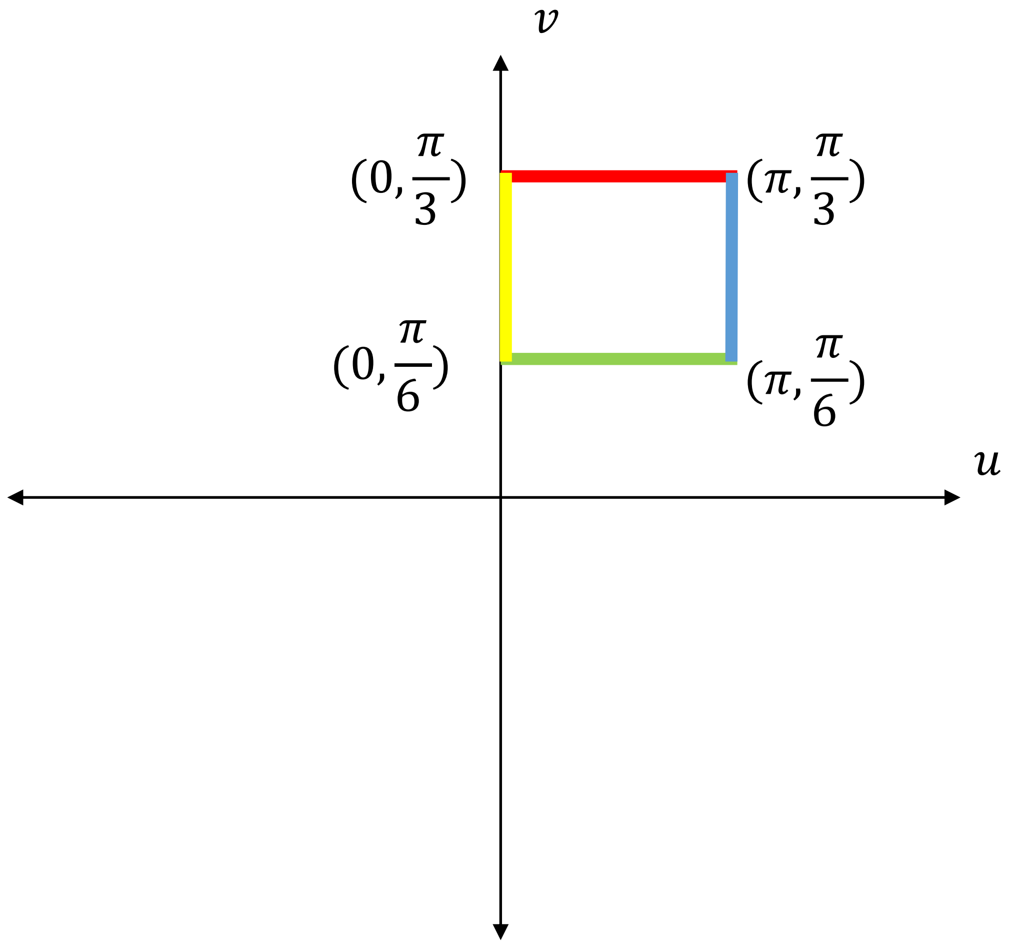
$$Length_{M_A}(c_A) = \frac{(5 + 3\sqrt{3})\pi}{6}$$

$$\mathrm{Area}_{\mathrm{M}_A}(\Omega_A) = \frac{(\sqrt{3} - 2)\pi}{2}$$

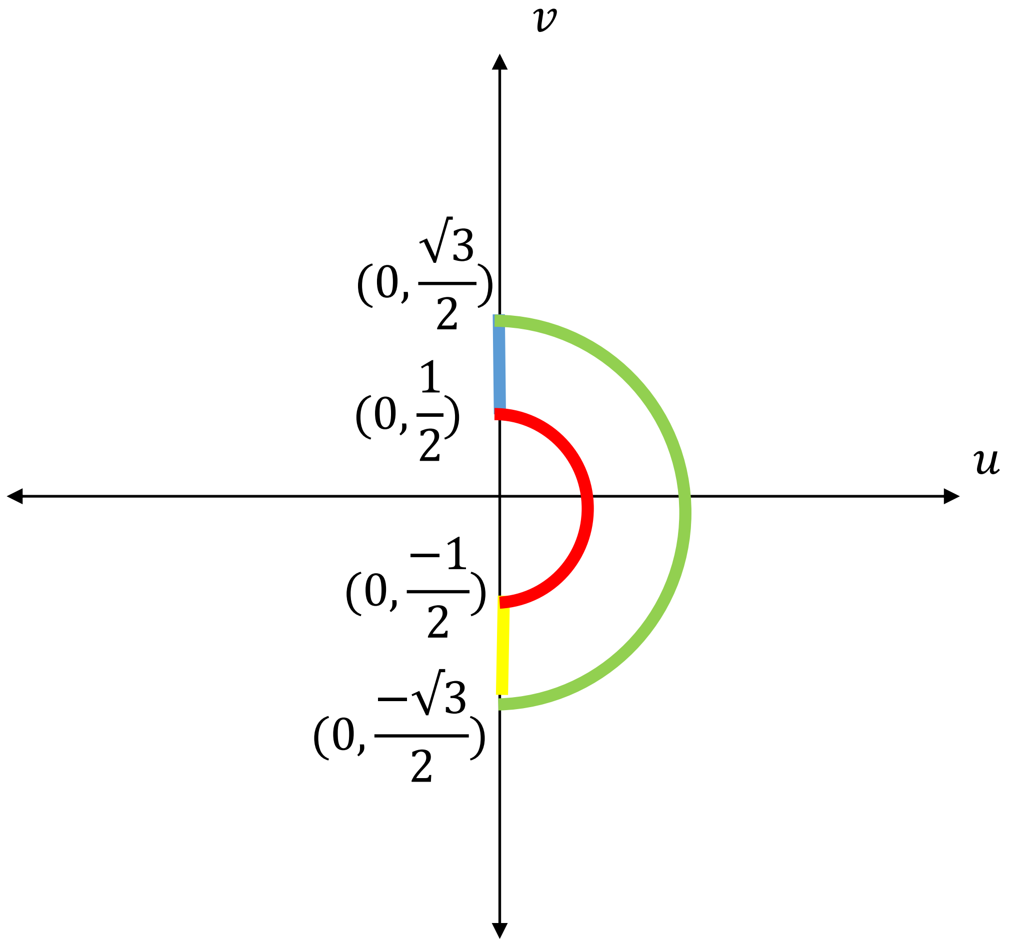
Is $(\Omega_A, M_A) \equiv (\Omega_B, M_B)$?

