Search and Intersection

O'Rourke, Chapter 7

de Berg *et al.*, Chapter 11

Announcements

- Assignment 3 web-page has been updated:
  - Additional extra credit
  - Hints for managing a dynamic half-edge representation

Outline

- Review
  - Duality
  - Linear Programming
- Half-Spaces and Convex Hulls (2D)

Duality

Definition:

Given a point $p = (\alpha, \beta)$ in the plane, define the *dual line* to be the (non-vertical) line with equation:

$$p^* = \{(x, y) | y = 2\alpha x - \beta\}$$

Given a line $L = \{(x, y) | y = mx + b\}$, define the *dual point* to be the point with coordinates:

$$L^* = \left(\frac{m}{2}, -b\right)$$

Duality

Properties:

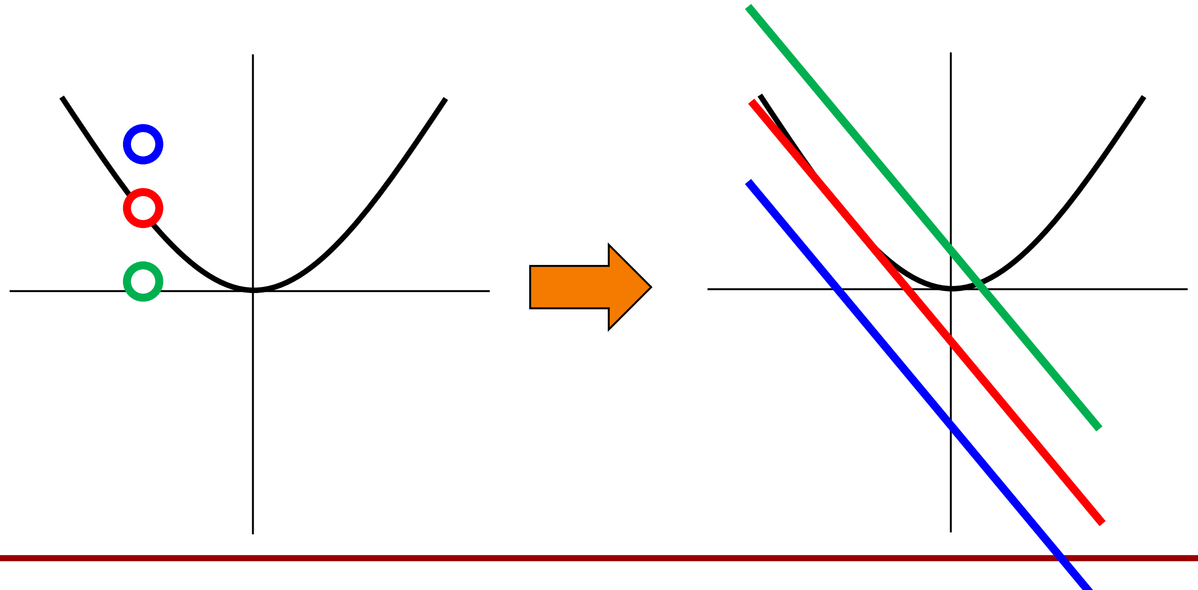
Given a point p and lines L, L_1, and L_2:

- $(p^*)^* = p$ and $(L^*)^* = L$.
- $p \in L$ iff. $L^* \in p^*$.
- $p \in L_1 \cap L_2$ iff. $L_1^*, L_2^* \in p^*$.
- L is below/above p iff. L^* is above/below p^*.
- p is on the parabola $y = x^2$ iff. p^* is tangent to the parabola at p.

Duality

Properties:

- Given a point $p = (\alpha, \beta)$:
  - The slope of p^* is the slope of the tangent to the parabola at (α, α^2).
  - p^* passes through the point $\left(\alpha, \alpha^2 + (\alpha^2 - \beta)\right)$.



