Lecture 1

Introduction to Geometry Processing

Spring 2017

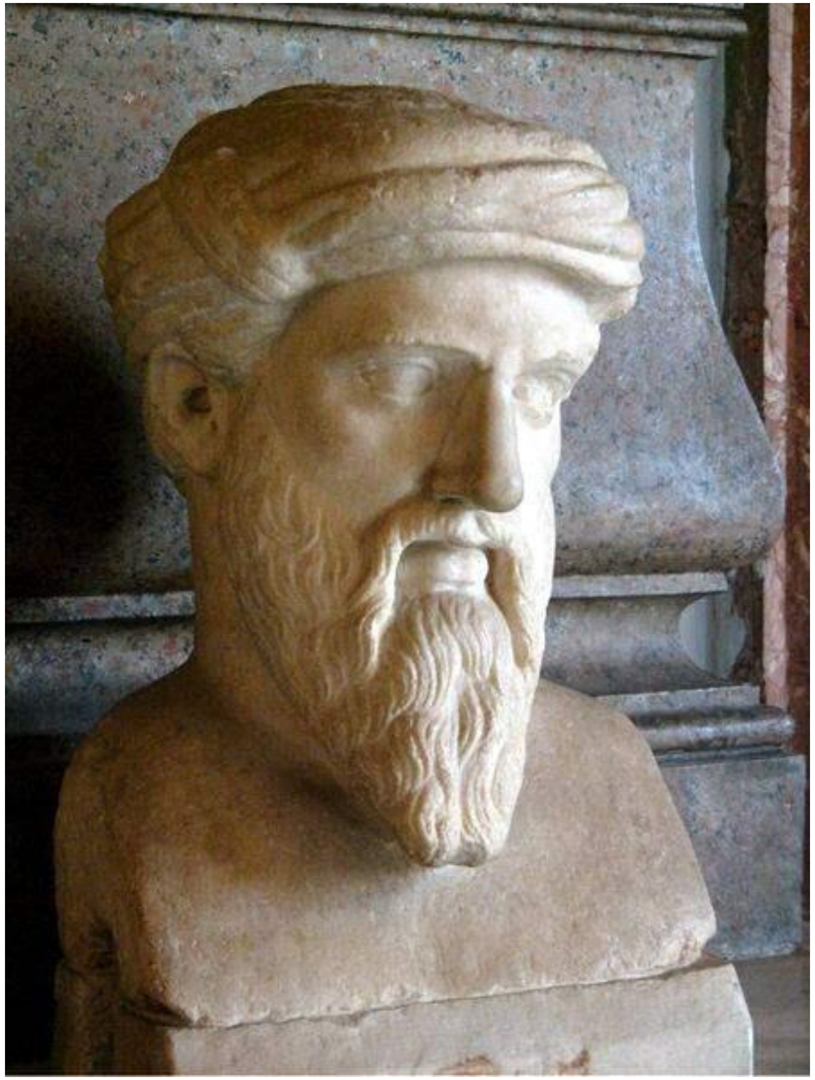
Johns Hopkins University

Let’s start from the beginning …

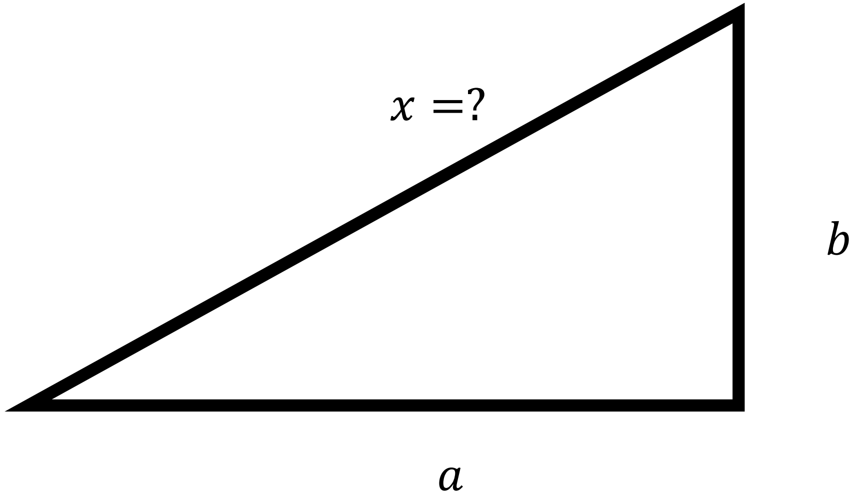
Γεωμετρία

Branch of mathematics concerned with questions of shape, size, relative position of figures, and the properties of space.

[Wikipedia]