$$Length_{M_B}(c_B) = \frac{(5 + 3\sqrt{3})\pi}{6}$$

$$\mathrm{Area}_{\mathrm{M}_B}(\Omega_B) = \frac{(\sqrt{3} - 2)\pi}{2}$$



$$M_A = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$



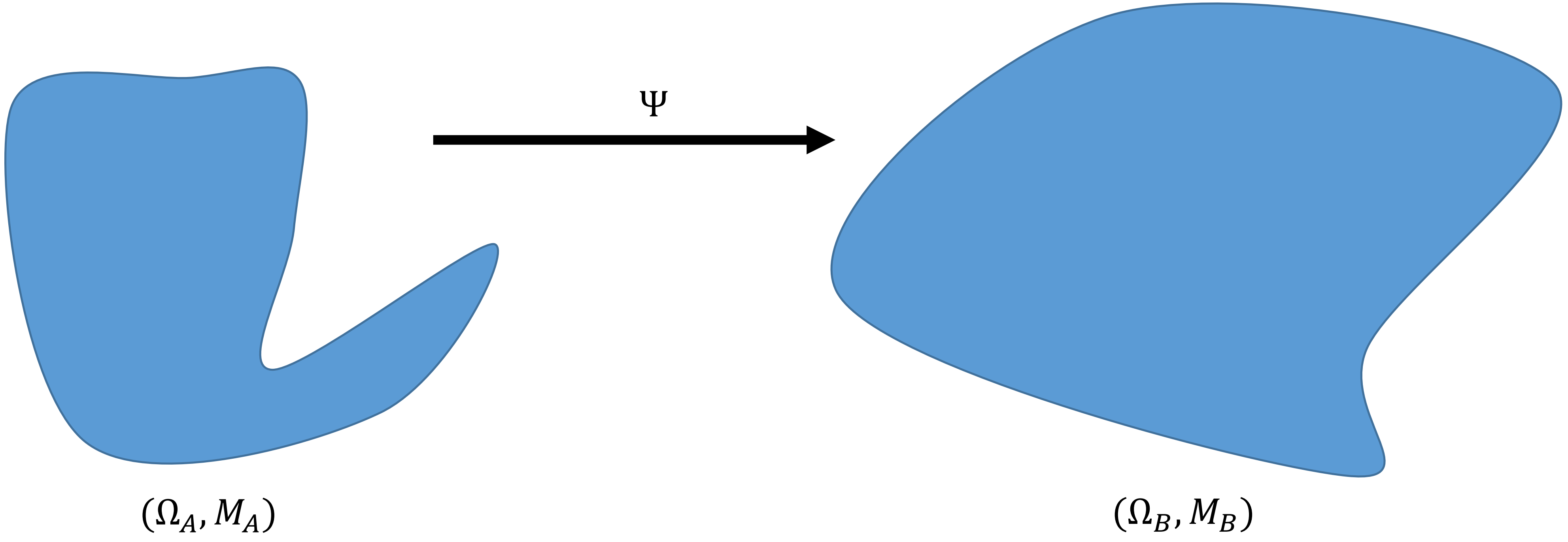
$$M_B = \frac{1}{1 - u^2 - v^2} \begin{pmatrix} 1 - v^2 & uv \\ uv & 1 - u^2 \end{pmatrix}$$

Consider $\Psi : \Omega_A \to \Omega_B$, $\Psi\,(u, v) \to (\cos v \sin u\,, -\cos v \cos u)$,

Prove that:

- $Length_{M_A}(c_A) = Length_{M_B}(\Psi(c_A))\,,\ \forall c_A \subset \Omega_A$
- $Area_{M_A}(\Gamma_A) = Area_{M_B}(\Psi(\Gamma_A))\,,\ \forall \Gamma_A \subset \Omega_A$

Remark:



Denote $\langle X, Y \rangle_M := X^\top M\, Y$.