Linear Programming

Goal:

Given a set of linear constraints:

$$C_i = \{p \mid \langle p, n_i \rangle \geq d_i\}$$

and a linear energy function:

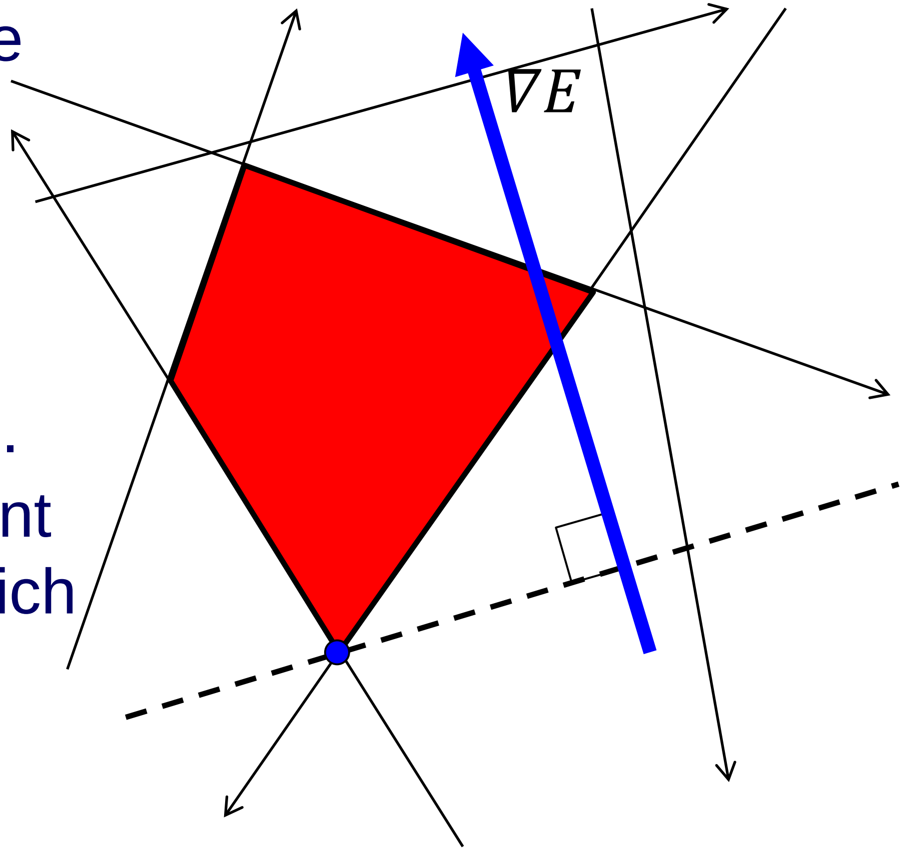
$$E(p) = \langle p, n \rangle + d$$

we would like to find the point p that satisfies the constraints and minimizes the energy.

Linear Programming

<u>Approach</u>:

- Since the constraints are linear, each one defines a half-space of valid solutions.
- The intersection of these half-spaces is convex.
- Since the energy is linear, it has a constant gradient ∇E pointing away from the minimum.
- The minimizer is the point in the convex region which is extreme along direction n.



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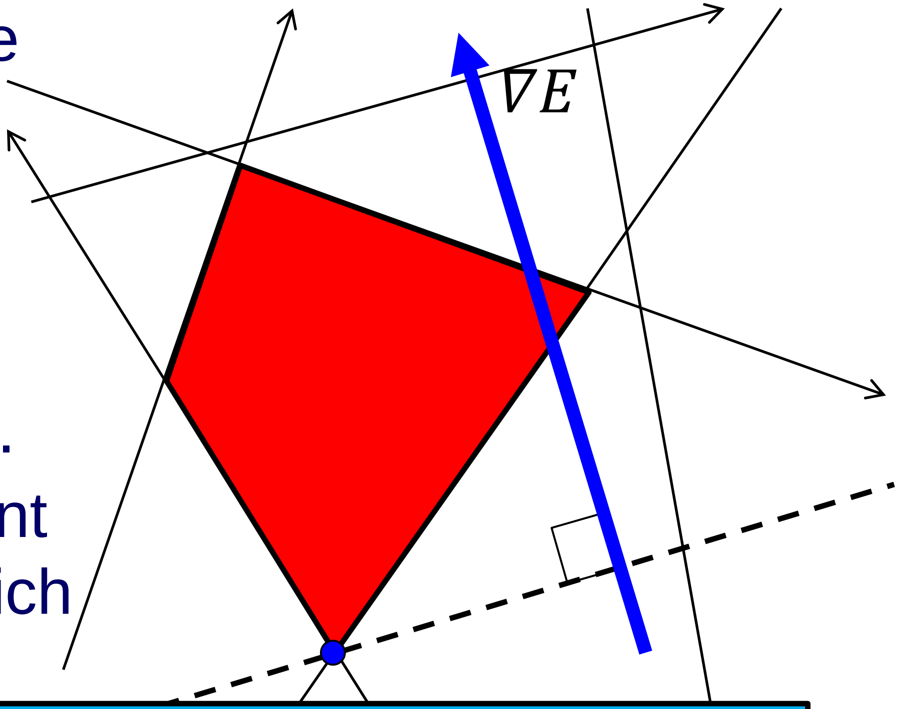


How do we compute the convex hull corresponding to the intersection of half spaces?

