Search and Intersection

O’Rourke, Chapter 7

Anouncements

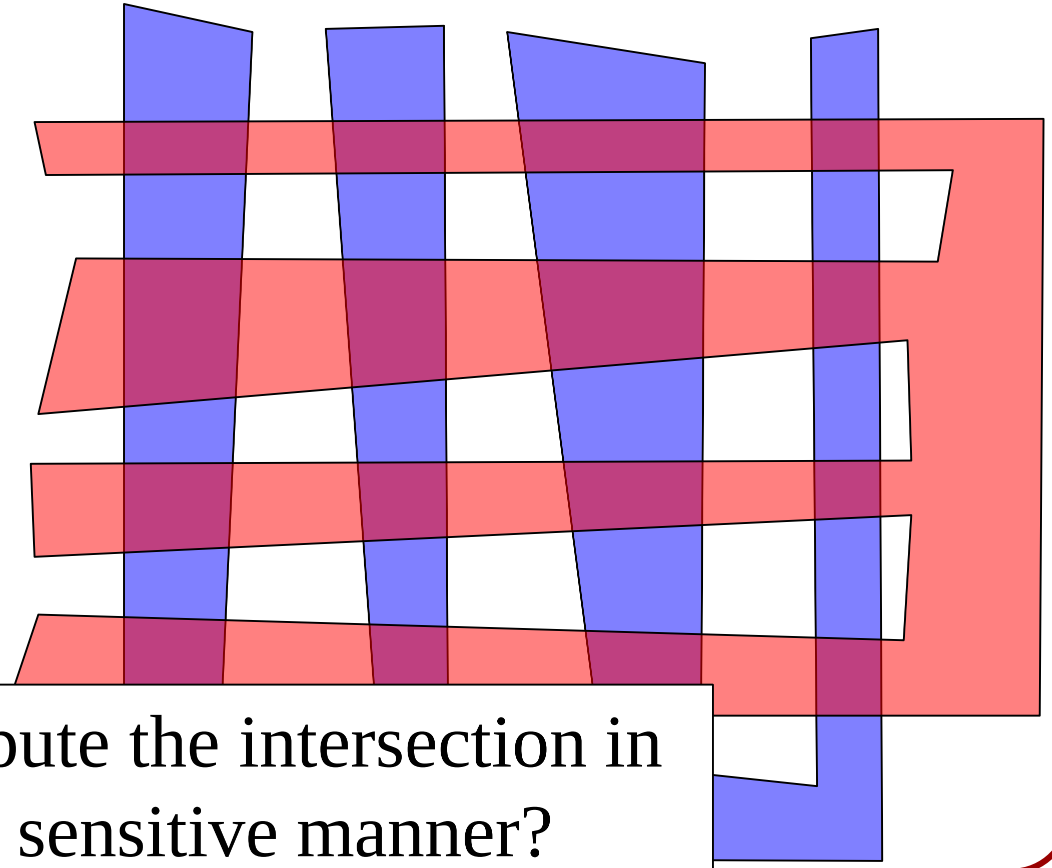
- Assignment 2 has been graded
- Assignment 3 has been posted

Outline

- Polygon Intersection
  - Segment Intersection
- Convex Polygon Intersection

Polygon Intersection

Given polygons P and Q, in the worst case they can intersect in $O(|P| \cdot |Q|)$ positions.



Can we compute the intersection in an output sensitive manner?

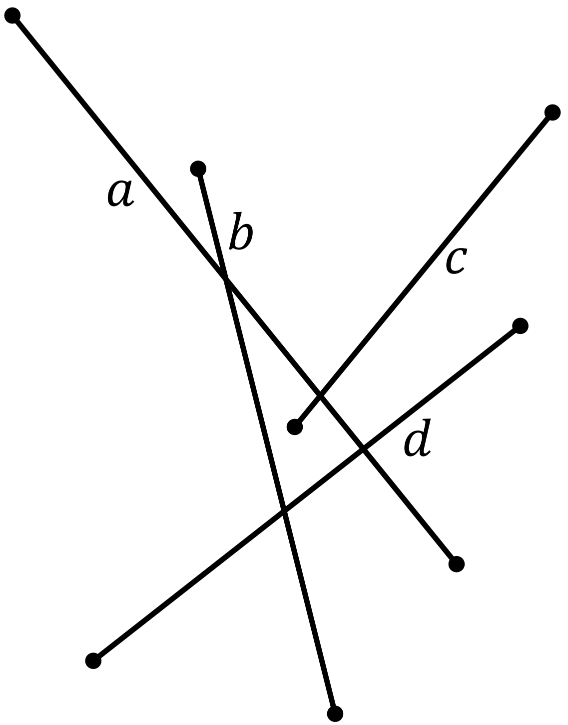
Segment Intersection

Given a set of line segments, find crossings.

Approach:

Assume general position.

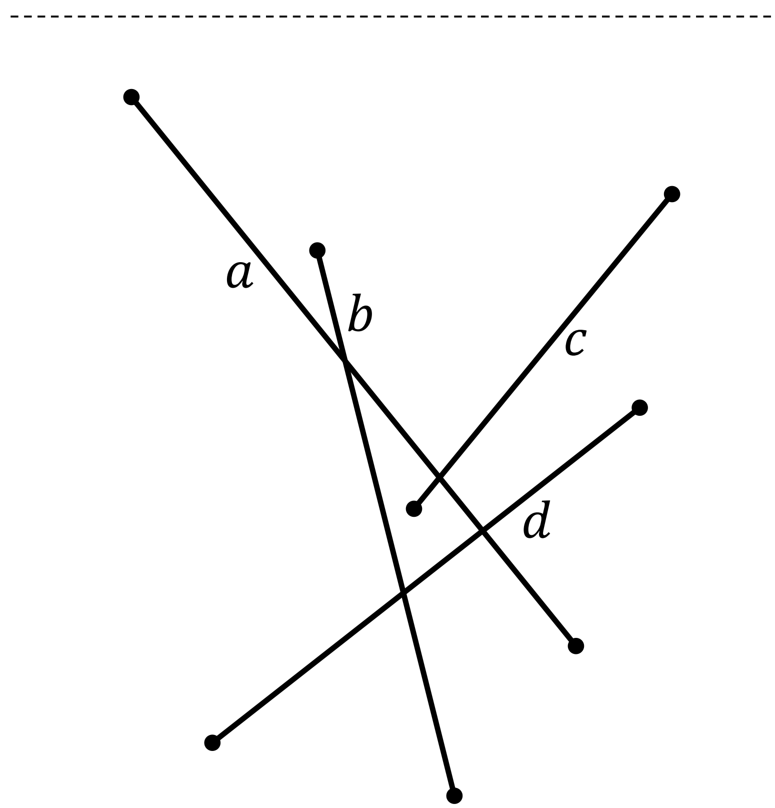
Use sweep line algorithm.



Segment Intersection

Sweep line algorithm:

- Initialize queue with end-points sorted by height.
- Initialize list of events.
- Advance:
  - Add/remove segments
  - Adjust event list
  - Test for neighboring intersections
  - Adjust queue

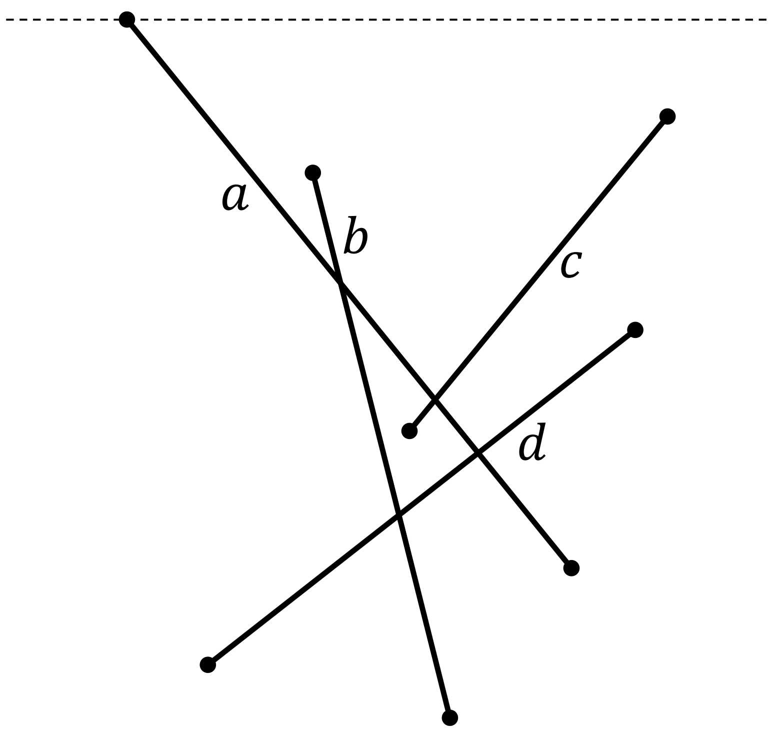


$Q = (a_1, c_1, b_1, d_1, c_2, a_2, d_2, b_2)$
$L = \emptyset$

Segment Intersection

Sweep line algorithm:

- Initialize queue with end-points sorted by height.
- Initialize list of events.
- Advance:
  - Add/remove segments
  - Adjust event list
  - Test for neighboring intersections
  - Adjust queue



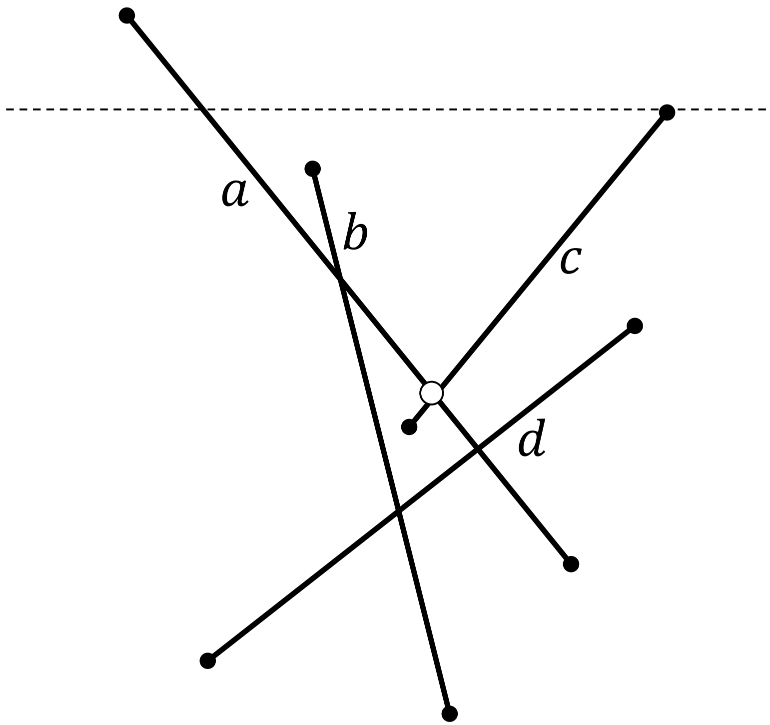
$Q = (c_1, b_1, d_1, c_2, a_2, d_2, b_2)$
$L = (a)$

Segment Intersection

Sweep line algorithm:

- Initialize queue with end-points sorted by height.
- Initialize list of events.
- Advance:
  - Add/remove segments
  - Adjust event list
  - Test for neighboring intersections
  - Adjust queue

$Q = (b_1, d_1, ac, c_2\ , a_2, d_2, b_2)$
$L = (a, c)$



Segment Intersection

Sweep line algorithm:

- Initialize queue with end-points sorted by height.
- Initialize list of events.
- Advance:
    - Add/remove segments
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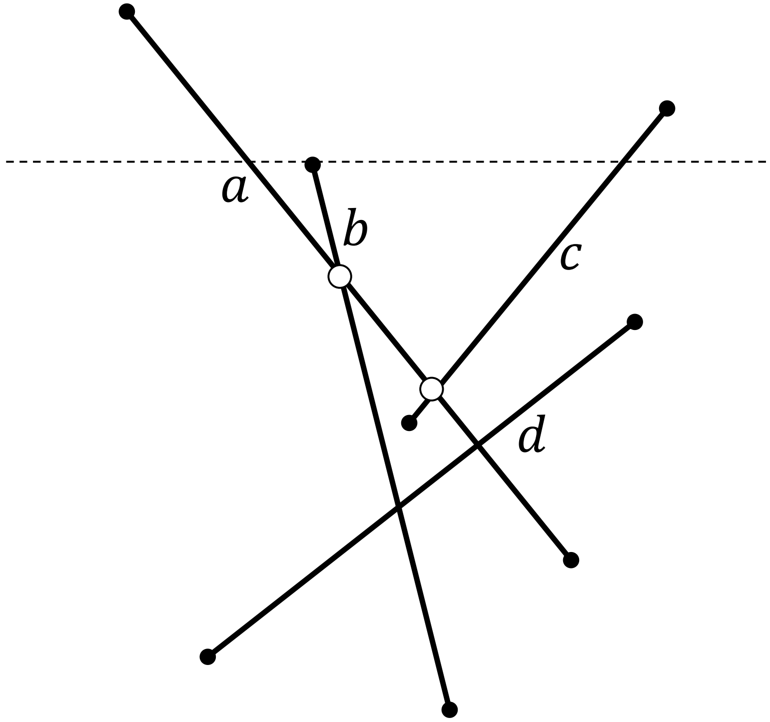
    - Adjust queue

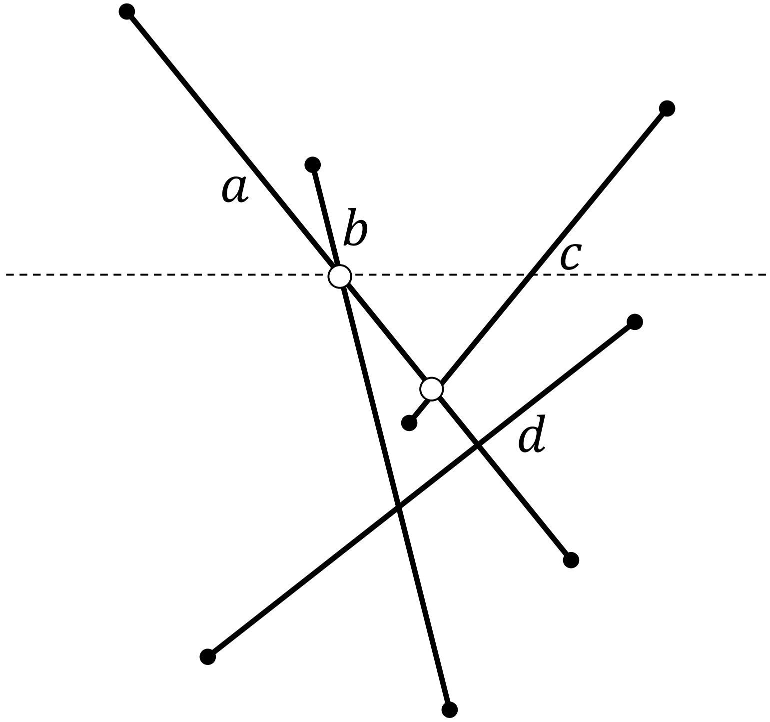


$Q = (ab, d_1, ac, c_2, a_2, d_2, b_2)$
$L = (a, b, c)$

Segment Intersection

Sweep line algorithm:

- Initialize queue with end-points sorted by height.
- Initialize list of events.
- Advance:
  - Add/remove segments
  - Adjust event list
  - Test for neighboring intersections
  - Adjust queue



$Q = (d_1, ac, c_2, a_2, d_2, b_2)$
$L = (b, a, c)$

Segment Intersection

Sweep line algorithm:

- Initialize queue with end-points sorted by height.
- Initialize list of events.
- Advance:
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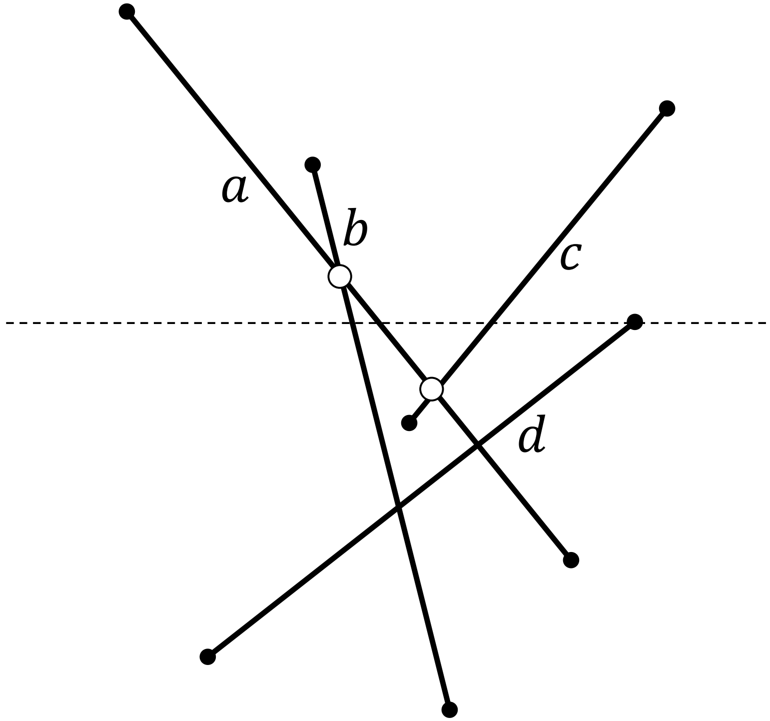
  - Adjust event list
  - Test for neighboring intersections
  - Adjust queue



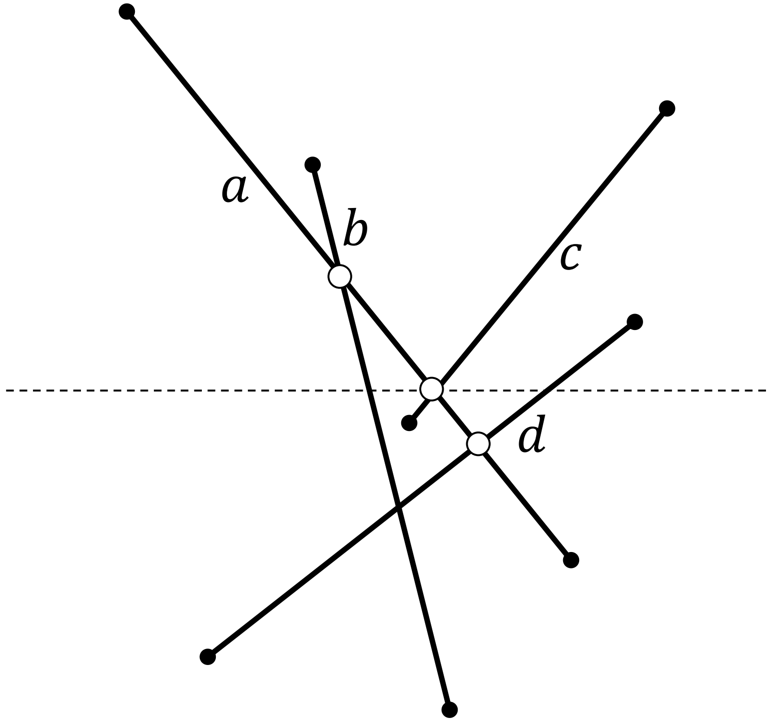
$Q = (\, ac, c_2, a_2, d_2, b_2)$

$L = (b, a, c, d)$

Segment Intersection

Sweep line algorithm:

- Initialize queue with end-points sorted by height.
- Initialize list of events.
- Advance:
  - Add/remove segments
  - Adjust event list
  - Test for neighboring intersections
  - Adjust queue

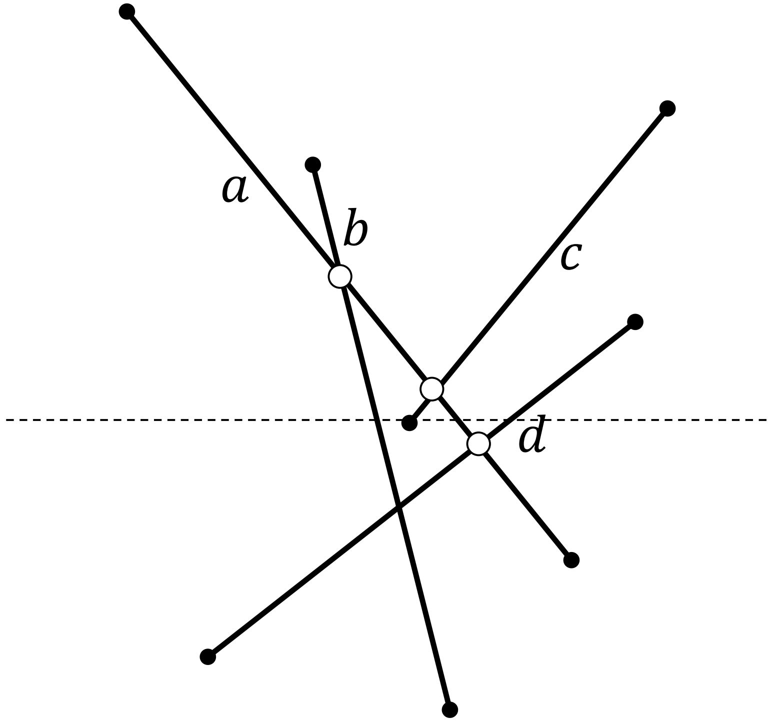


$$Q = (\, c_2, ad, a_2, d_2, b_2)$$
$$L = (b, c, a, d)$$

Segment Intersection

Sweep line algorithm:

- Initialize queue with end-points sorted by height.
- Initialize list of events.
- Advance:
  - Add/remove segments
  - Adjust event list
  - Test for neighboring intersections
  - Adjust queue



$Q = (ad, a_2, d_2, b_2)$

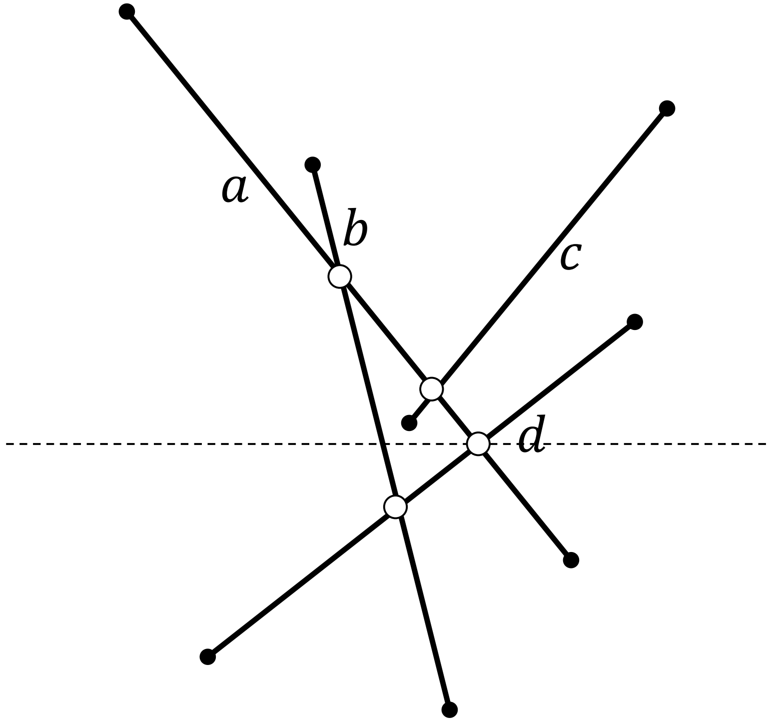
$L = (b, a, d)$

Segment Intersection

Sweep line algorithm:

- Initialize queue with end-points sorted by height.
- Initialize list of events.
- Advance:
  - Add/remove segments
  - Adjust event list
  - Test for neighboring intersections
  - Adjust queue

$Q = (bd, a_2, d_2, b_2)$
$L = (b, d, a)$

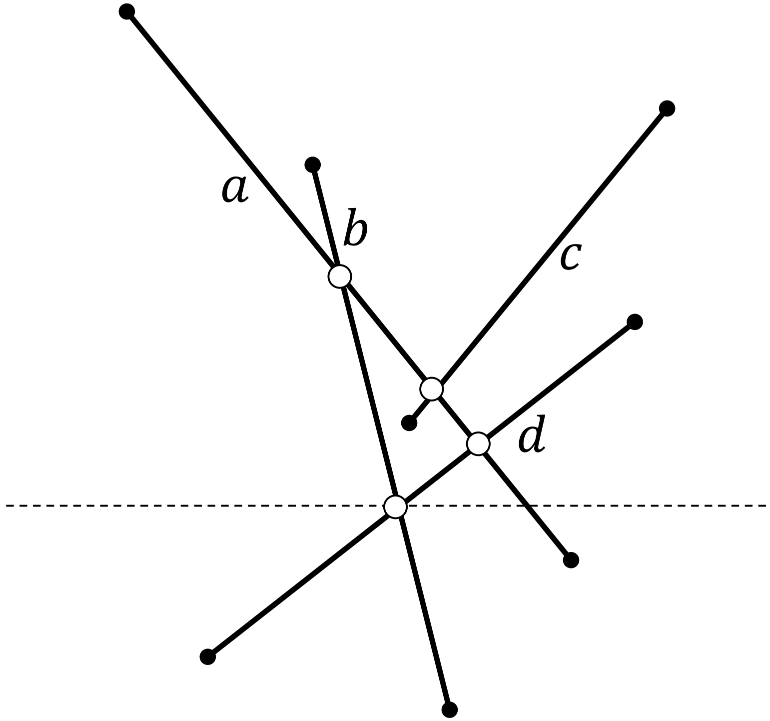


Segment Intersection

<u>Sweep line algorithm</u>:

- Initialize queue with end-points sorted by height.
- Initialize list of events.
- Advance:
  - Add/remove segments
  - Adjust event list
  - Test for <u>neighboring</u> intersections
  - Adjust queue

$Q = (\, a_2, d_2, b_2)$
$L = (d, b, a)$

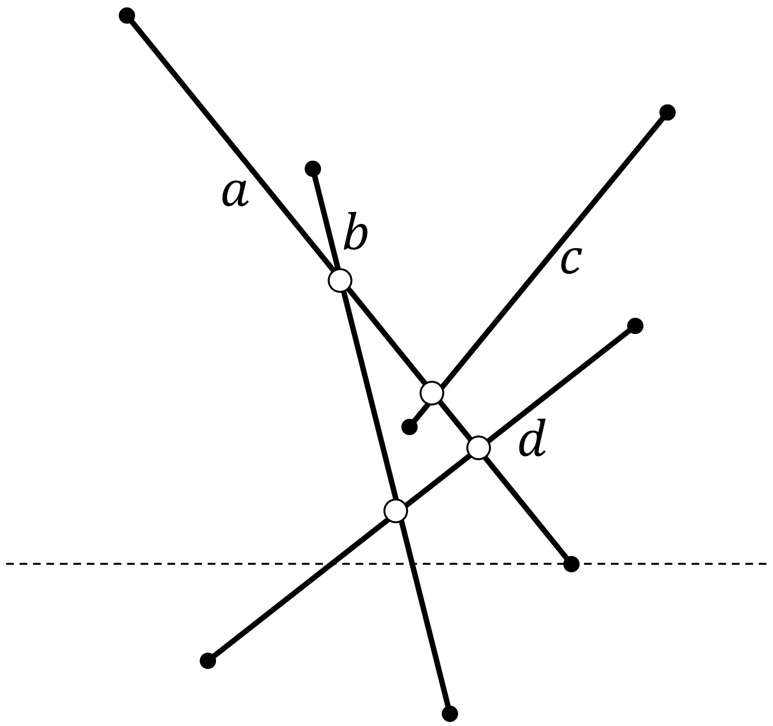


Segment Intersection

Sweep line algorithm:

- Initialize queue with end-points sorted by height.
- Initialize list of events.
- Advance:
  - Add/remove segments
  - Adjust event list
  - Test for neighboring intersections
  - Adjust queue

$Q = (d_2, b_2)$
$L = (d, b)$



Segment Intersection

Sweep line algorithm:

- Initialize queue with end-points sorted by height.
- Initialize list of events.
- Advance:
  - Add/remove segments
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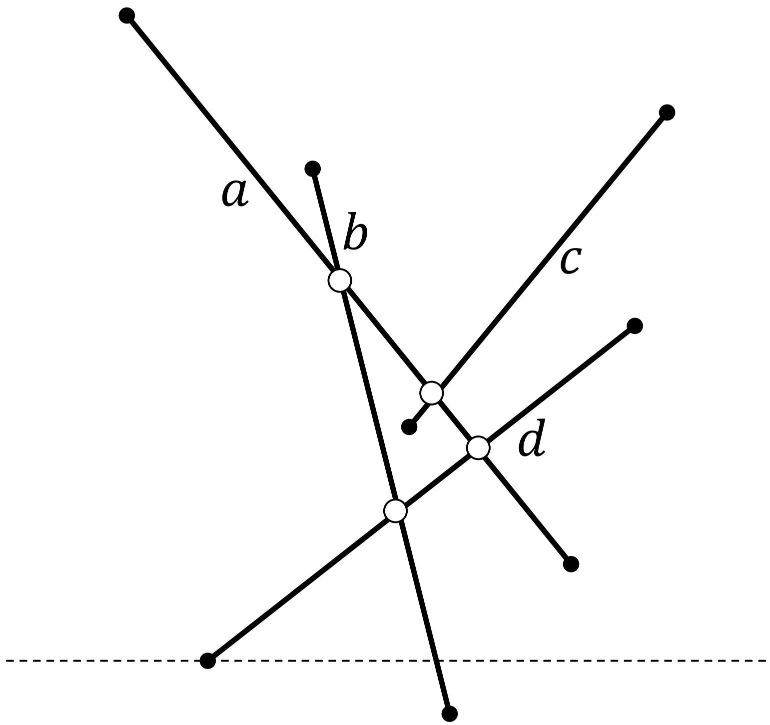
  - Test for neighboring intersections
  - Adjust queue



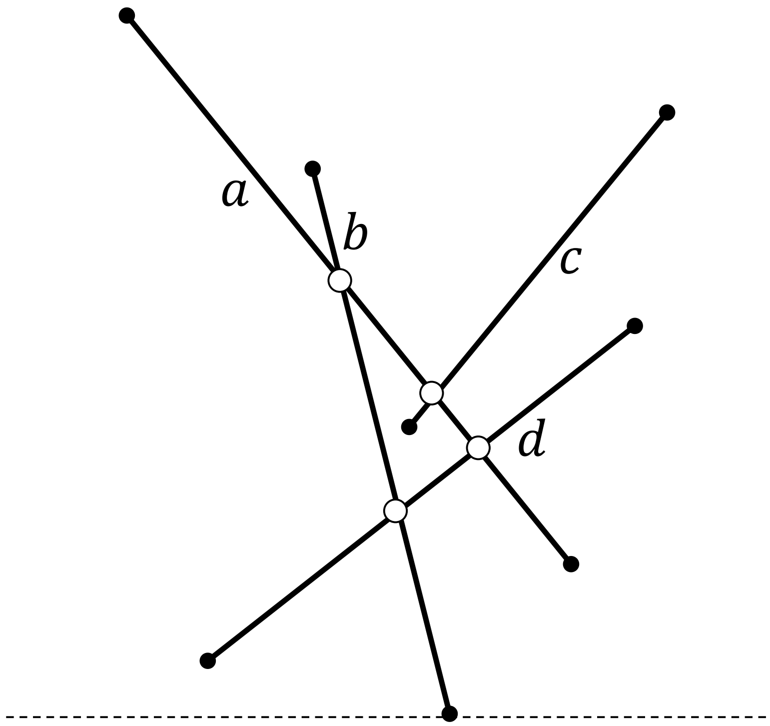
$Q = (b_2)$
$L = (b)$

Segment Intersection

Sweep line algorithm:

- Initialize queue with end-points sorted by height.
- Initialize list of events.
- Advance:
  - Add/remove segments
  - Adjust event list
  - Test for neighboring intersections
  - Adjust queue

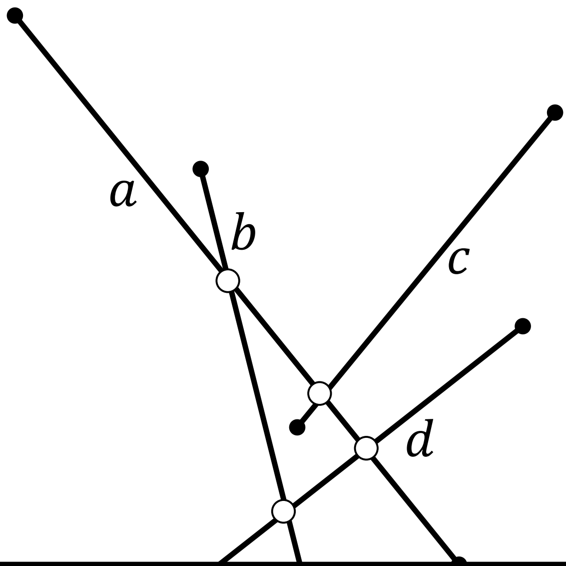
$Q = \emptyset$
$L = \emptyset$



Segment Intersection

<u>Sweep line algorithm</u>:

- Initialize queue with end-points sorted by height.
- Initialize list of events.
- Advance:
  - Add/remove segments
  - Adjust event list
  - Test for <u>neighboring</u> intersections
  - Adjust queue



With the right data-structures, this has complexity $O((n+k)\log n)$, with k the number of intersections.