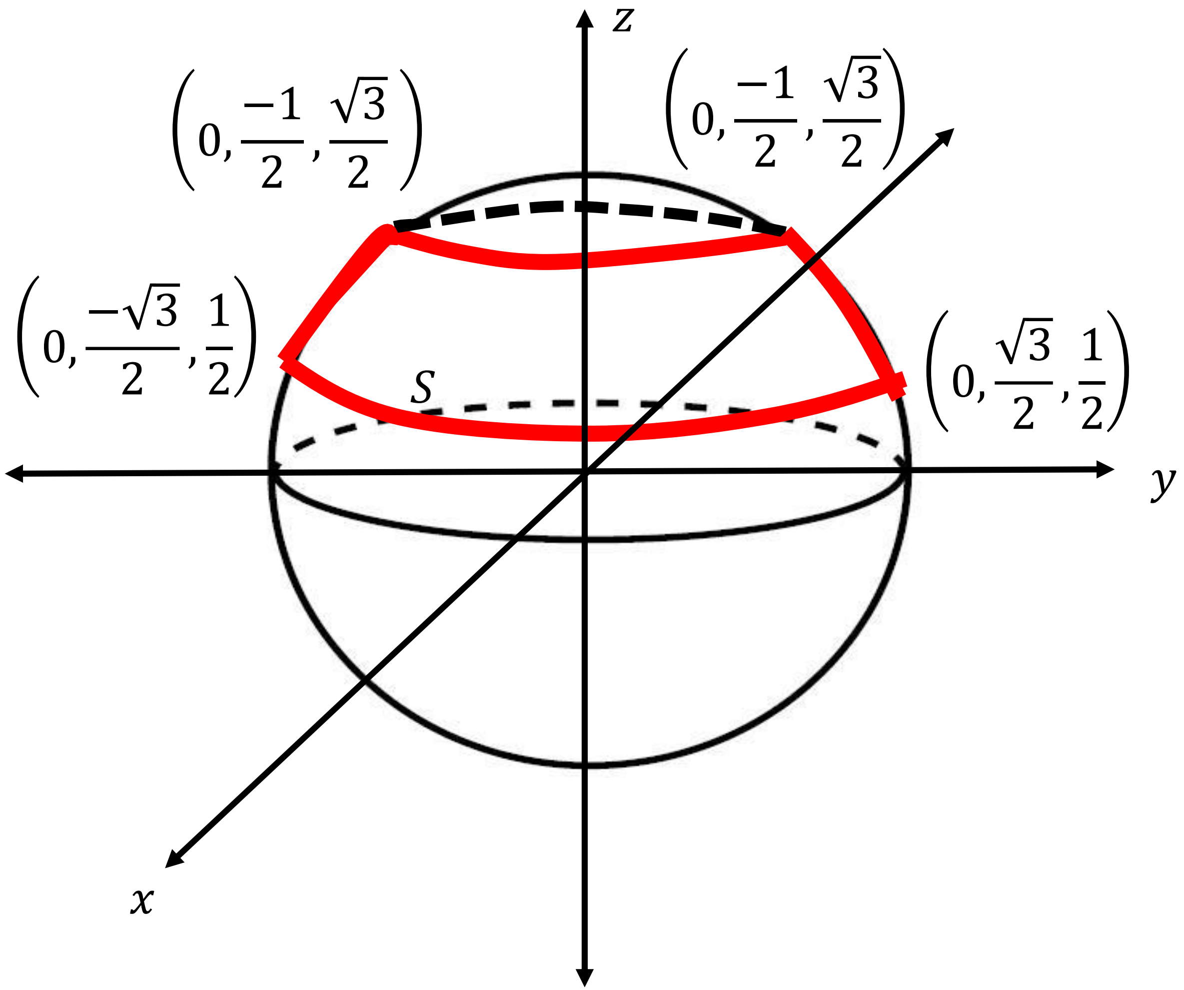
If $\exists \Psi : \Omega_A \to \Omega_B$, such that,

$$\langle X, Y \rangle_{M_A} = \langle d\Psi X, d\Psi Y \rangle_{M_B} \; \forall\, X, Y \in T\Omega_A(p),$$