Outline

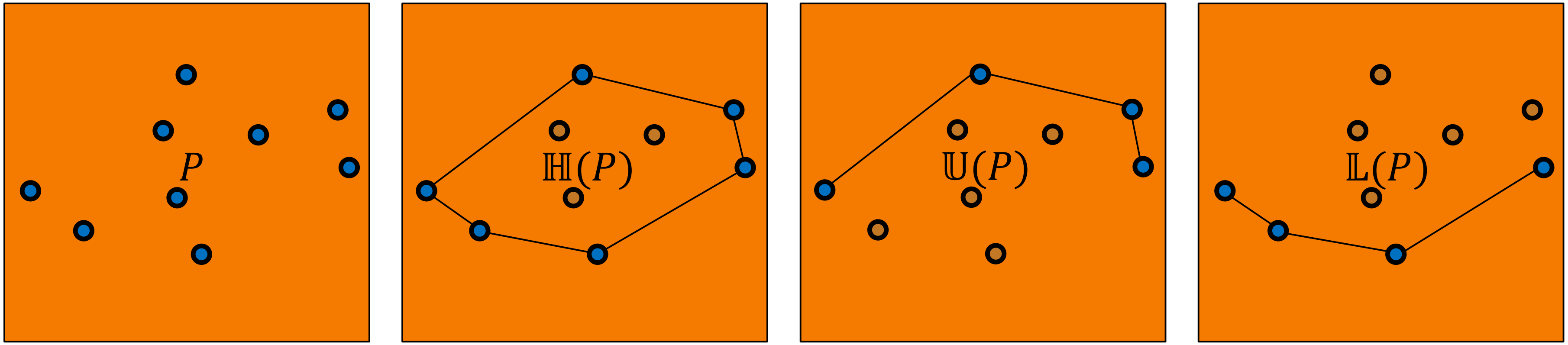
- Review
- Half-Spaces and Convex Hulls (2D)

Half-Spaces and Convex Hulls (2D)

Notation:

Given a set of points, $P = \{p_1, \dots, p_n\}$:

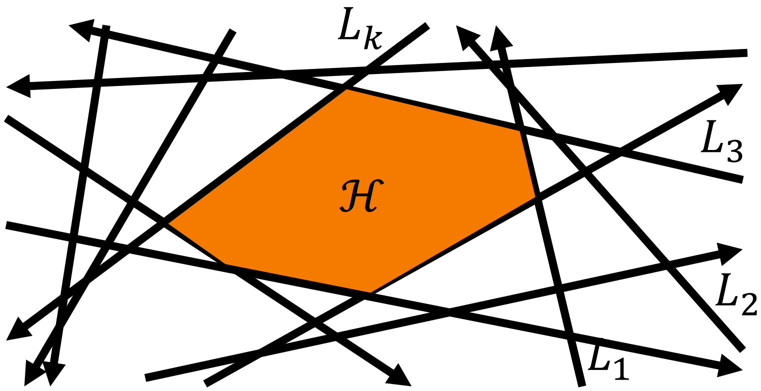
- Denote the convex hull of the points as: $\mathbb{H}(P)$
- Denote the upper hull of the points as: $\mathbb{U}(P)$
- Denote the lower hull of the points as: $\mathbb{L}(P)$



Half-Spaces and Convex Hulls (2D)

Goal:

Given a set of half-spaces, represented by directed lines $\{L_1, \ldots, L_n\}$ compute the convex hull, $\mathcal{H}$, corresponding to the boundary of their intersection.