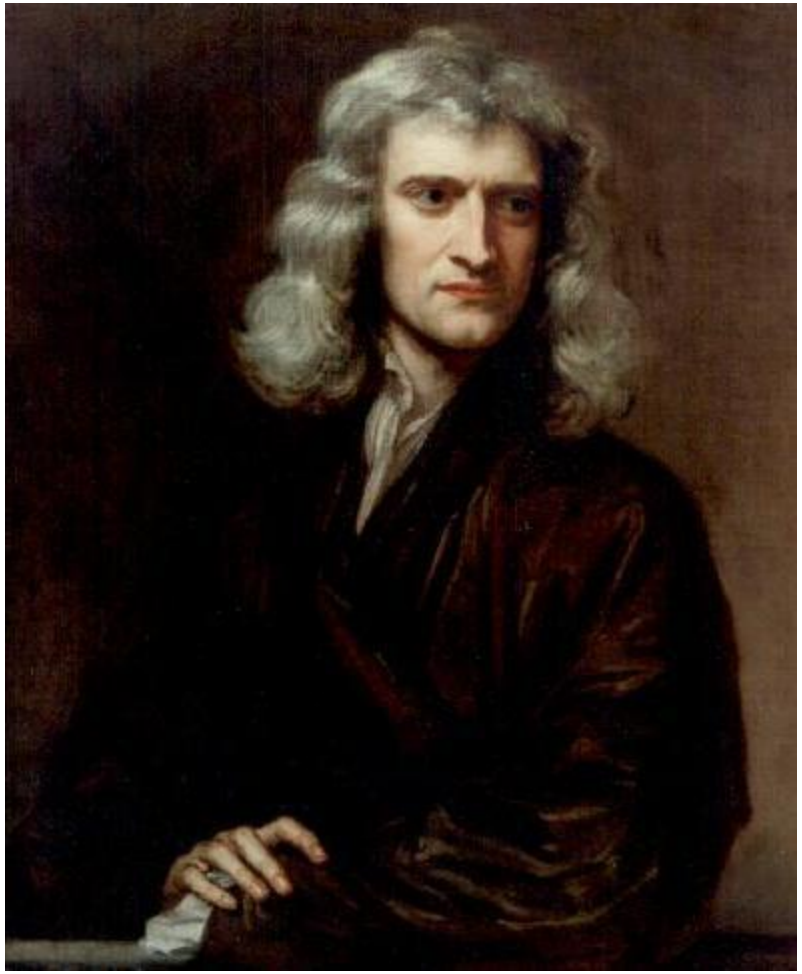


Pythagoras
(570 – c. 495 BC)

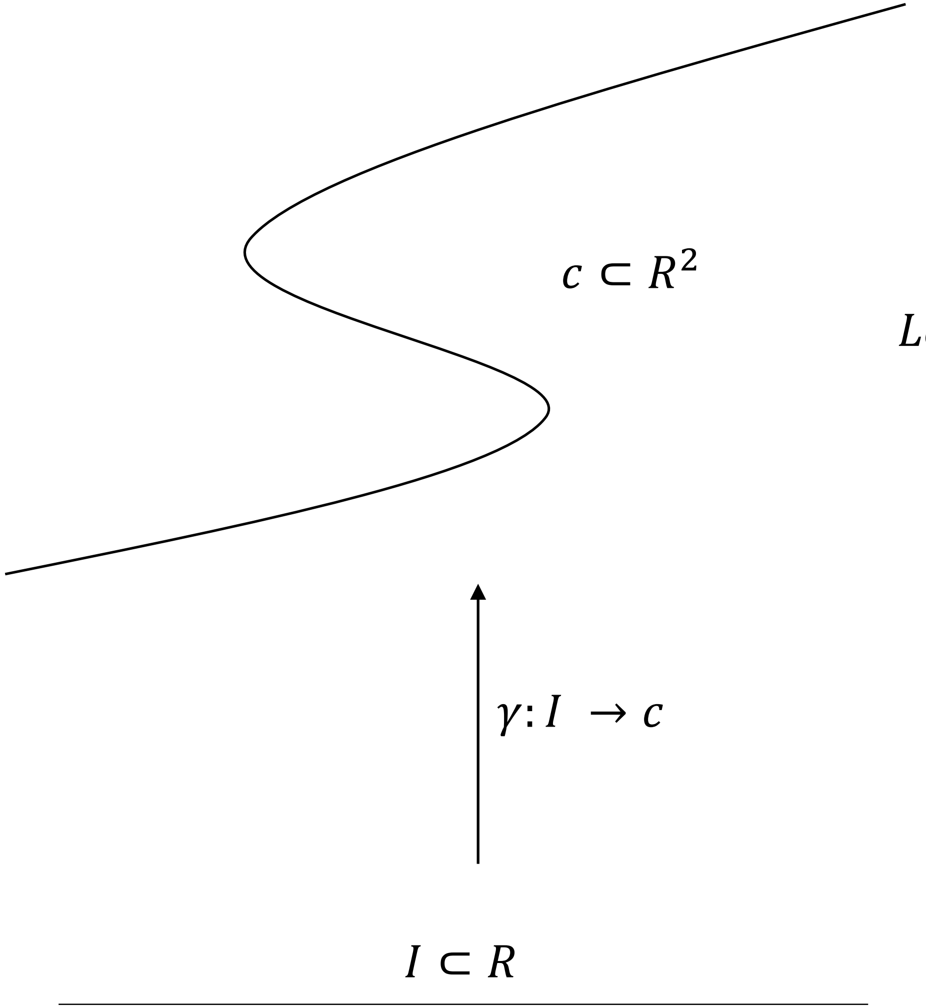


$$x = \sqrt{a^2 + b^2}$$

Prove it!



Isaac Newton
(1642 - 1727)



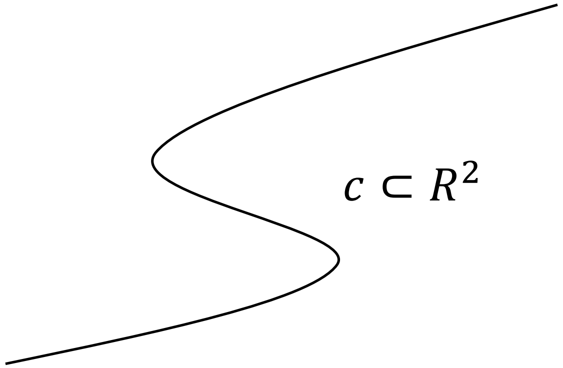
$Length(c) = ?$

$$Length(c) = \int_I \sqrt{\left(\frac{d\gamma}{dt}\right)^{\top}\left(\frac{d\gamma}{dt}\right)}\, dt = \int_I \sqrt{\left(\frac{d\gamma^u}{dt}\right)^2 + \left(\frac{d\gamma^v}{dt}\right)^2}\, dt$$

Prove it!