then,

$$(\Omega_A, M_A) \equiv (\Omega_B, M_B)$$

Embedding

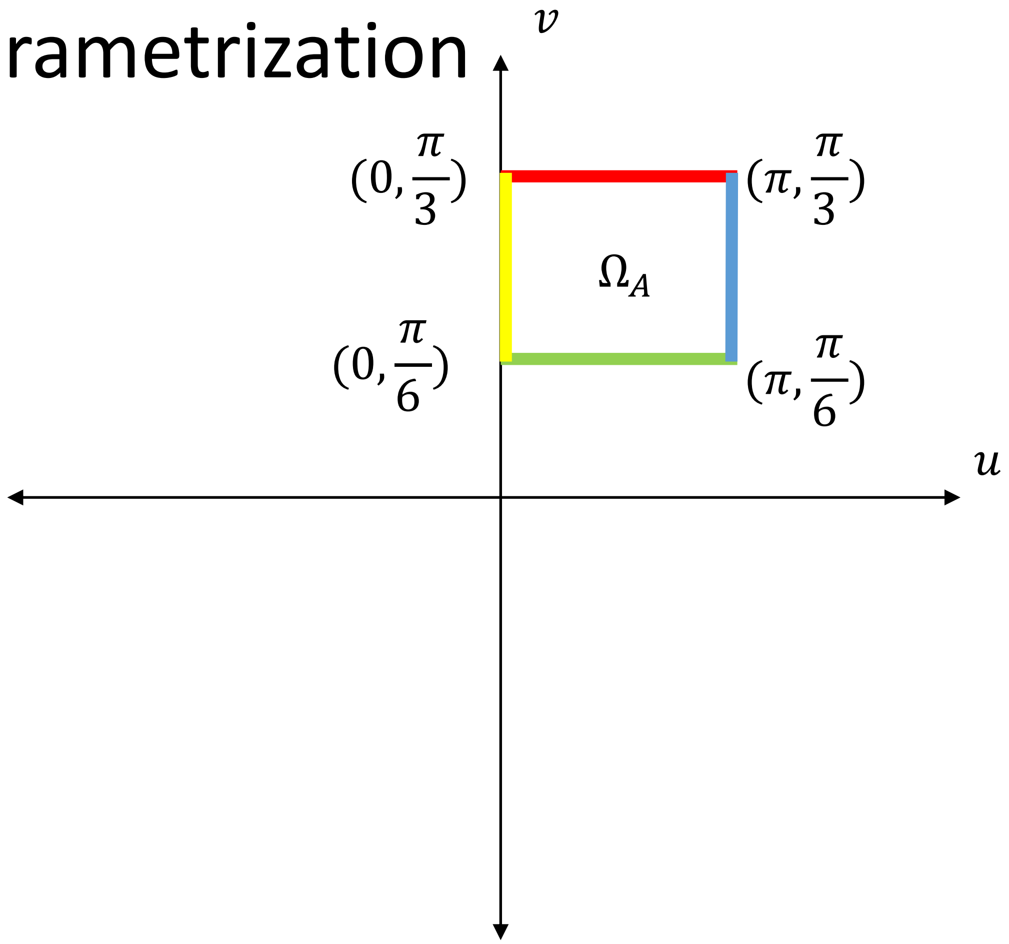


$$S = \{(x, y, z): x^2 + y^2 + z^2 = 1, x \geq 0, \frac{1}{2} \leq z \leq \frac{\sqrt{3}}{2}\}$$

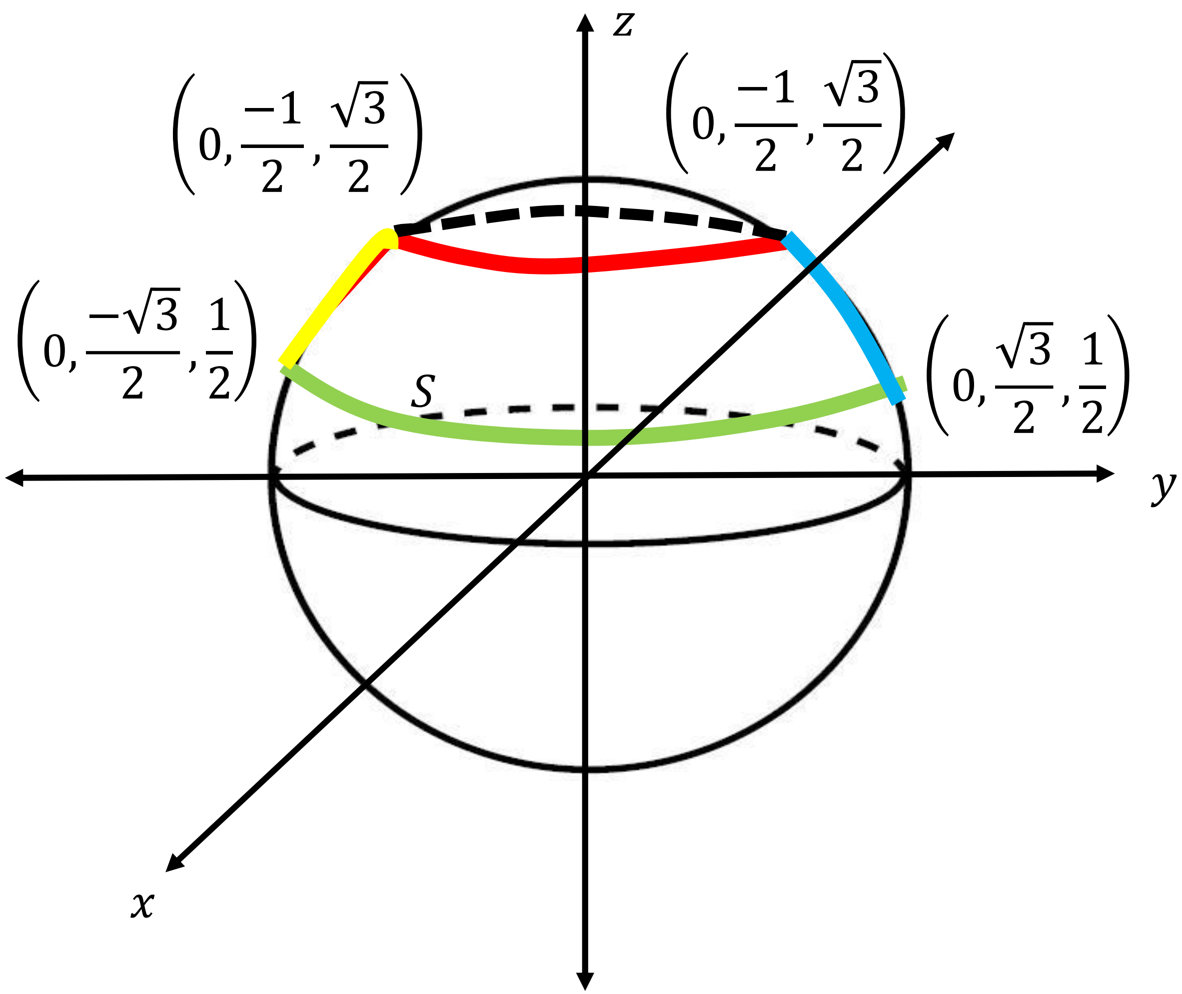
What is the perimeter of S?