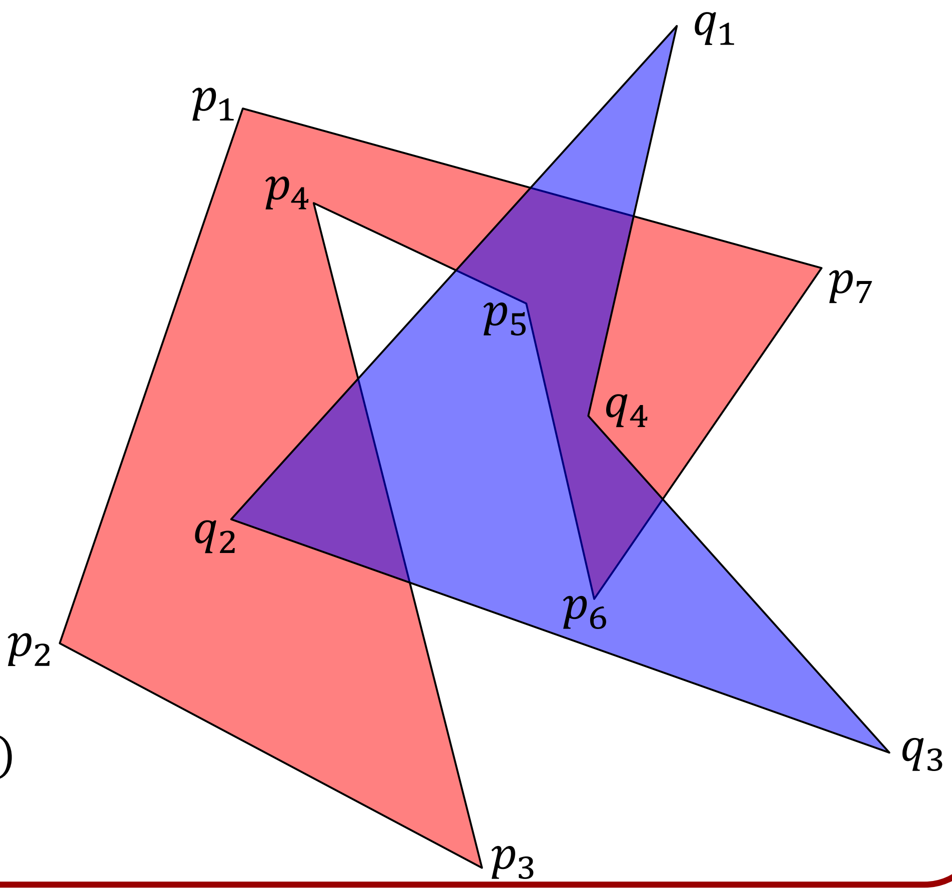
Polygon Intersection

Sweep line algorithm:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$Q = (q_1, p_1, p_4, p_7, p_5, q_4, q_2, p_6, p_2, q_3, p_3)$

$L = \emptyset$

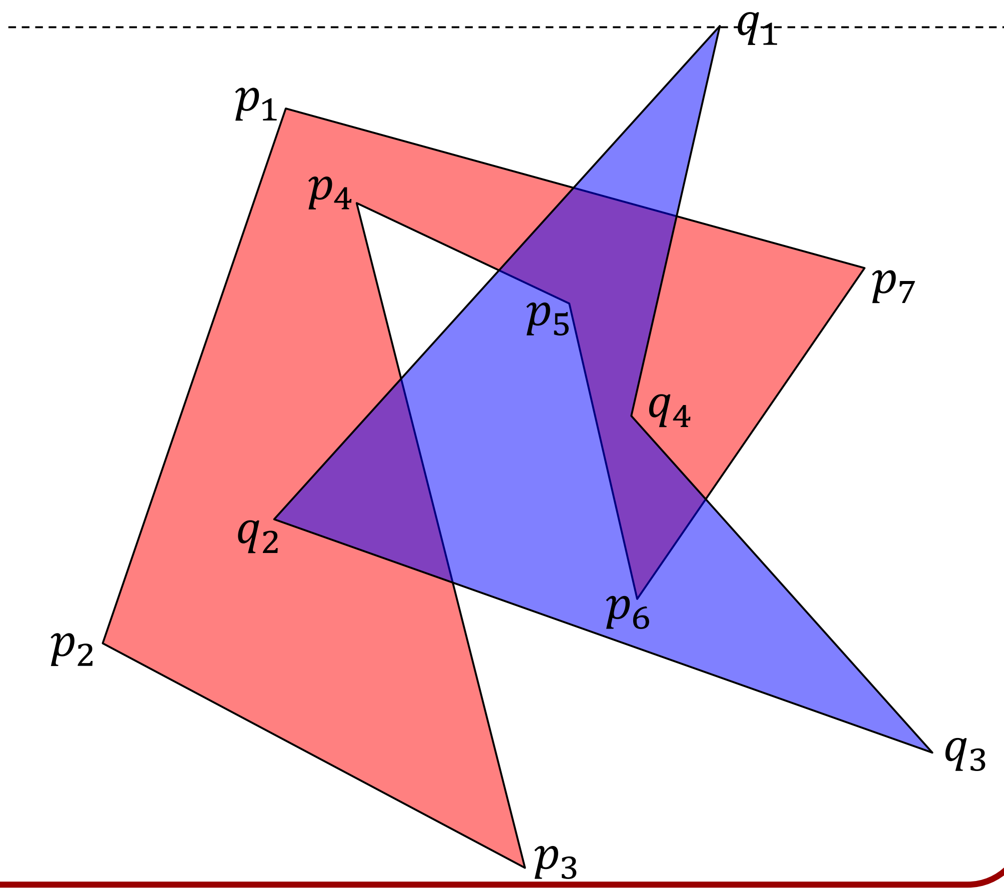
Polygon Intersection

<u>Sweep line algorithm</u>:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$Q = (p_1, p_4, p_7, p_5, q_4, q_2, p_6, p_2, q_3, p_3)$
$L = (\emptyset - q_{12} - Q - q_{41} - \emptyset)$

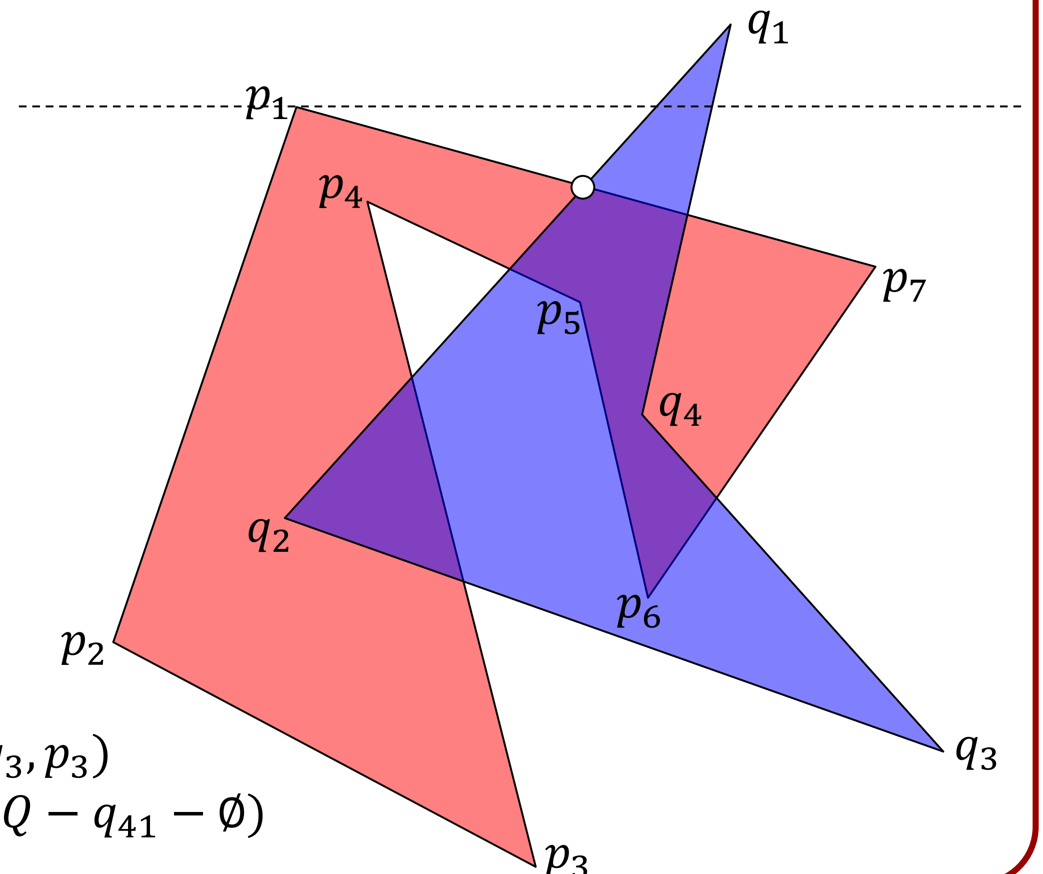
Polygon Intersection

<u>Sweep line algorithm</u>:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$$Q = (p_{71}q_{12}, p_4, p_7, p_5, q_4, q_2, p_6, p_2, q_3, p_3)$$

$$L = (\emptyset - p_{12} - P - p_{71} - \emptyset - q_{12} - Q - q_{41} - \emptyset)$$

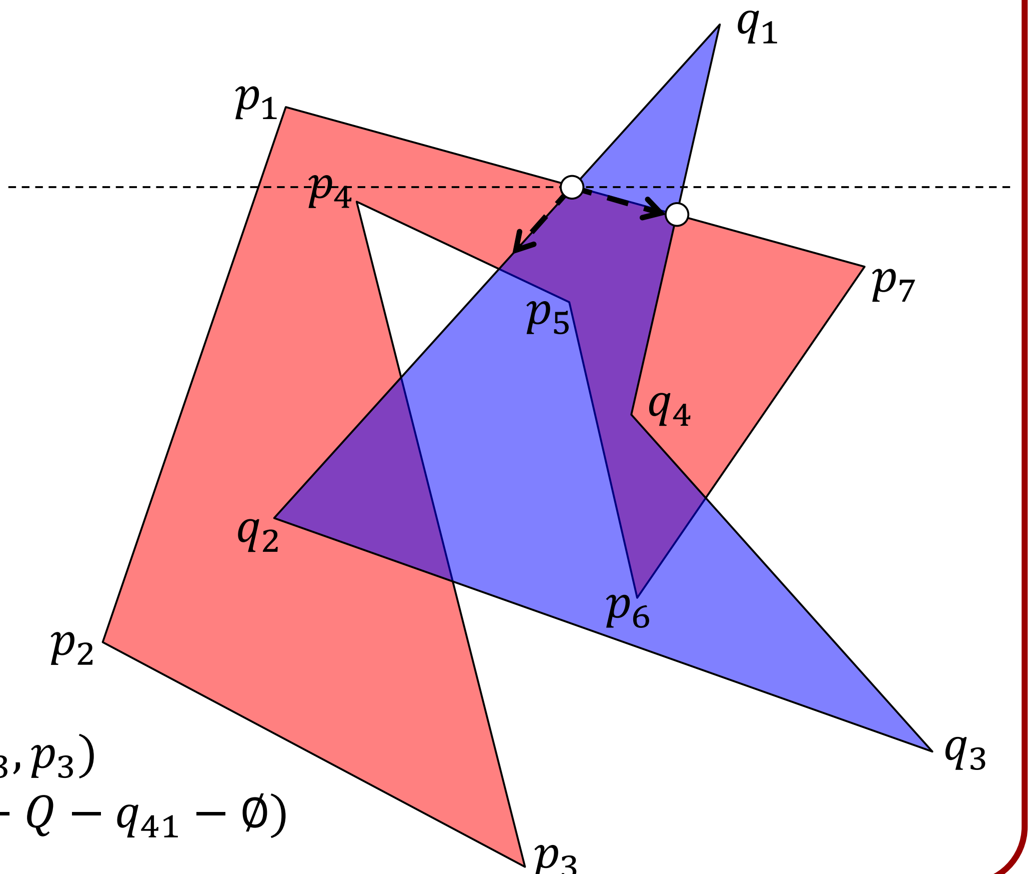
Polygon Intersection

<u>Sweep line algorithm</u>:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$$Q = (p_4, p_{71} q_{41}, p_7\ , p_5, q_4, q_2, p_6, p_2, q_3, p_3)$$