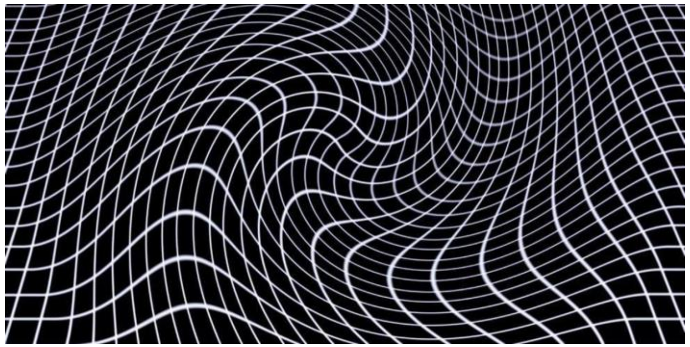
Bernhard Riemann

(1826 - 1866)



$c \subset R^2$

,



Metric(M)

$Length_M(c) =?$

Metric

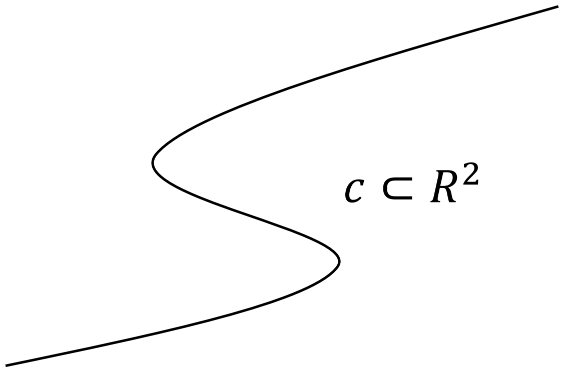
$$M\colon R^2 \rightarrow L_{2\times 2}$$

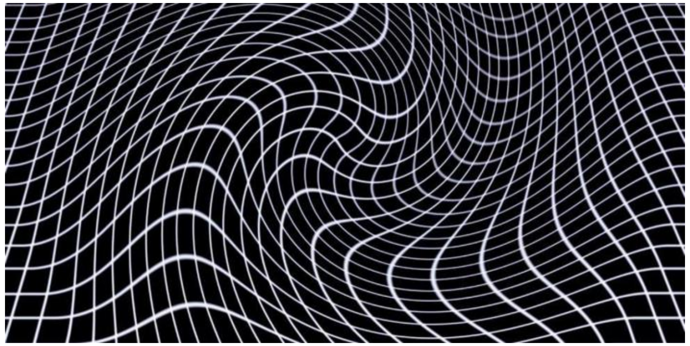
Example:

$$M_1(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$M_2(u,v) = \begin{pmatrix} |u|+1 & 1 \\ 1 & |v|+1 \end{pmatrix}$$

$$M_3(u,v) = \begin{pmatrix} u^2 & 0 \\ 0 & v^2 \end{pmatrix}$$





Metric(M)

$\gamma: I \to c$

$I \subset R$

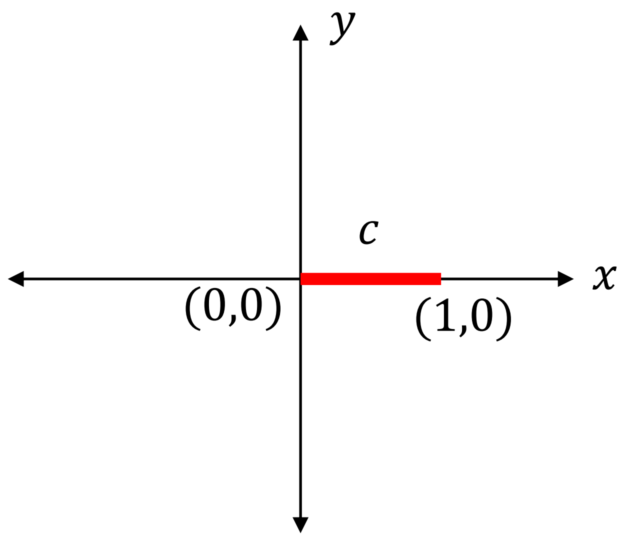
$$Length_M(c) = \int_I \sqrt{\left(\frac{d\gamma}{dt}\right)^{\top} M \left(\frac{d\gamma}{dt}\right)}\,dt = \int_I \sqrt{\begin{pmatrix} \frac{d\gamma^u}{dt} \\ \frac{d\gamma^v}{dt} \end{pmatrix}^{\top} M \begin{pmatrix} \frac{d\gamma^u}{dt} \\ \frac{d\gamma^v}{dt} \end{pmatrix}}\,dt$$

Bernhard Riemann

(1826 - 1866)

Length

$$Length_M(c) = \int_I \sqrt{\left(\frac{d\gamma}{dt}\right)^\top M \left(\frac{d\gamma}{dt}\right)} dt$$