What is the area of S?

Parametrization



$$M_A = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

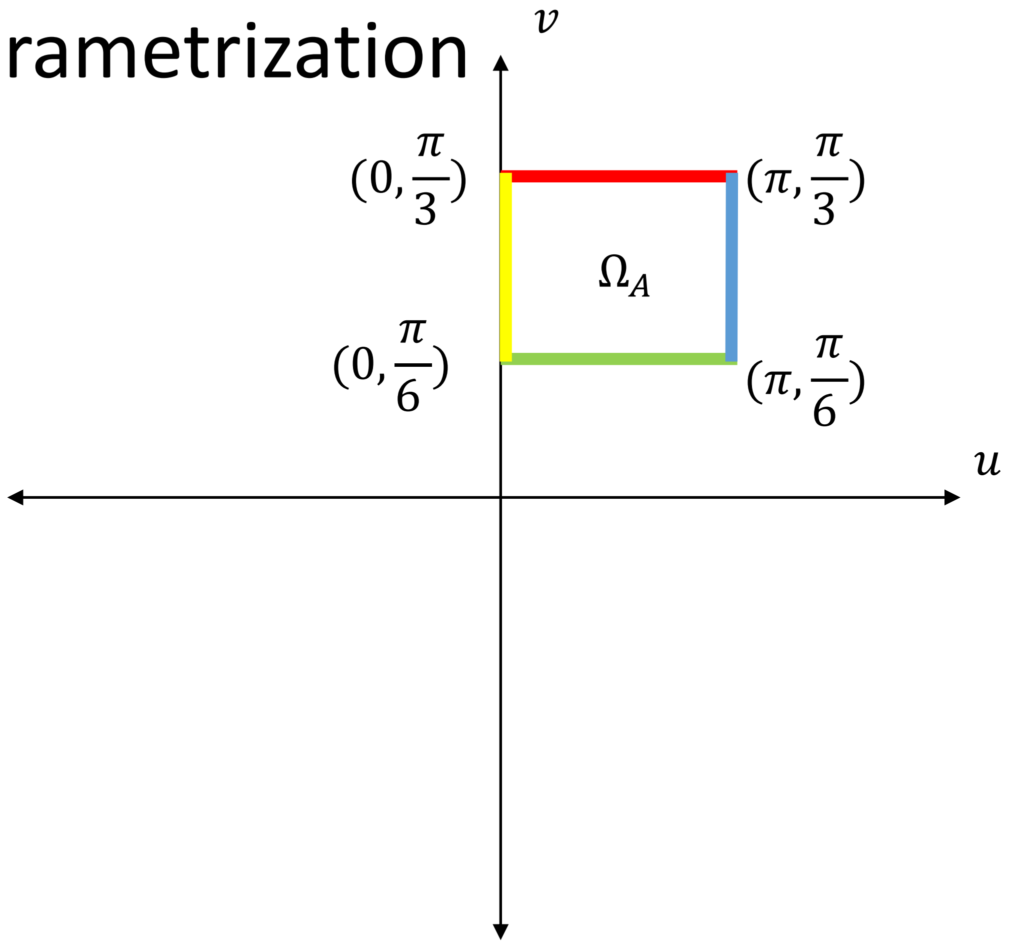


$$S = \{(x,y,z): x^2+y^2+z^2 = 1, x \geq 0, \frac{1}{2} \leq z \leq \frac{\sqrt{3}}{2}\}$$