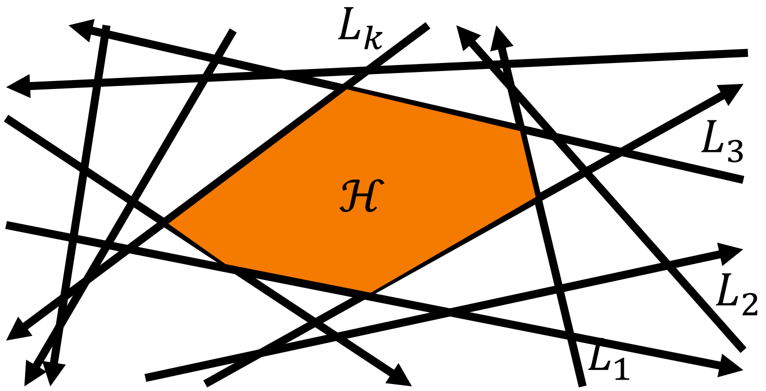


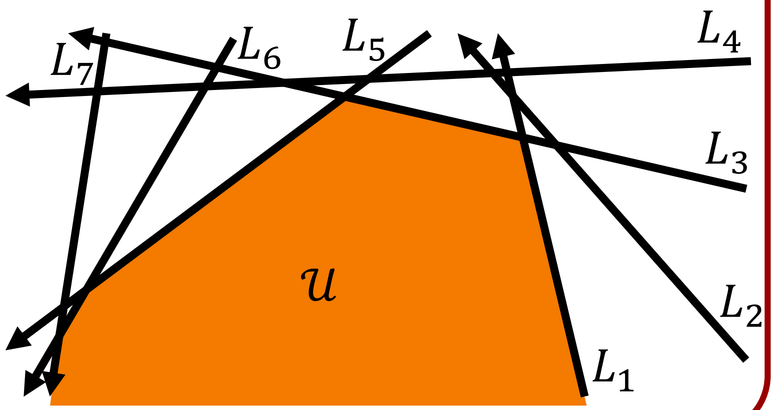
Half-Spaces and Convex Hulls (2D)

Approach:

Consider the upper/lower hulls, $\mathcal{U}/\mathcal{L}$, independently.

- For the upper (resp. lower) part, assume all line segments have $(0,-\infty)$ (resp. $(0,\infty)$) to their left.





Half-Spaces and Convex Hulls (2D)

Approach:

Consider the upper/lower hulls, $\mathcal{U}/\mathcal{L}$, independently.

An edge $e \subset \mathcal{U}$ is the set of points on a line that are below all the other lines:

$$e \subset L_i \quad \text{and} \quad \text{Below}(v, L_k)\ \forall k \neq i\,, v \in e$$

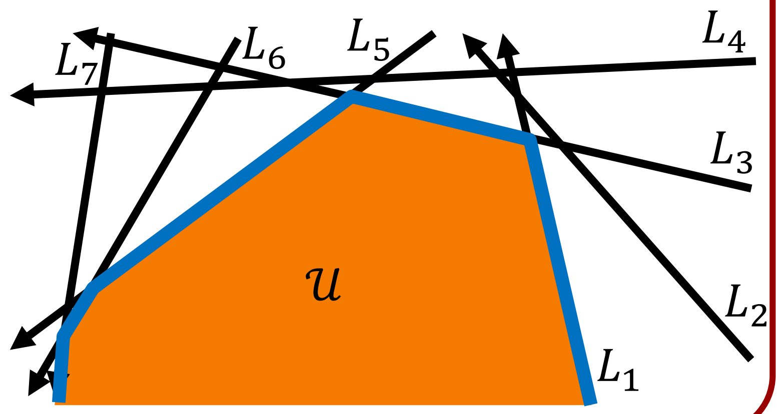
Dually:

$$L_i^* \in v^* \quad \text{and} \quad \text{Below}(L_k^*, v^*)\ \forall k \neq i\,, v \in e$$

⇔ Lines v^*, with $v \in e$, pass through L_i^* and have all other $\{L_1^*, \ldots, L_n^*\}$ below.