$$M_1(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

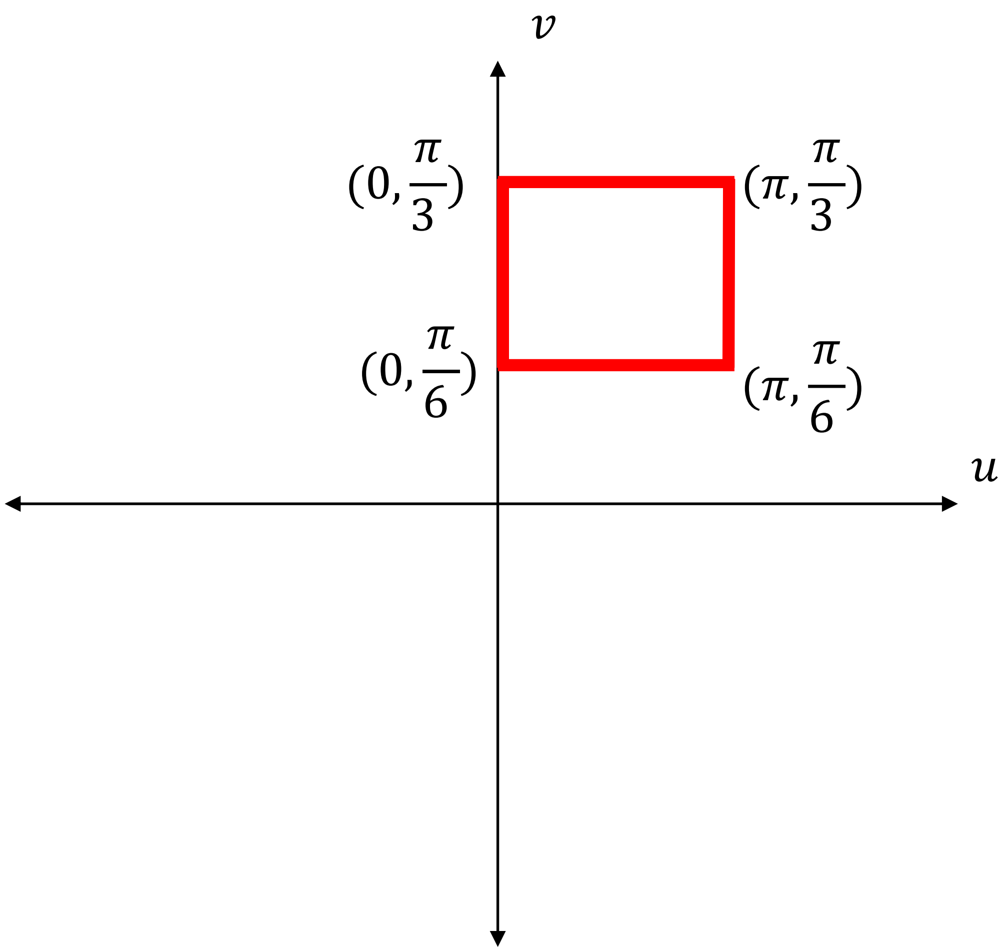
$$Length_{M_1}(c) = ?$$

$$M_2(u,v) = \begin{pmatrix} |u|+1 & 1 \\ 1 & |v|+1 \end{pmatrix}$$

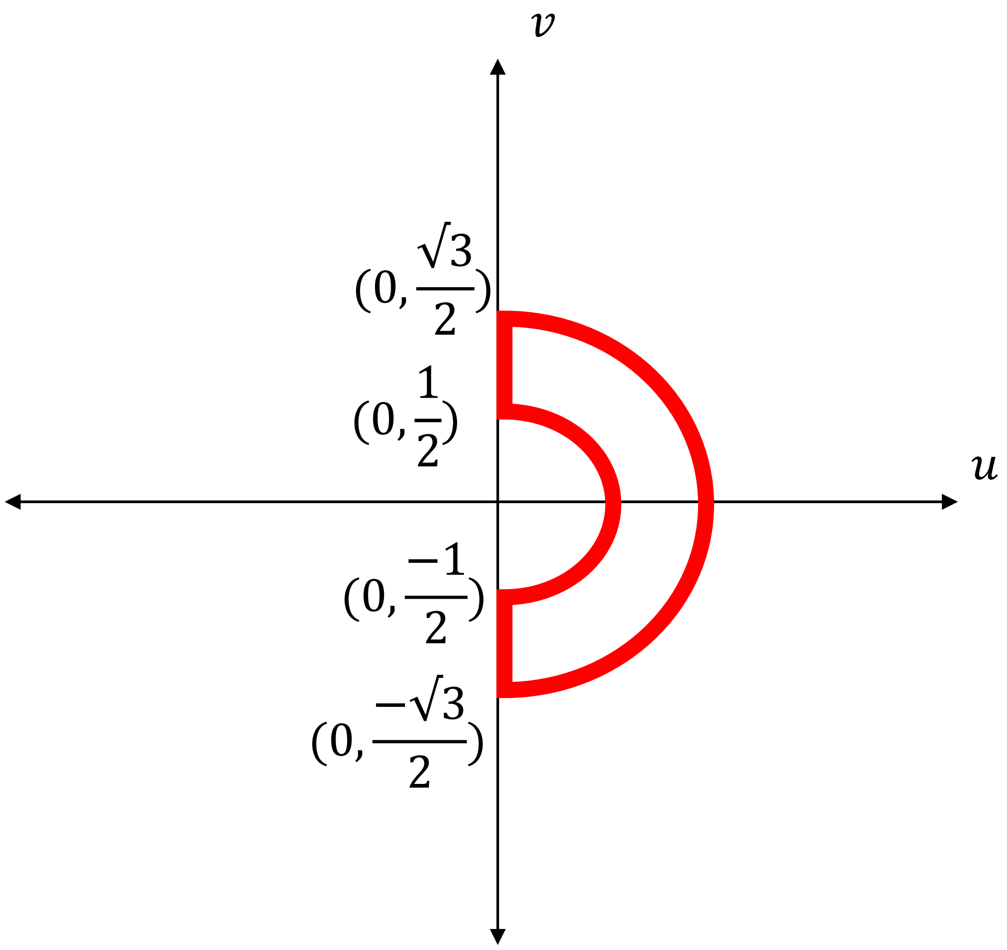
$$Length_{M_2}(c) = ?$$

$$M_3(u,v) = \begin{pmatrix} u^2 & 0 \\ 0 & v^2 \end{pmatrix}$$