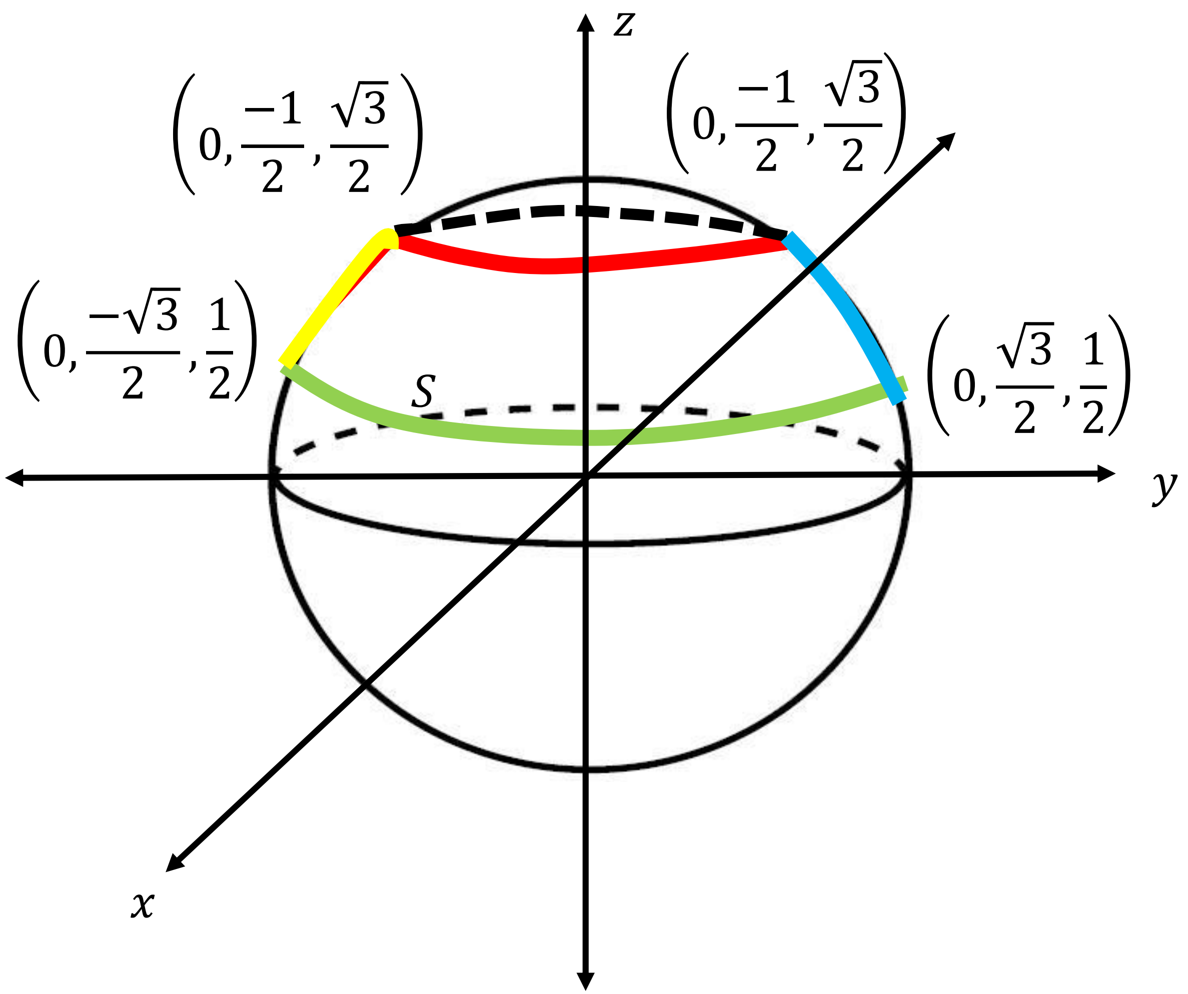
Let $f_A\colon \Omega_A \to S$, $f_A(u,v) = (\cos v \sin u\,, -\cos v \cos u, \sin v)$.

Compute $df_A^\top df_A$.

Parametrization



$$M_A = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$



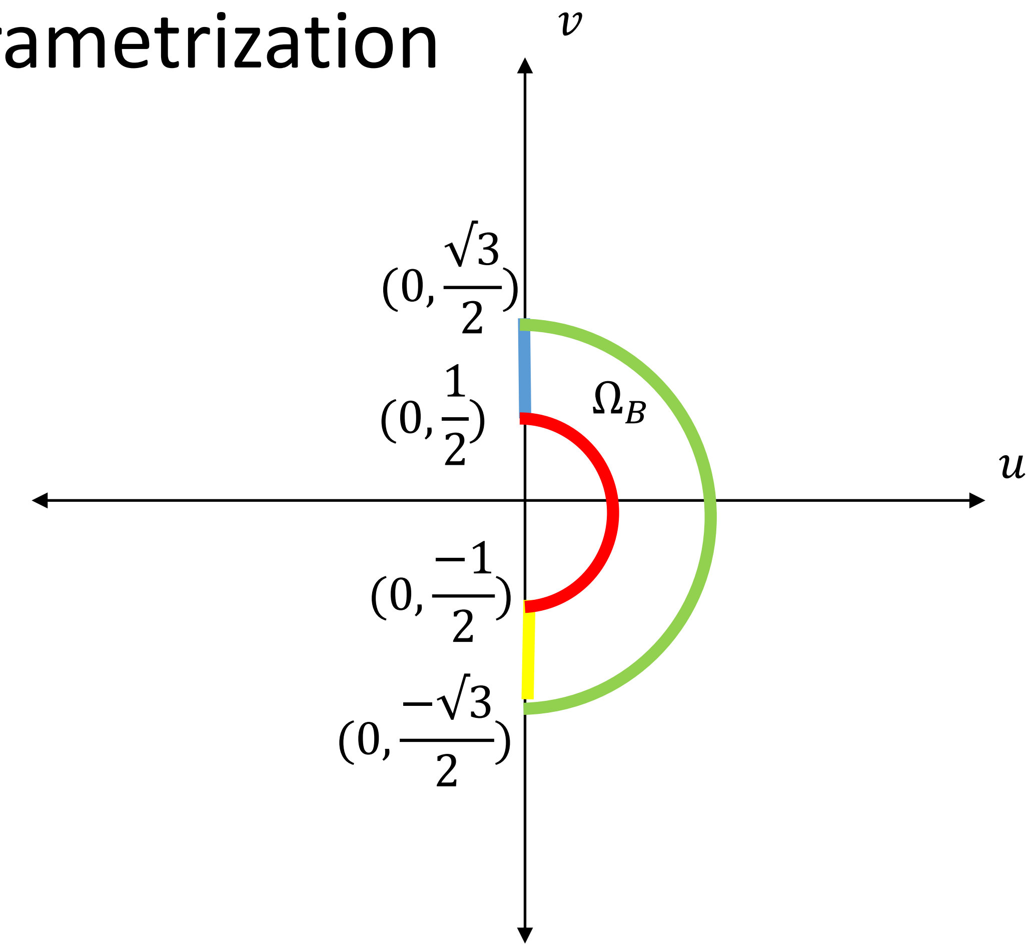
$$S = \{(x, y, z): x^2 + y^2 + z^2 = 1, x \geq 0, \frac{1}{2} \leq z \leq \frac{\sqrt{3}}{2}\}$$