$$L = (\emptyset - p_{12} - P - q_{12} - PQ - p_{71} - Q - q_{41} - \emptyset)$$

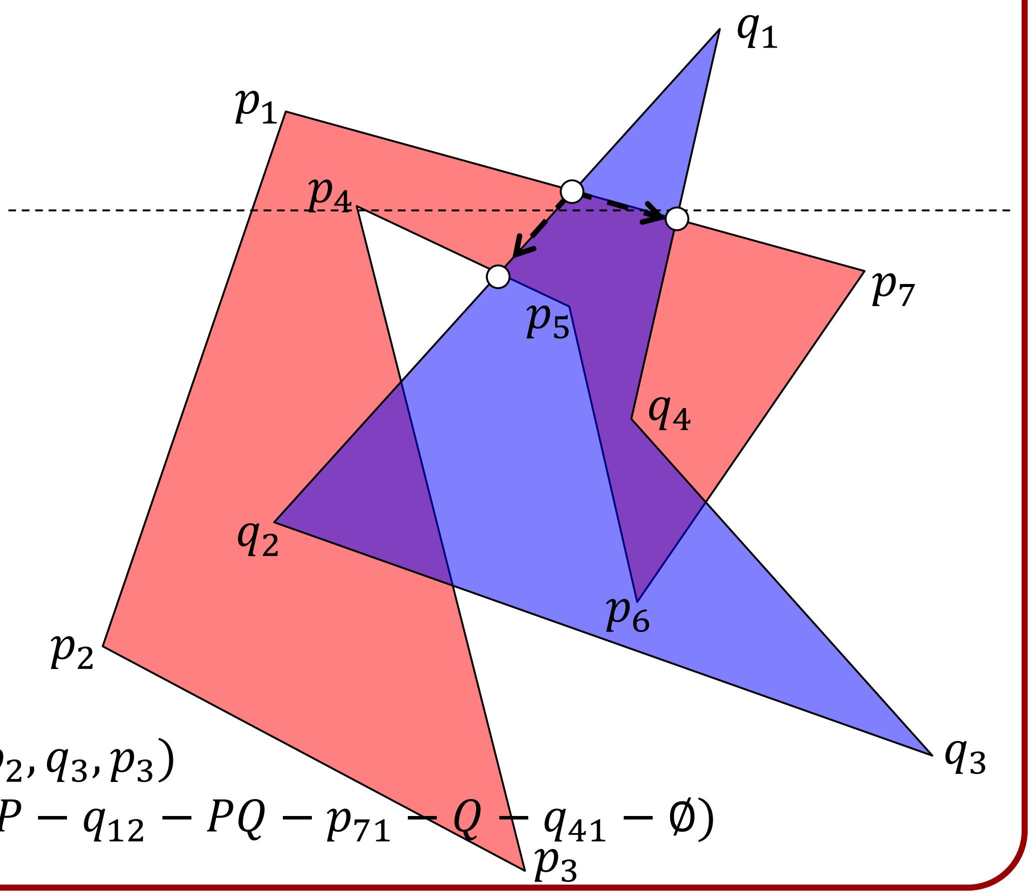
Polygon Intersection

Sweep line algorithm:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$$Q = (p_{71}q_{41}, p_7, p_{45}q_{12}, p_5, q_4, q_2, p_6, p_2, q_3, p_3)$$
$$L = (\emptyset - p_{12} - P - p_{34} - \emptyset - p_{45} - P - q_{12} - PQ - p_{71} - Q - q_{41} - \emptyset)$$

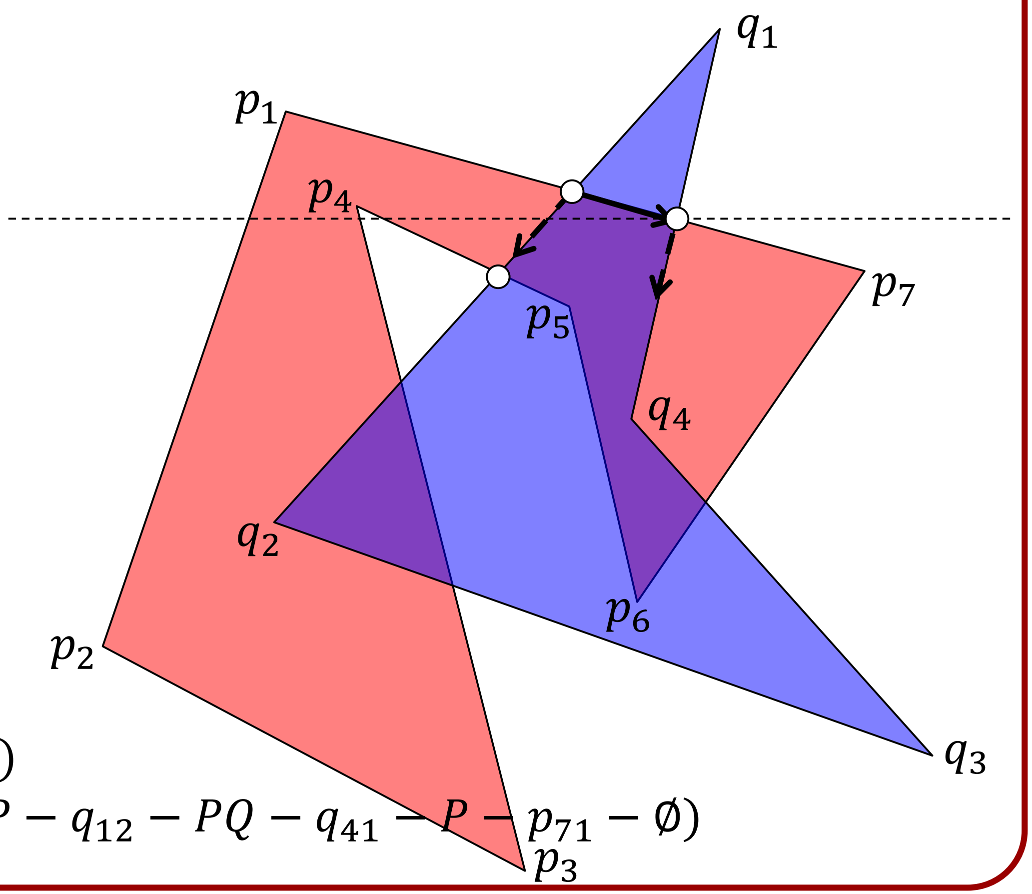
Polygon Intersection

<u>Sweep line algorithm</u>:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$$Q = (p_7, p_{45} q_{12}, p_5, q_4, q_2, p_6, p_2, q_3, p_3)$$

$$L = (\emptyset - p_{12} - P - p_{34} - \emptyset - p_{45} - P - q_{12} - PQ - q_{41} - P - p_{71} - \emptyset)$$

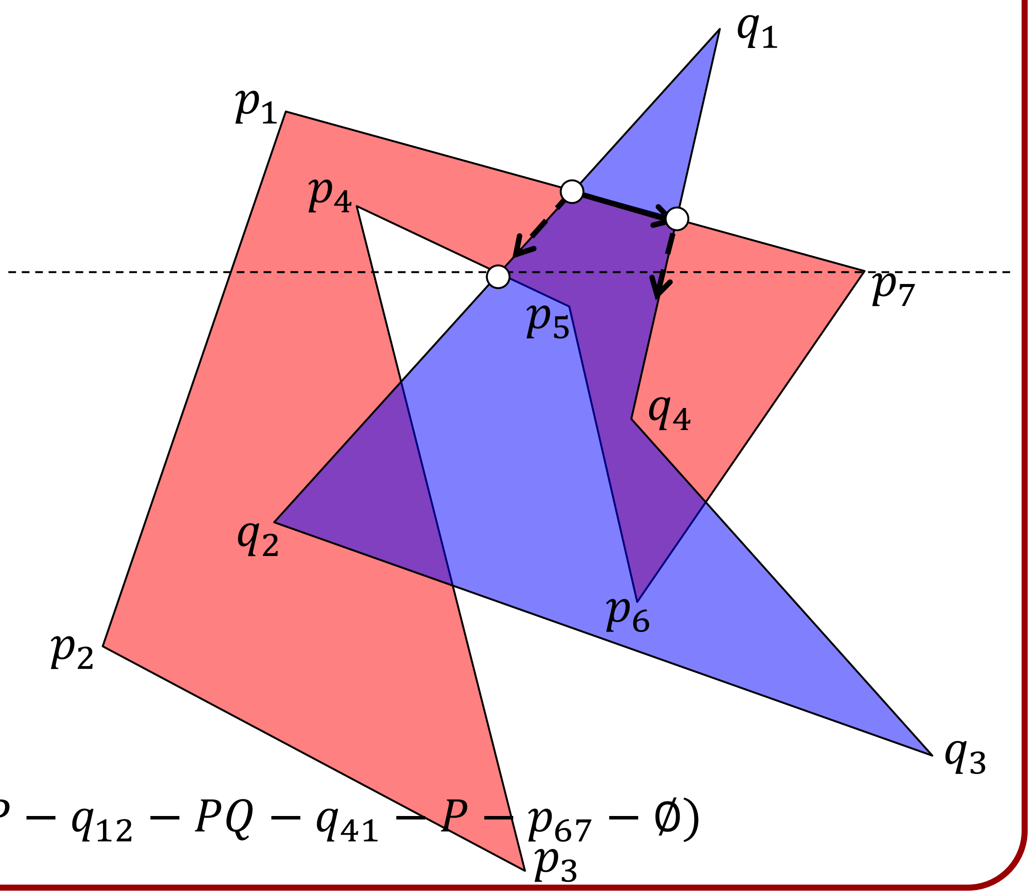
Polygon Intersection

<u>Sweep line algorithm</u>:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$Q = (p_{45}q_{12}, p_5, q_4, q_2, p_6, p_2, q_3, p_3)$

$L = (\emptyset - p_{12} - P - p_{34} - \emptyset - p_{45} - P - q_{12} - PQ - q_{41} - P - p_{67} - \emptyset)$

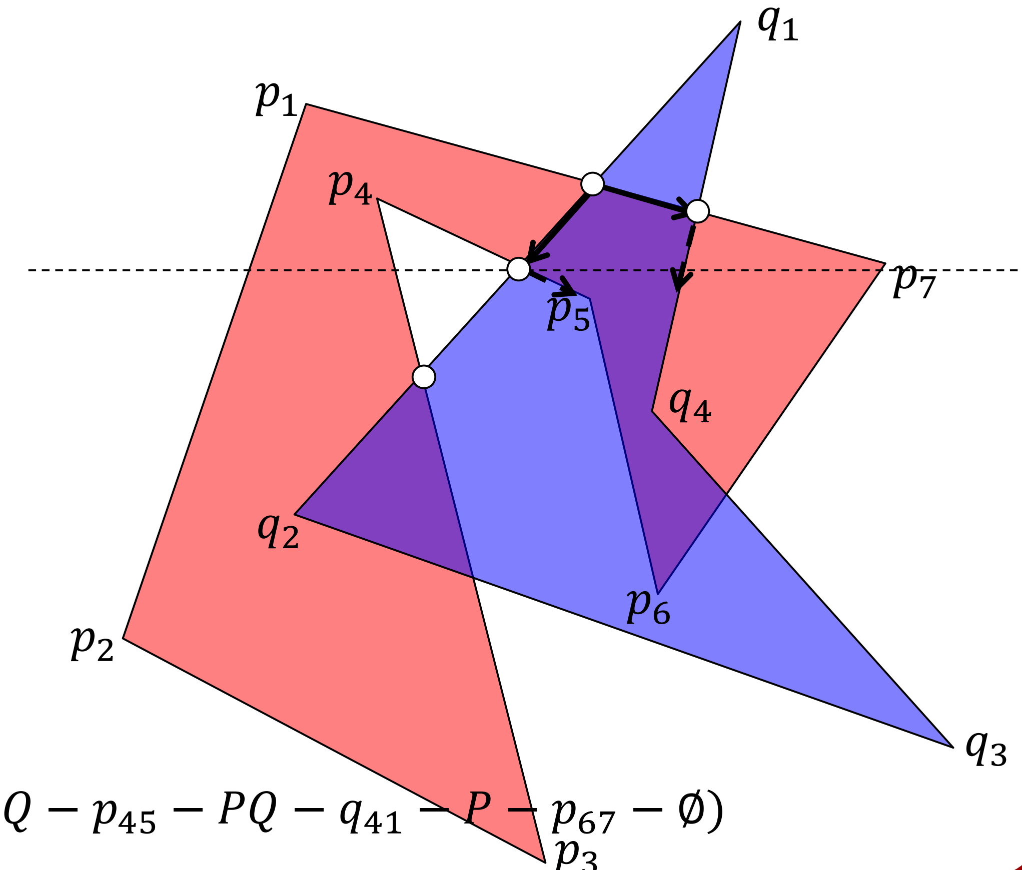
Polygon Intersection

<u>Sweep line algorithm</u>:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$Q = (p_5, p_{34}q_{12}, q_4, q_2, p_6, p_2, q_3, p_3)$