$$Length_{M_3}(c) = ?$$



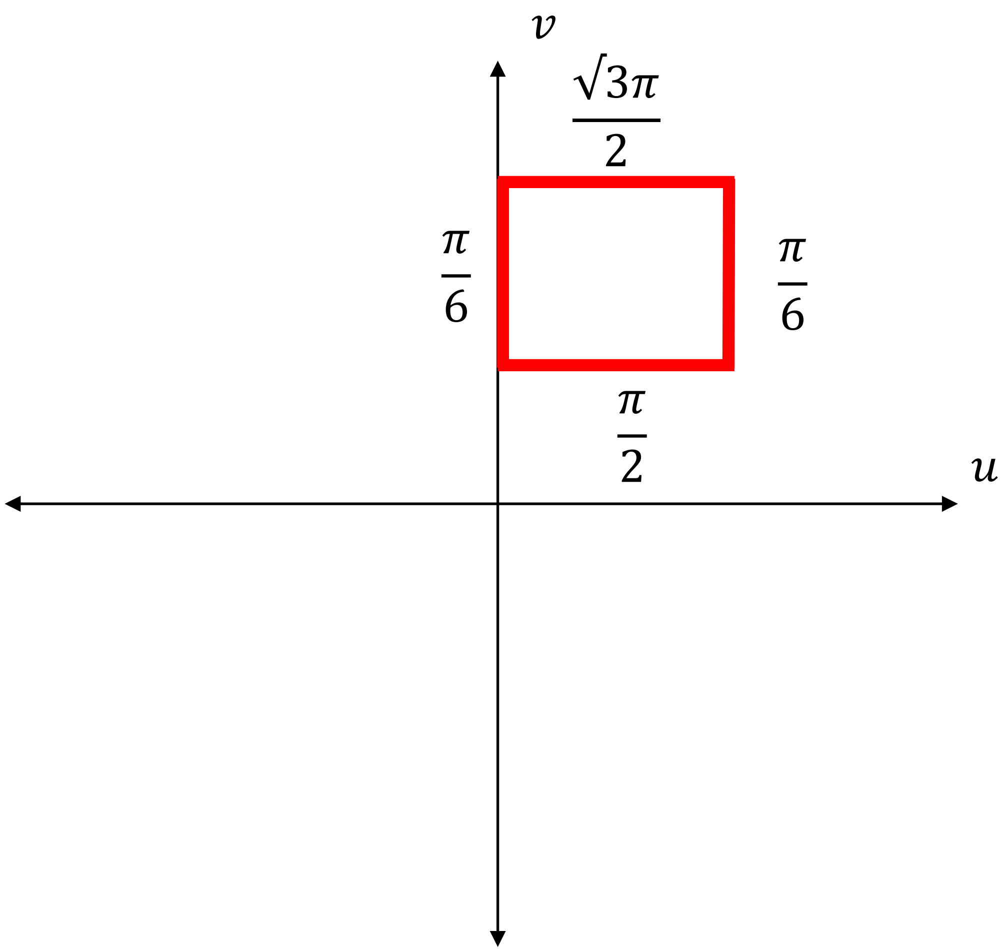
$$M_A = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$



$$M_B = \frac{1}{1 - u^2 - v^2} \begin{pmatrix} 1 - v^2 & uv \\ uv & 1 - u^2 \end{pmatrix}$$

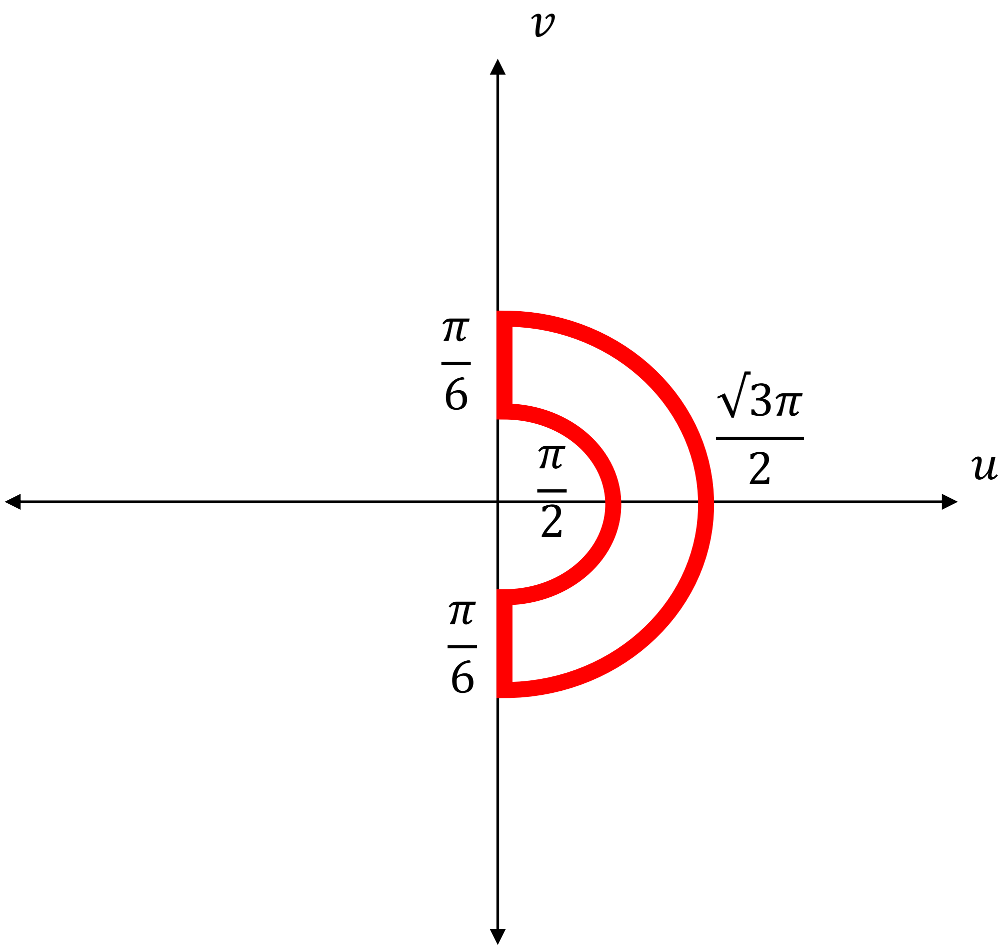
Compute the perimeter of the curve in red for the respective metrics.