⇔ L_i^* is a vertex of:

$$\mathcal{U}^* = \mathbb{U}(L_1^*, \ldots, L_n^*)$$



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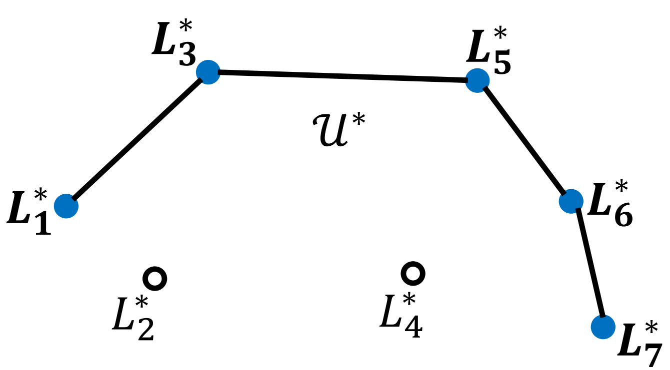
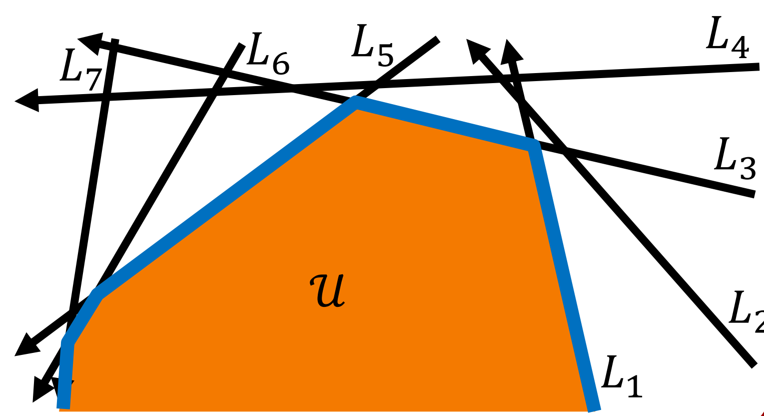




Half-Spaces and Convex Hulls (2D)

Approach:

Consider the upper/lower hulls, $\mathcal{U}/\mathcal{L}$, independently.

A vertex $v \in \mathcal{U}$ lies on the intersection of two lines and is below all the other lines:

$$v \in L_i \cap L_j \quad \text{and} \quad \text{Below}(v, L_k)\ \forall k \neq i, j$$

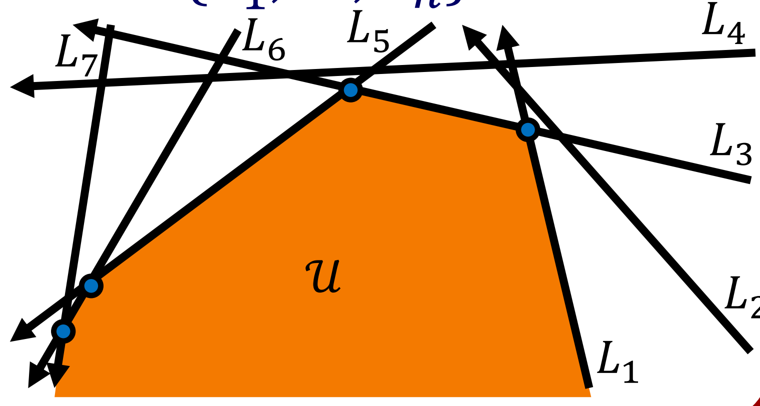
Dually:

$$L_i^*, L_j^* \in v^* \quad \text{and} \quad \text{Below}(L_k^*, v^*)\ \forall k \neq i, j$$

$\Leftrightarrow$ The line segment $\overline{L_i^* L_j^*}$ has all other $\{L_1^*, \ldots, L_n^*\}$ below.