$L = (\emptyset - p_{12} - P - p_{34} - \emptyset - q_{12} - Q - p_{45} - PQ - q_{41} - P - p_{67} - \emptyset)$

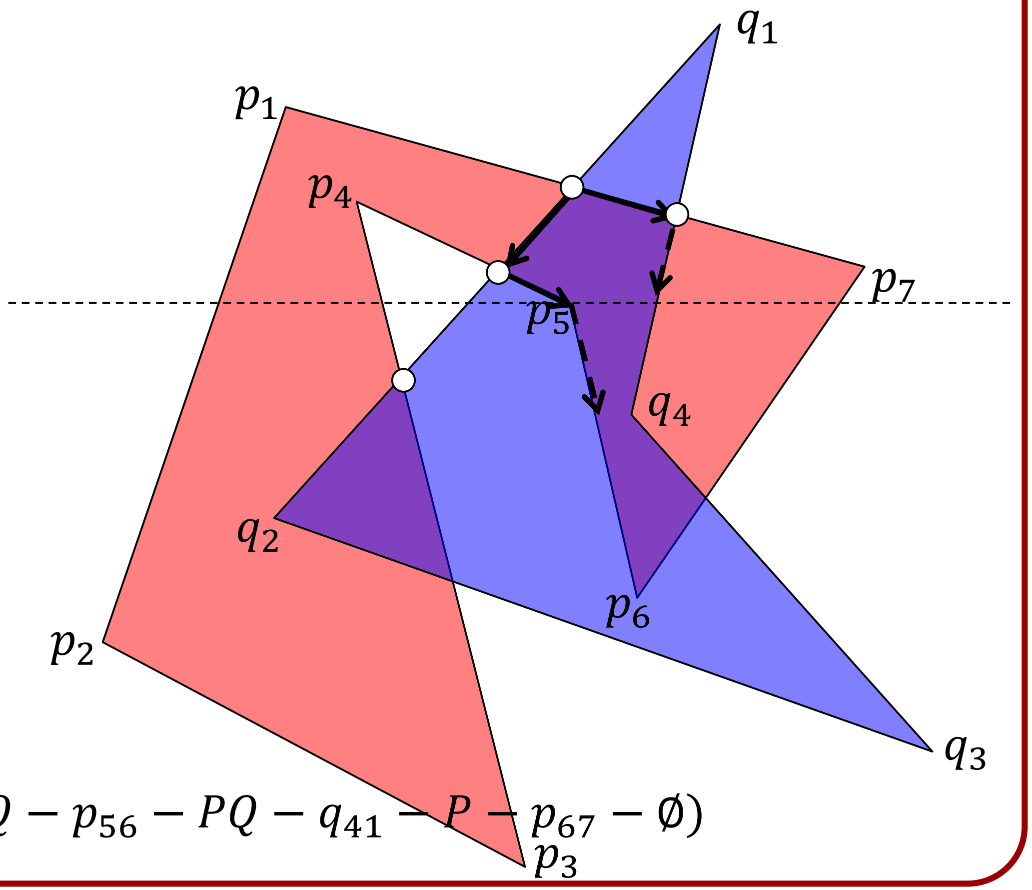
Polygon Intersection

<u>Sweep line algorithm</u>:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$Q = (p_{34}q_{12}, q_4, q_2, p_6, p_2, q_3, p_3)$

$L = (\emptyset - p_{12} - P - p_{34} - \emptyset - q_{12} - Q - p_{56} - PQ - q_{41} - P - p_{67} - \emptyset)$

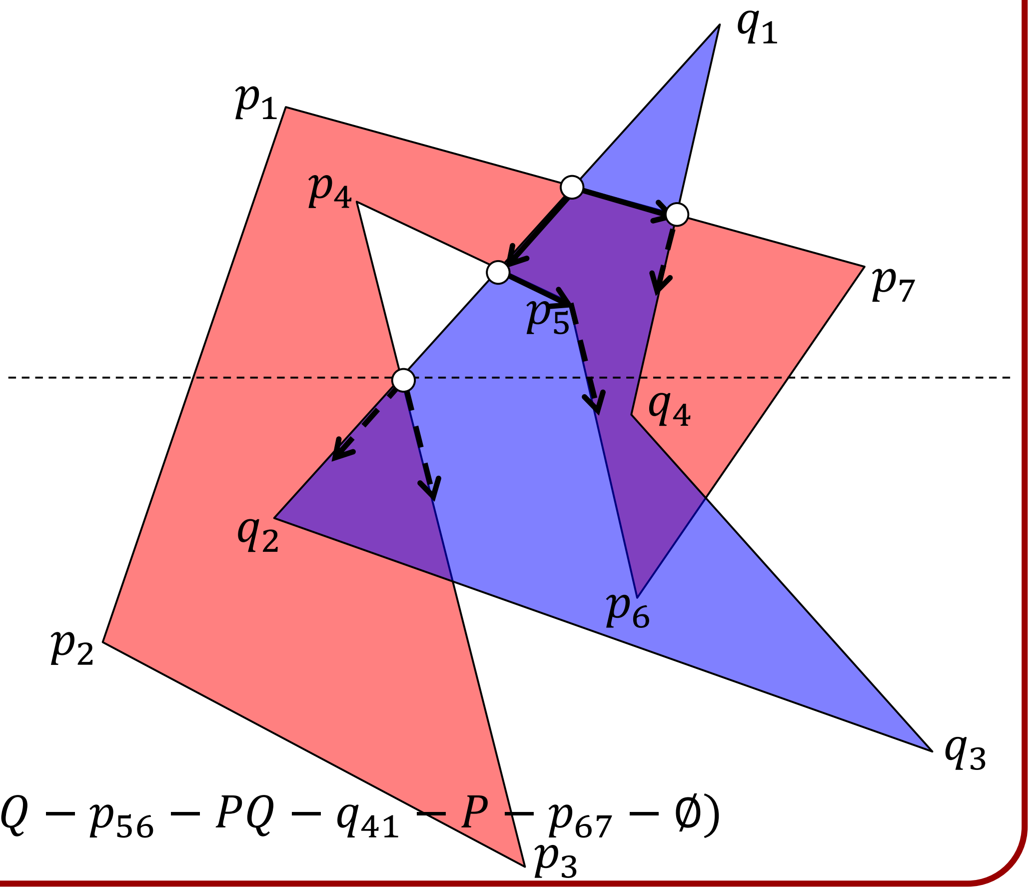
Polygon Intersection

<u>Sweep line algorithm</u>:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$Q = (q_4, q_2, p_6, p_2, q_3, p_3)$

$L = (\emptyset - p_{12} - P - q_{12} - PQ - p_{34} - Q - p_{56} - PQ - q_{41} - P - p_{67} - \emptyset)$

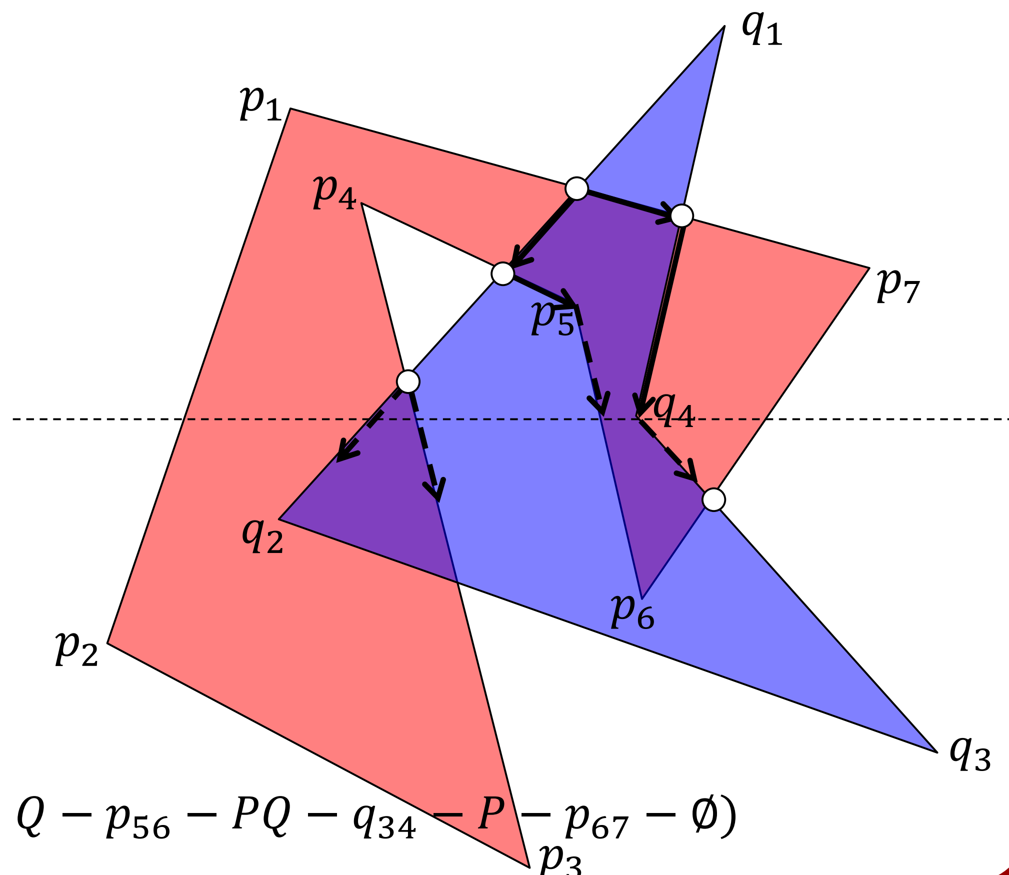
Polygon Intersection

<u>Sweep line algorithm</u>:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$Q = (p_{67} q_{34}, q_2, p_6, p_2, q_3, p_3)$

$L = (\emptyset - p_{12} - P - q_{12} - PQ - p_{34} - Q - p_{56} - PQ - q_{34} - P - p_{67} - \emptyset)$