$$Length_M(c) = \int_I \sqrt{\left(\frac{d\gamma}{dt}\right)^{\top} M \left(\frac{d\gamma}{dt}\right)}\, dt$$



$$M_A = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

$$Length_{M_A}(c_A) = \frac{(5+3\sqrt{3})\pi}{6}$$

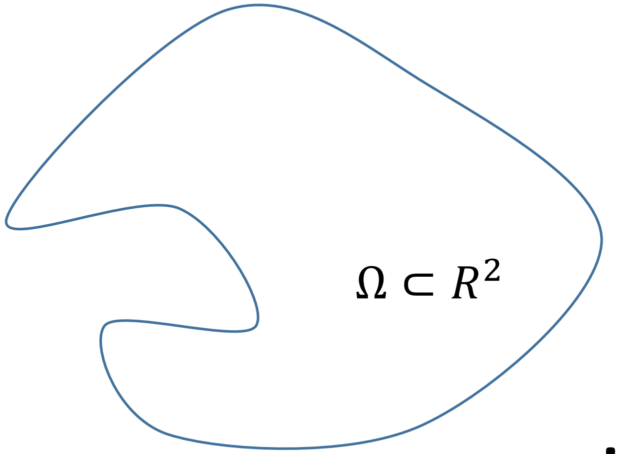


$$M_B = \frac{1}{1-u^2-v^2}\begin{pmatrix} 1-v^2 & uv \\ uv & 1-u^2 \end{pmatrix}$$

$$Length_{M_B}(c_B) = \frac{(5+3\sqrt{3})\pi}{6}$$

Area

Bernhard Riemann
(1826 - 1866)