$\Leftrightarrow$ $\overline{L_i^* L_j^*}$ is an edge of

$$\mathcal{U}^* = \mathbb{U}(L_1^*, \ldots, L_n^*)$$



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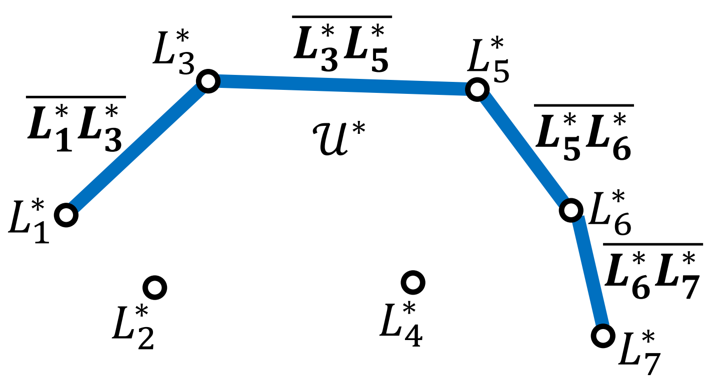
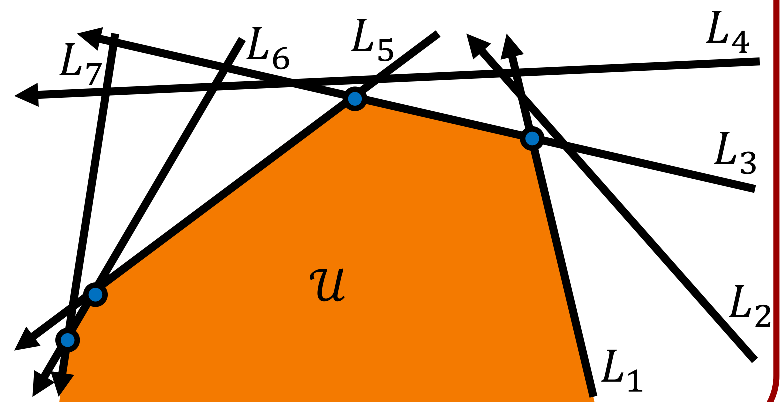




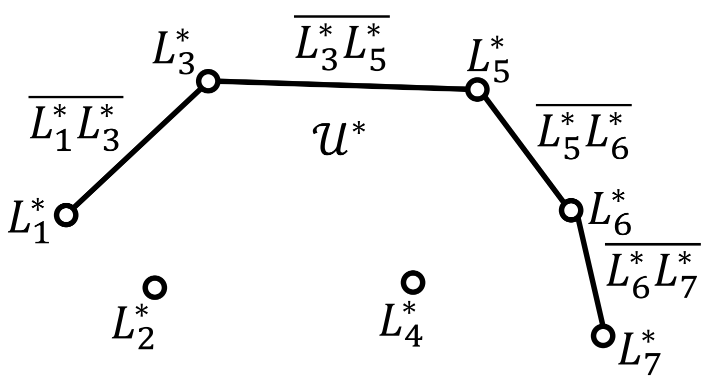
Half-Spaces and Convex Hulls (2D)

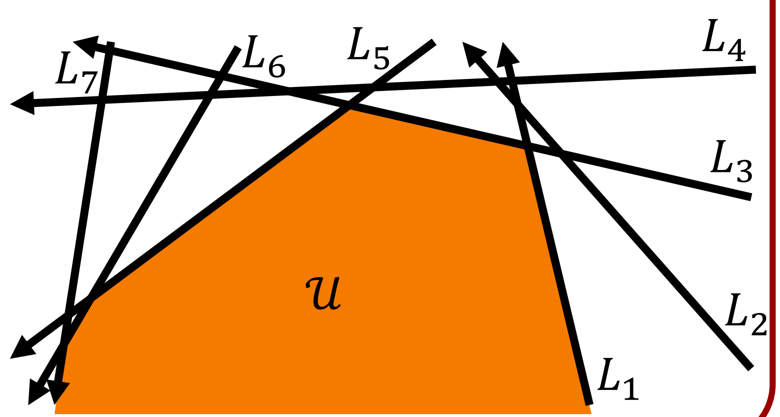
<u>Implementation</u>:

UpperHull($\{L_1, \dots, L_n\}$)

- $\{v_1, \dots, v_m\} \leftarrow$ UpperHull($\{L_1^*, \dots, L_n^*\}$)
- return $\{v_m^*, \dots, v_1^*\}$

This gives the lines in the (CCW) order in which they appear on the intersection of half-spaces.





Half-Spaces and Convex Hulls (2D)