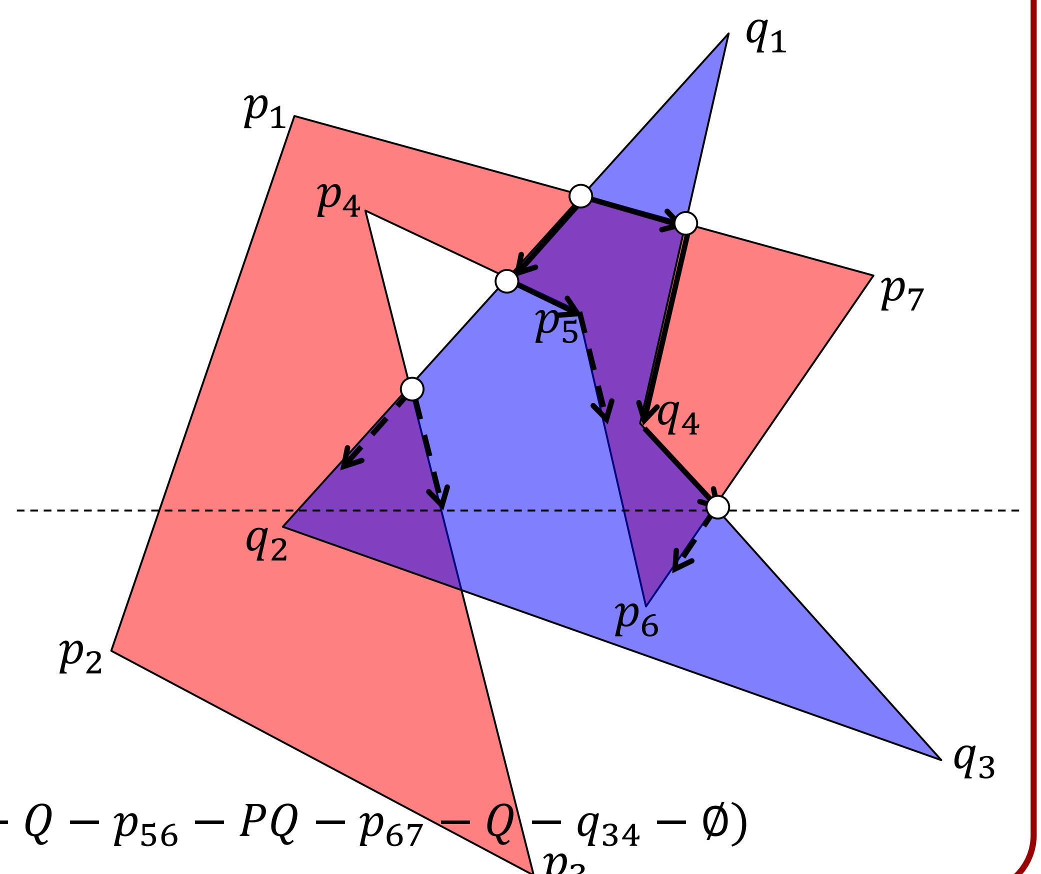
Polygon Intersection

<u>Sweep line algorithm</u>:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$Q = (q_2, p_6, p_2, q_3, p_3)$

$L = (\emptyset - p_{12} - P - q_{12} - PQ - p_{34} - Q - p_{56} - PQ - p_{67} - Q - q_{34} - \emptyset)$

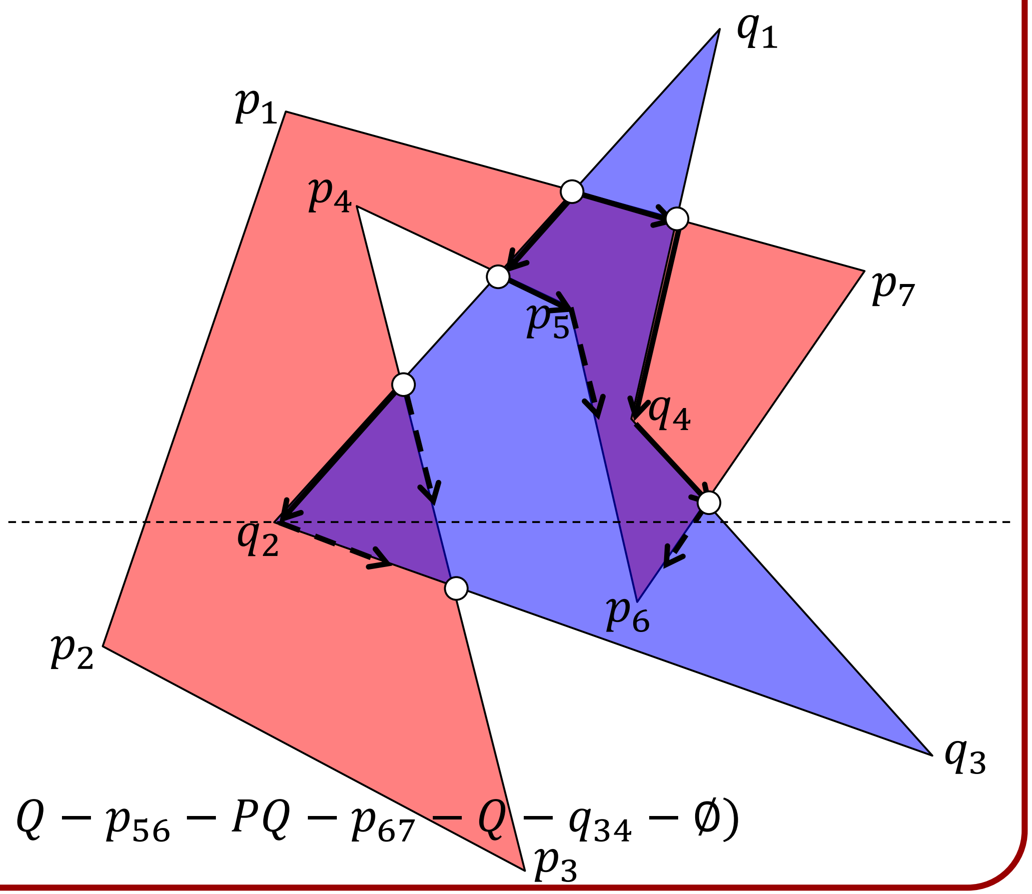
Polygon Intersection

<u>Sweep line algorithm</u>:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$Q = (p_{34}q_{23}, p_6, p_2, q_3, p_3)$

$L = (\emptyset - p_{12} - P - q_{23} - PQ - p_{34} - Q - p_{56} - PQ - p_{67} - Q - q_{34} - \emptyset)$

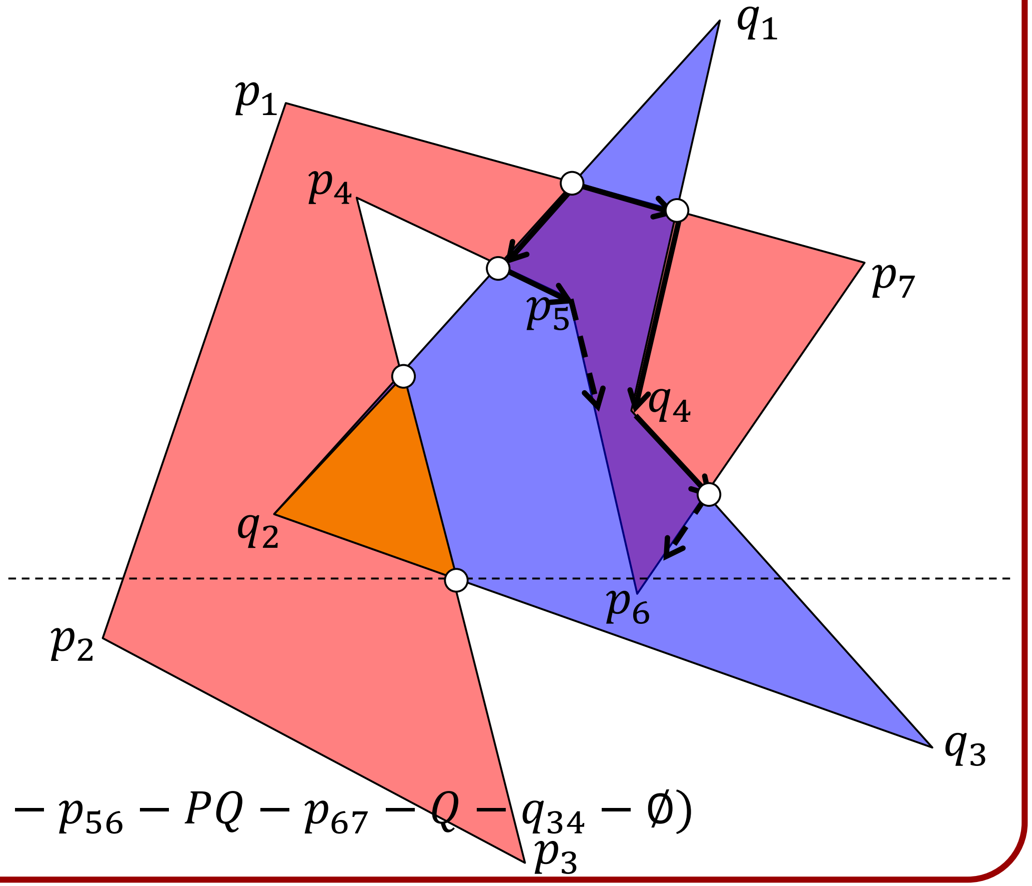
Polygon Intersection

<u>Sweep line algorithm</u>:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$Q = (p_6, p_2, q_3, p_3)$

$L = (\emptyset - p_{12} - P - p_{34} - \emptyset - q_{23} - Q - p_{56} - PQ - p_{67} - Q - q_{34} - \emptyset)$

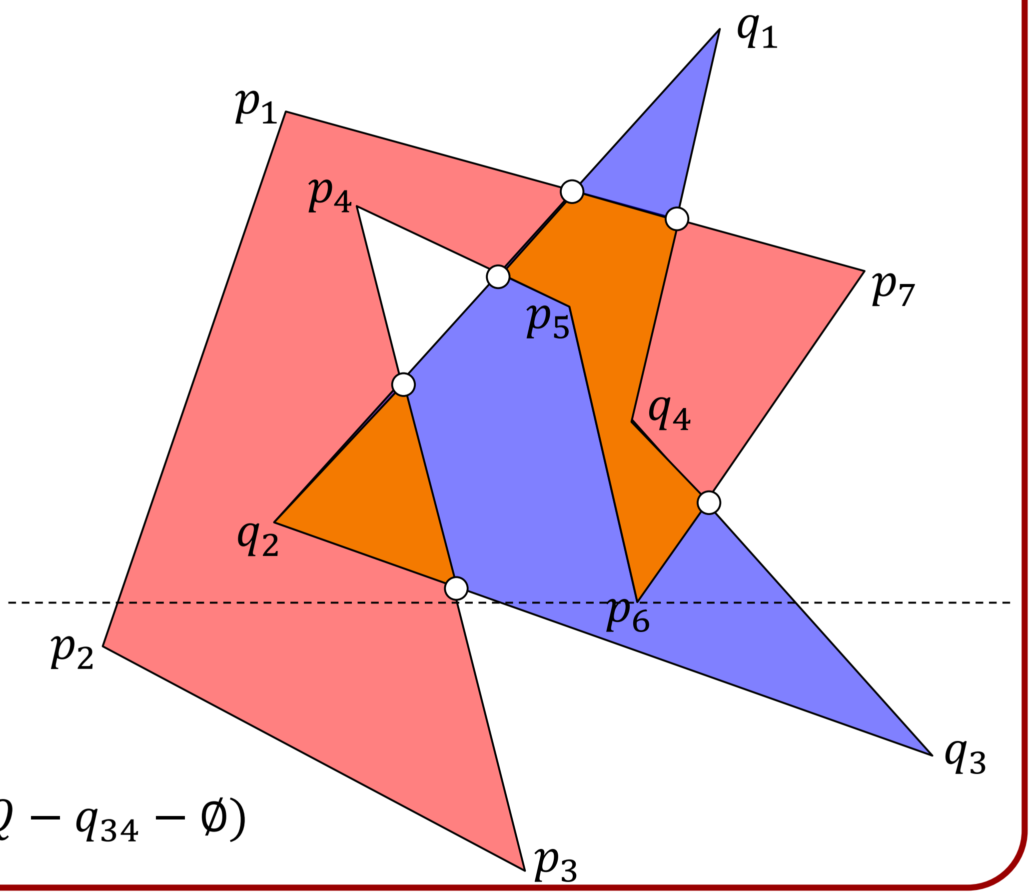
Polygon Intersection

Sweep line algorithm:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$Q = (p_2, q_3, p_3)$

$L = (\emptyset - p_{12} - P - p_{34} - \emptyset - q_{23} - Q - q_{34} - \emptyset)$