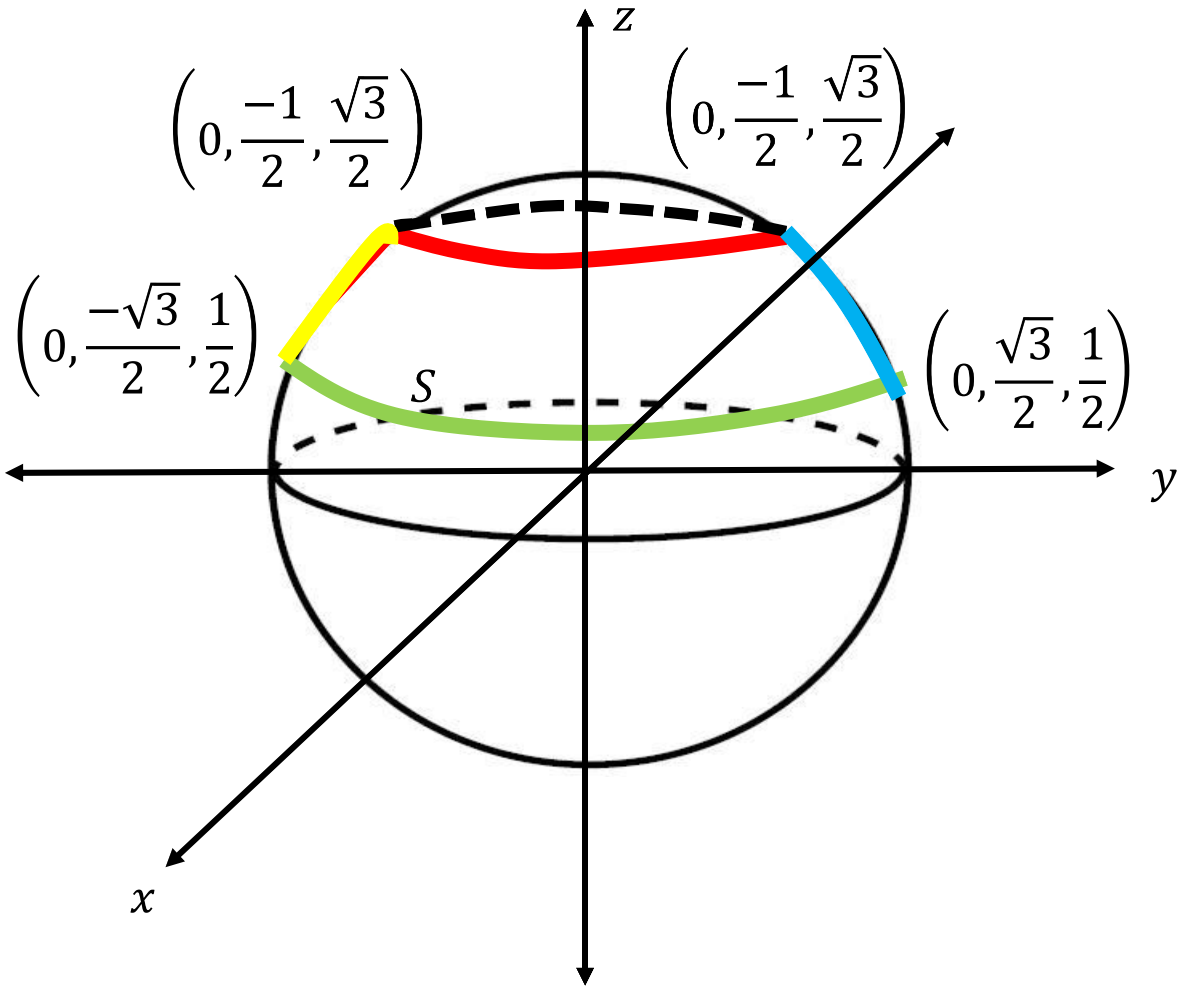
$$f_A(u, v) = (\cos v \sin u\,, -\cos v \cos u, \sin v).$$

Prove that:

- $Length_{M_A}(c_A) = Length_{E_3}(\,f_A(c_A))\ \forall c_A \subset \Omega_A$
- $Area_{M_A}(\Gamma_A) = Area_{E_3}(\,f_A(\Gamma_A))\ \forall \Gamma_A \subset \Omega_A$