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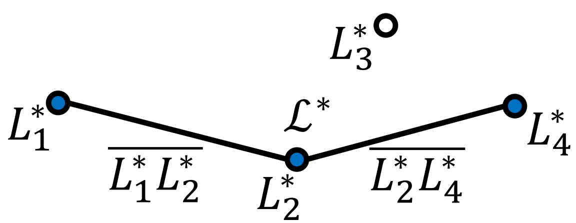
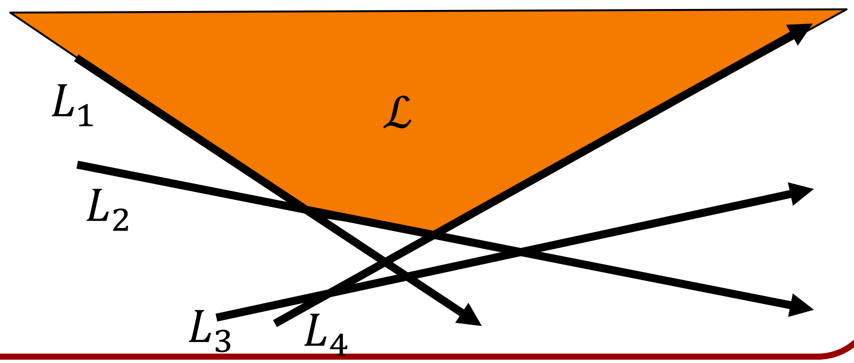
UpperHull($\{L_1, \dots, L_n\}$)

- $\{v_1, \dots, v_m\} \leftarrow$ UpperHull($\{L_1^*, \dots, L_n^*\}$)
- return $\{v_m^*, \dots, v_1^*\}$

LowerHull($\{L_1, \dots, L_n\}$)

- $\{v_1, \dots, v_m\} \leftarrow$ LowerHull($\{L_1^*, \dots, L_n^*\}$)
- return $\{v_1^*, \dots, v_m^*\}$





Half-Spaces and Convex Hulls (2D)

Implementation:

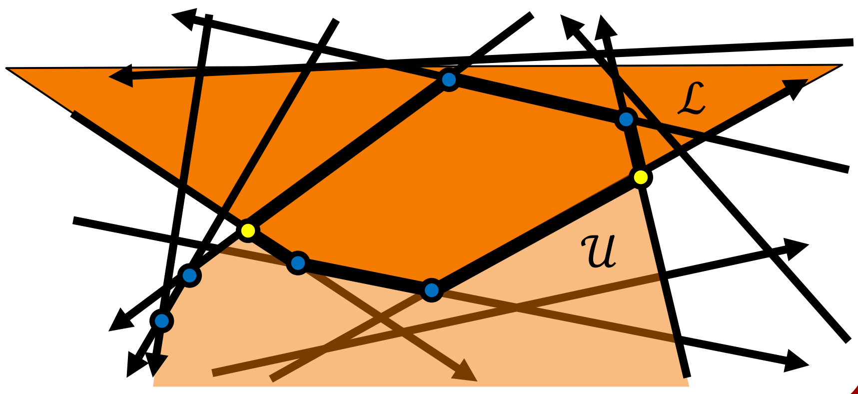
UpperHull($\{L_1, \ldots, L_n\}$)

- $\{v_1, \ldots, v_m\} \leftarrow$ UpperHull($\{L_1^*, \ldots, L_n^*\}$)
- return $\{v_m^*, \ldots, v_1^*\}$

LowerHull($\{L_1, \ldots, L_n\}$)

- $\{v_1, \ldots, v_m\} \leftarrow$ LowerHull($\{L_1^*, \ldots, L_n^*\}$)
- return $\{v_1^*, \ldots, v_m^*\}$

Taking the intersection (in linear time), we get the convex hull.



Half-Spaces and Convex Hulls (2D)

Implementation:

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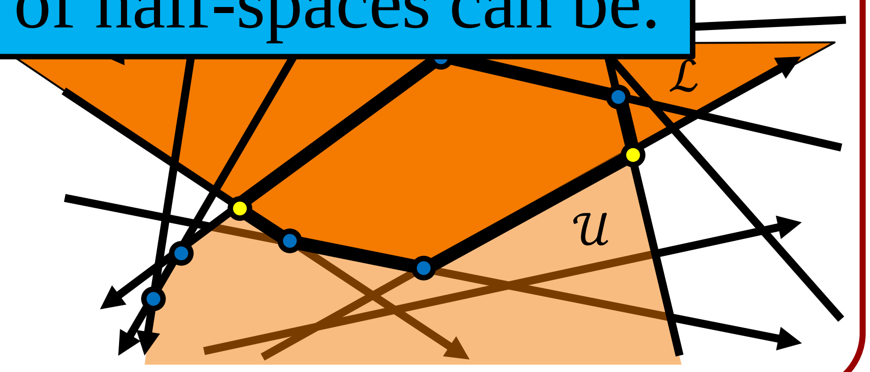
We have to separately compute the upper and lower hulls because the dual map is undefined (discontinuous) as lines approach vertical.