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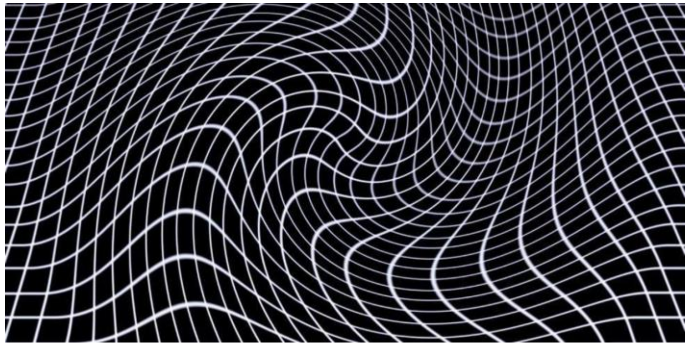
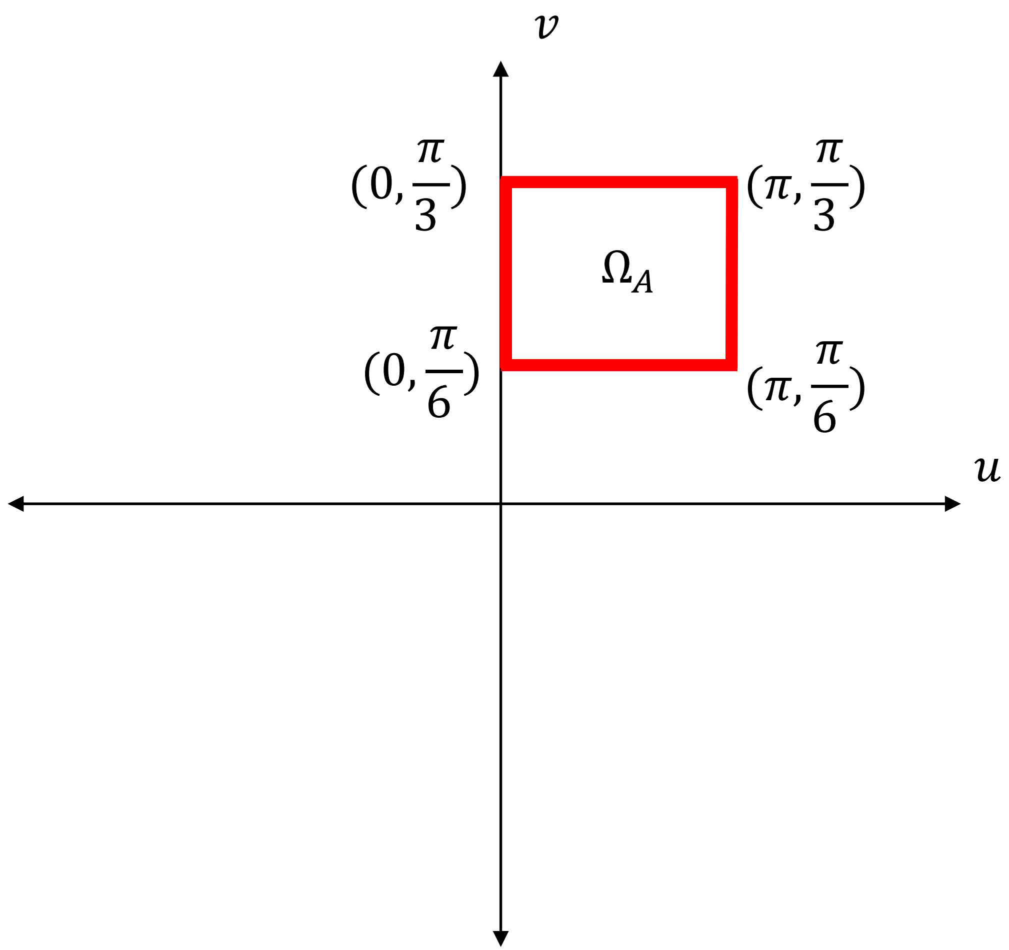
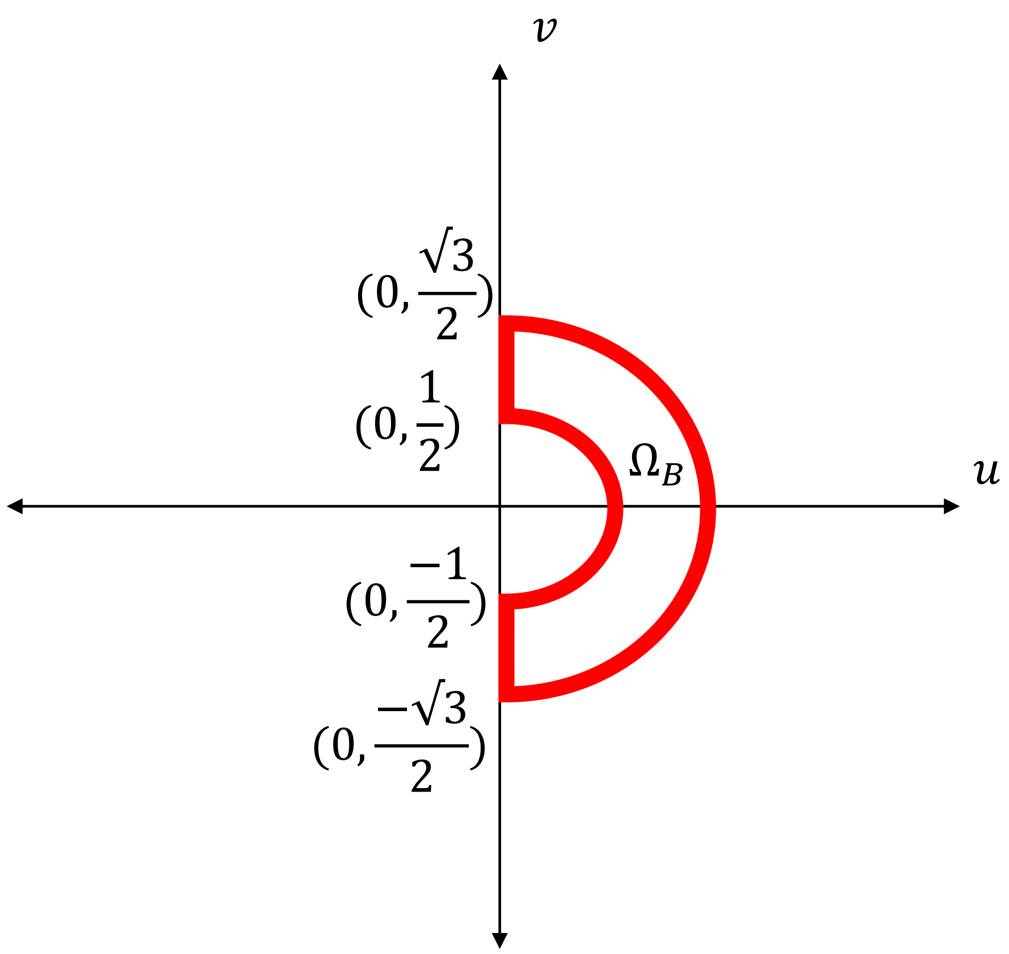


Metric(M)