Parametrization



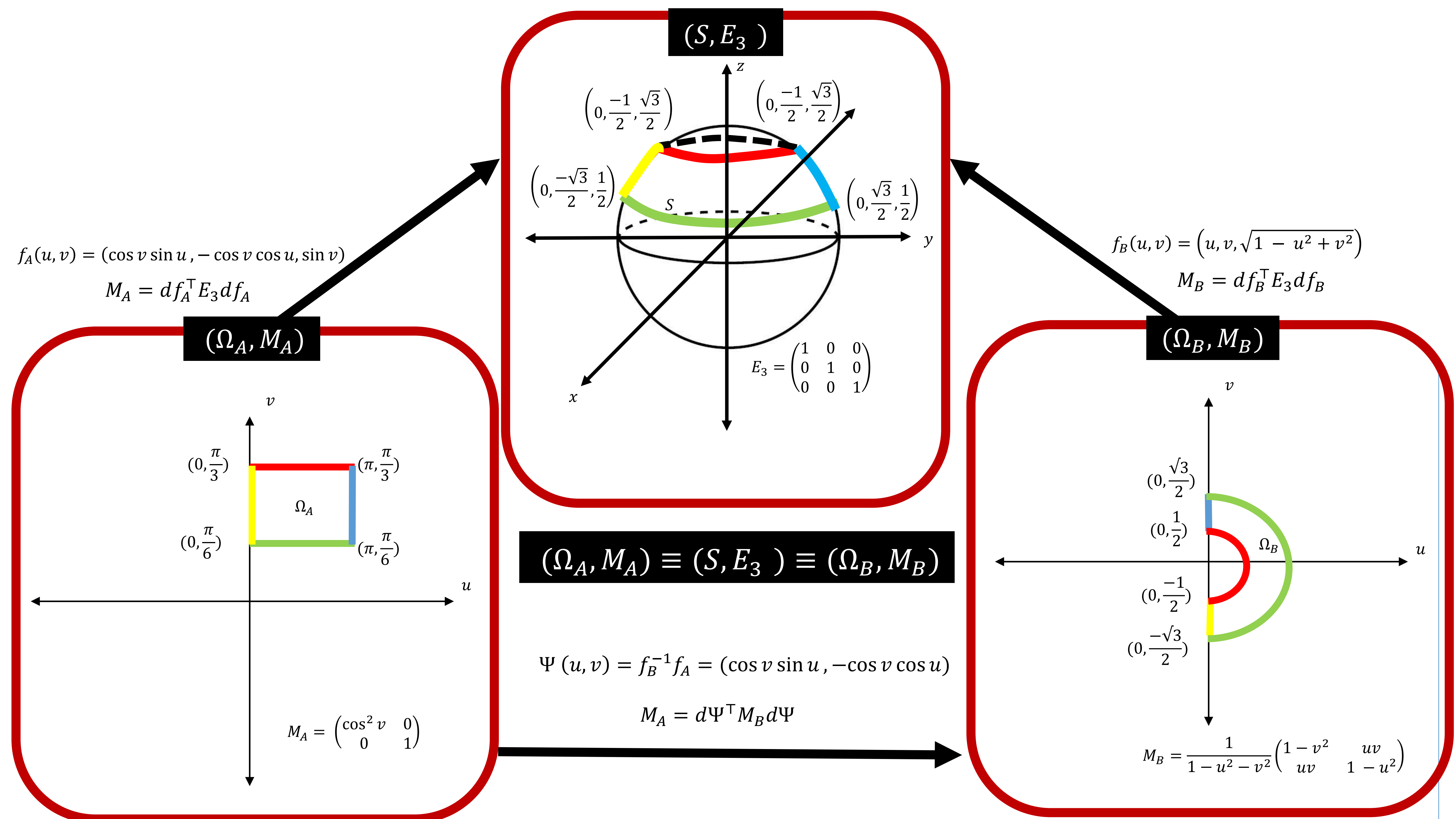


$$M_B = \frac{1}{1-u^2-v^2}\begin{pmatrix} 1-v^2 & uv \\ uv & 1-u^2 \end{pmatrix}$$

$$S = \{(x, y, z): x^2 + y^2 + z^2 = 1, x \geq 0, \frac{1}{2} \leq z \leq \frac{\sqrt{3}}{2}\}$$

Let $f_B: \Omega_B \to S$, $f_B(u, v) = \left(u, v, \sqrt{1 - u^2 + v^2}\right)$.

Compute $df_B^\top df_B$.



(S, E_3)
$\left(0, \frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(0, \frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(0, \frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$
$\left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
S
z
y
x
$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$f_A(u, v) = (\cos v \sin u, -\cos v \cos u, \sin v)$
$M_A = df_A^\top E_3 df_A$
$f_B(u, v) = \left(u, v, \sqrt{1 - u^2 + v^2}\right)$
$M_B = df_B^\top E_3 df_B$
(Ω_A, M_A)
v
u
$(0, \frac{\pi}{3})$
$(\pi, \frac{\pi}{3})$
Ω_A
$(0, \frac{\pi}{6})$
$(\pi, \frac{\pi}{6})$
$M_A = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$
$(\Omega_A, M_A) \equiv (S, E_3) \equiv (\Omega_B, M_B)$
$\Psi(u, v) = f_B^{-1} f_A = (\cos v \sin u, -\cos v \cos u)$
$M_A = d\Psi^\top M_B d\Psi$
(Ω_B, M_B)
v
u
$(0, \frac{\sqrt{3}}{2})$
$(0, \frac{1}{2})$
Ω_B
$(0, \frac{-1}{2})$
$(0, \frac{-\sqrt{3}}{2})$
$M_B = \frac{1}{1 - u^2 - v^2} \begin{pmatrix} 1 - v^2 & uv \\ uv & 1 - u^2 \end{pmatrix}$