Is there an alternate definition of duality that would allow us to compute the hulls simultaneously?

No! The convex hull of the dual cannot be empty, but the intersection of half-spaces can be.

Ta... (in linear time), we get the convex hull.



Half-Spaces and Convex Hulls

Writing $p \in \mathbb{R}^n$ as:

$$p = (\alpha_1, \dots, \alpha_{n-1}, \beta) = (\vec{\alpha}, \beta)$$

we can define duality in n-dimensional space.

Given a point $p = (\vec{\alpha}, \beta)$, define the *dual hyper-plane* to be the (non-vertical) plane:

$$p^* = \{(\vec{x}, y) | y = 2\langle \vec{x}, \vec{\alpha} \rangle - \beta\}$$

Given a plane $H = \{(\vec{x}, y) | y = \langle \vec{m}, \vec{x} \rangle + b\}$, define the *dual point* to be the point with coordinates:

$$H^* = \left(\frac{\vec{m}}{2}, -b\right)$$

Half-Spaces and Convex Hulls

Writing $p \in \mathbb{R}^n$ as:

$$p \equiv (\alpha_1, \ldots, \alpha_{n-1}, \beta) \equiv (\vec{\alpha}, \beta)$$

The same properties hold in n-dimensions:

- $(p^*)^* = p$ and $(H^*)^* = H$.
- $p \in H$ iff. $H^* \in p^*$.
- $p \in H_1 \cap H_2$ iff. $H_1^*, H_2^* \in p^*$.
- H is below/above p iff. H^* is above/below p^*.
- $p = (\vec{x}, y)$ is on the parabola $y = \|\vec{x}\|^2$ iff. p^* is tangent to the parabola at p.

Given a plane $H = \{(\vec{x}, y) | y = \langle \vec{m}, \vec{x} \rangle + b\}$, define the *dual point* to be the point with coordinates:

$$H^* = \left(\frac{\vec{m}}{2}, -b\right)$$

Half-Spaces and Convex Hulls

In 3D:

Let $\{H_1,\ldots,H_m\}$ be oriented hyper-planes in $\mathbb{R}^3$ with $(0,0,-\infty)$ to the left and let $\mathcal{U}$ be the intersection of the associate half-spaces.

$$\{H_i^*,H_j^*,H_k^*\}\text{ is a triangle of }\mathbb{U}(\{H_1^*,\ldots,H_m^*\})$$
$$\Updownarrow$$
$$H_i\cap H_j\cap H_k\text{ is a vertex of }\mathcal{U}.$$

Half-Spaces and Convex Hulls

In 3D:

Let $\{H_1, \ldots, H_m\}$ be oriented hyper-planes in $\mathbb{R}^3$ with $(0, 0, -\infty)$ to the left and let $\mathcal{U}$ be the intersection of the associate half-spaces.

$$\{H_i^*, H_j^*\} \text{ is an edge of } \mathbb{U}(\{H_1^*, \ldots, H_m^*\})$$

$$\Updownarrow$$

$$H_i \cap H_j \text{ contains one edge of } \mathcal{U}.$$

Half-Spaces and Convex Hulls

<u>In 3D</u>:

Let $\{H_1, \ldots, H_m\}$ be oriented hyper-planes in $\mathbb{R}^3$ with $(0, 0, -\infty)$ to the left and let $\mathcal{U}$ be the intersection of the associate half-spaces.

$$H_i^* \text{ is a vertex of } \mathbb{U}(\{H_1^*, \ldots, H_m^*\})$$
$$\Updownarrow$$
$$H_i \text{ contains one face of } \mathcal{U}.$$

Half-Spaces and Convex Hulls

<u>More Generally</u>:

Let $\{H_1, \ldots, H_m\}$ be oriented hyper-planes in $\mathbb{R}^n$ with $(0, \ldots, 0, -\infty)$ to the left and let $\mathcal{U}$ be the intersection of the associate half-spaces.

$$\left\{H_{i_1}^*, \ldots, H_{i_k}^*\right\} \text{ is a } k\text{-simplex of } \mathbb{U}(\{H_1^*, \ldots, H_m^*\})$$

$$\Updownarrow$$

$$\cap_{j=1}^{k} H_{i_j} \text{ contains one } (n-d)\text{-dimensional face of } \mathcal{U}.$$