$$\text{Area}_{\text{M}}(\Omega) = \int_{\Omega} \sqrt{\det(M)}\, du\, dv$$



$$M_A = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

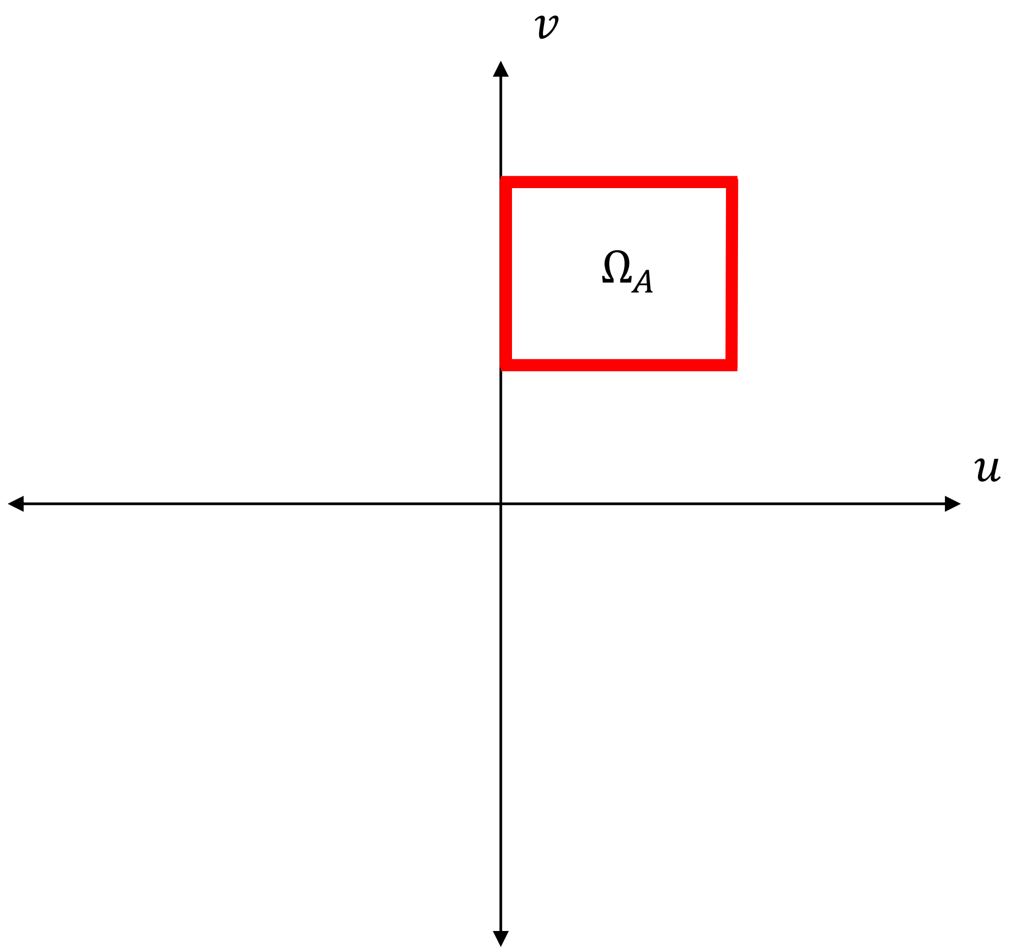


$$M_B = \frac{1}{1 - u^2 - v^2} \begin{pmatrix} 1 - v^2 & uv \\ uv & 1 - u^2 \end{pmatrix}$$

Compute the area of the region enclosed by the curve

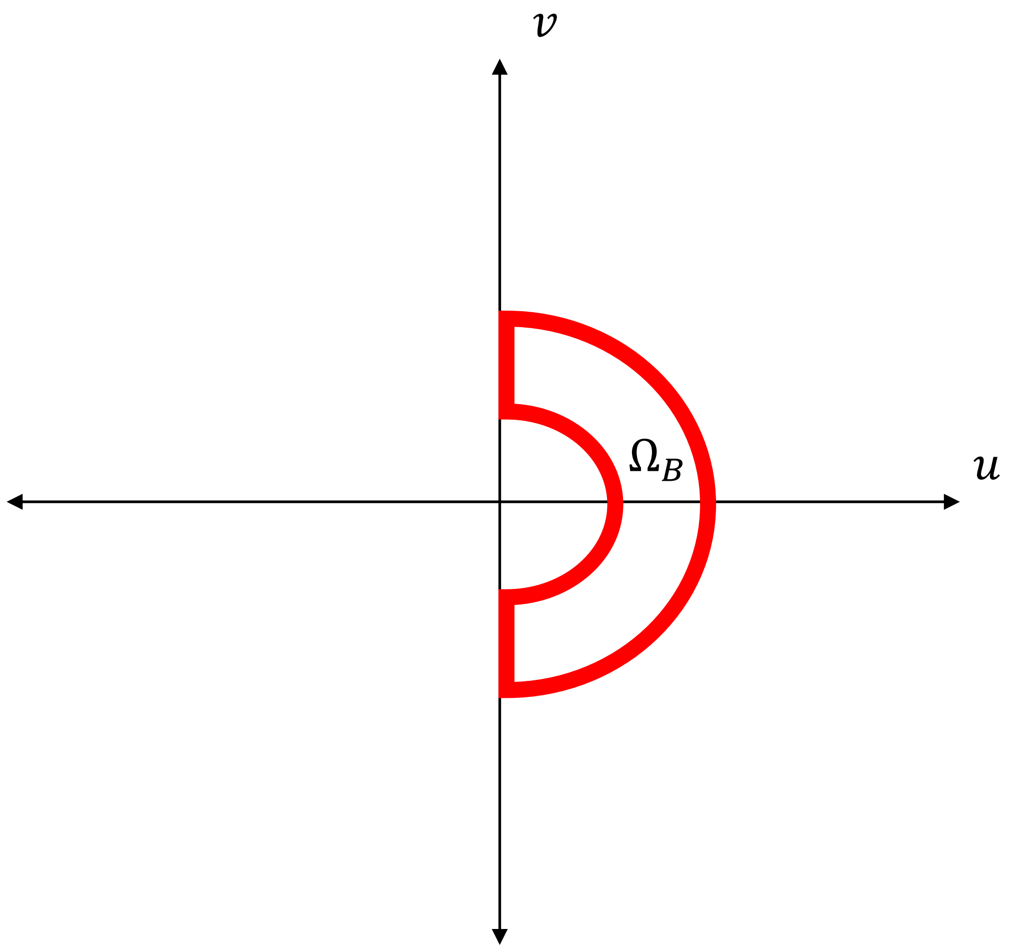
$$\mathrm{Area_M}(\Omega) = \int_{\Omega} \sqrt{\det(M)}dudv$$

Hint: Use the change of coordinates,
$u = rcos(\theta)$,
$v = rsin(\theta)$,
$dudv = rdrd\theta$



$$M_A = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{Area}_{\mathrm{M}_A}(\Omega_A) = \frac{(\sqrt{3}-2)\pi}{2}$$



$$M_B = \frac{1}{1-u^2-v^2}\begin{pmatrix} 1-v^2 & uv \\ uv & 1-u^2 \end{pmatrix}$$

$$\mathrm{Area}_{\mathrm{M}_B}(\Omega_B) = \frac{(\sqrt{3}-2)\pi}{2}$$