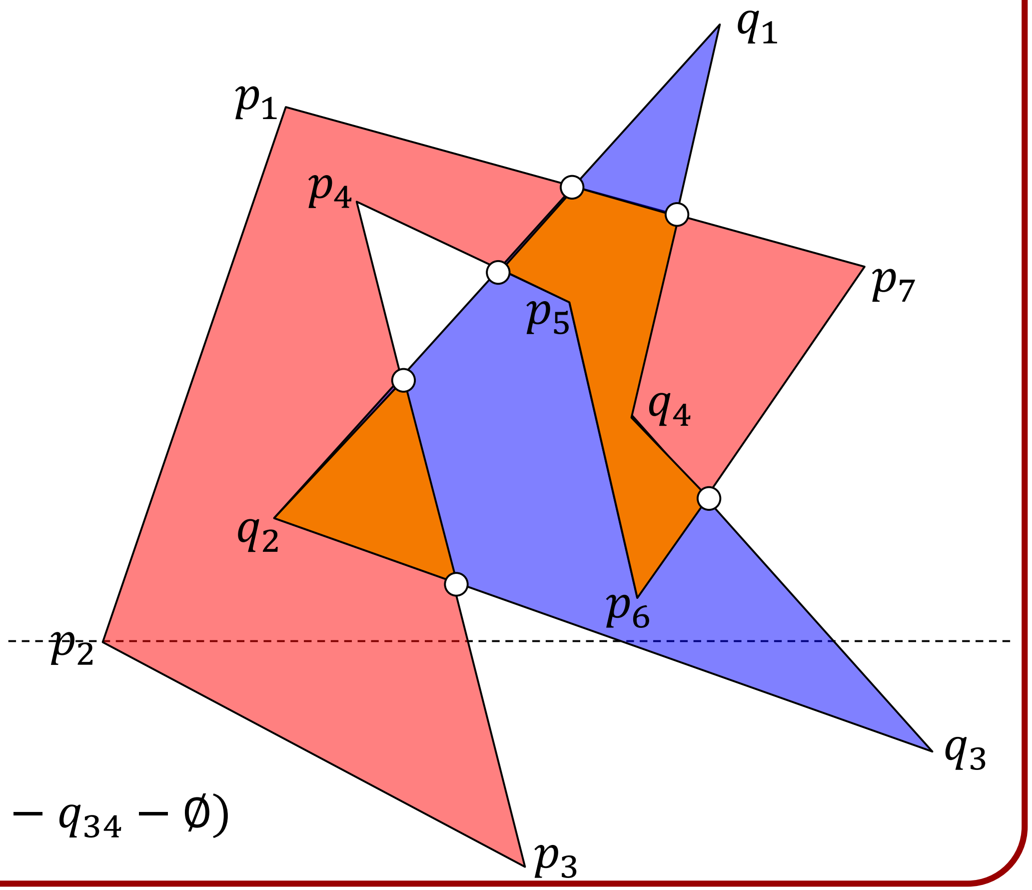
Polygon Intersection

<u>Sweep line algorithm</u>:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$Q = (q_3, p_3)$
$L = (\emptyset - p_{23} - P - p_{34} - \emptyset - q_{23} - Q - q_{34} - \emptyset)$

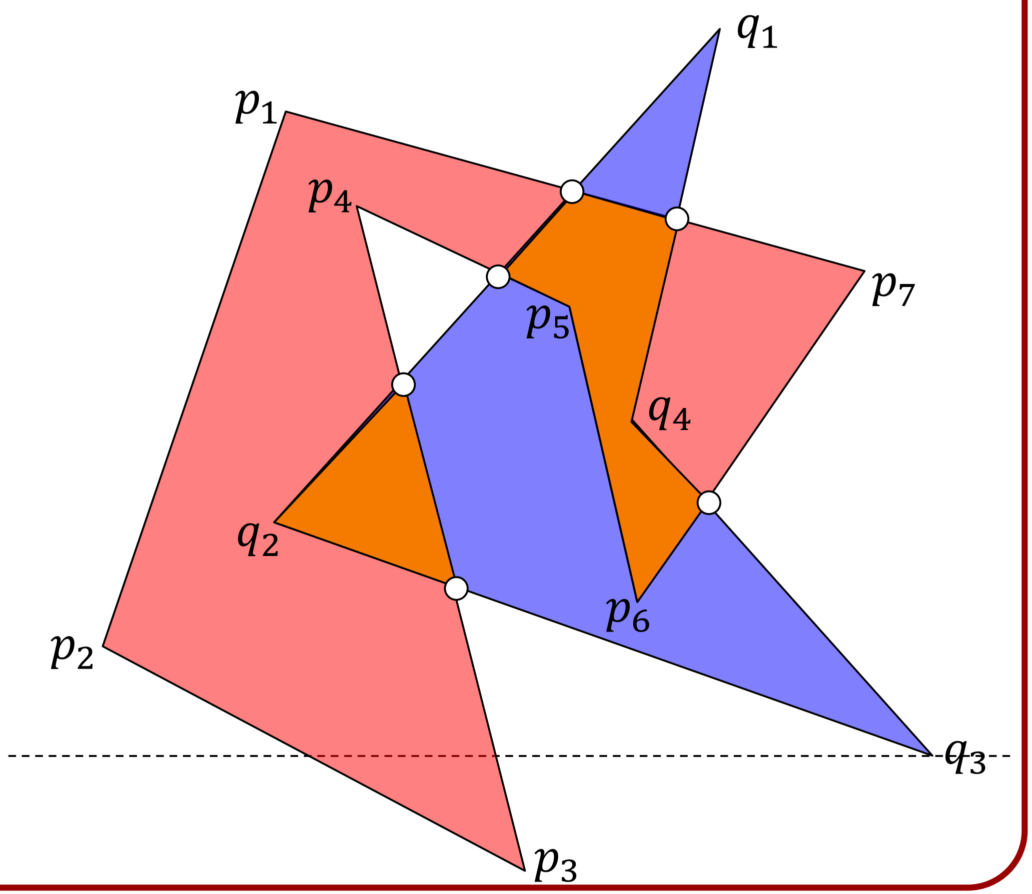
Polygon Intersection

<u>Sweep line algorithm</u>:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$Q = (\, p_3)$

$L = (\emptyset - p_{23} - P - p_{34} - \emptyset)$

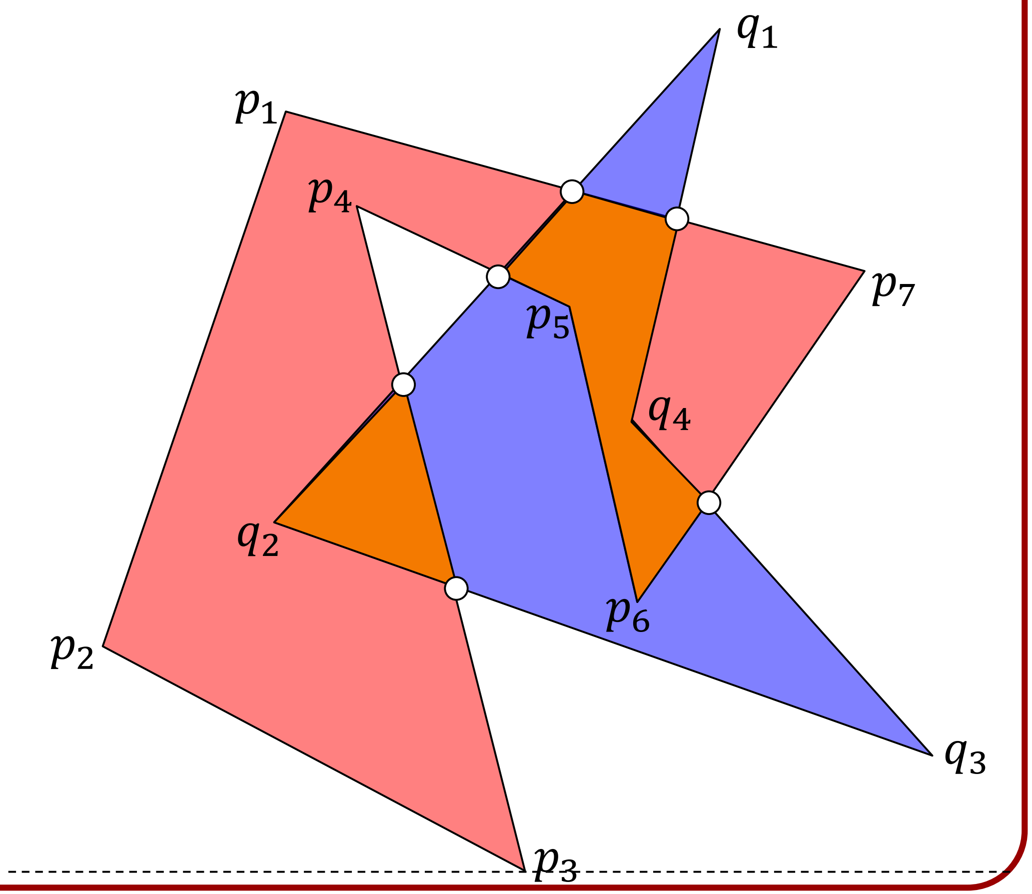
Polygon Intersection

<u>Sweep line algorithm</u>:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



$Q = \emptyset$
$L = \emptyset$

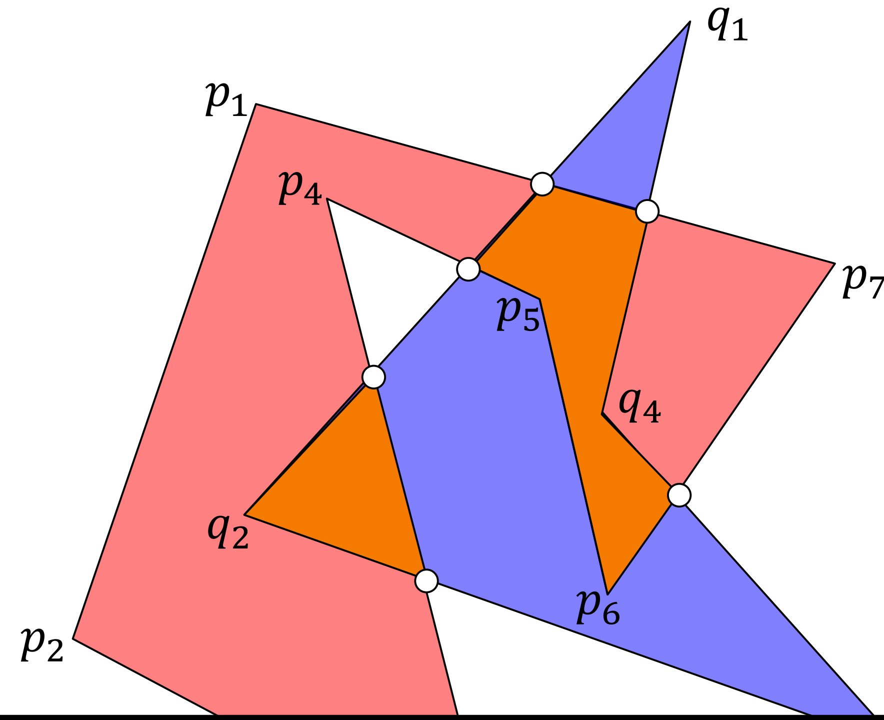
Polygon Intersection

<u>Sweep line algorithm</u>:

A similar approach gives polygon intersection.

Need to track:

- Span labels
- Polygon chains



This has complexity $O((|P| + |Q| + k)\log(|P| + |Q|))$, with k the number of intersections.

Outline

- Polygon Intersection
- Convex Polygon Intersection

Convex Polygon Intersection

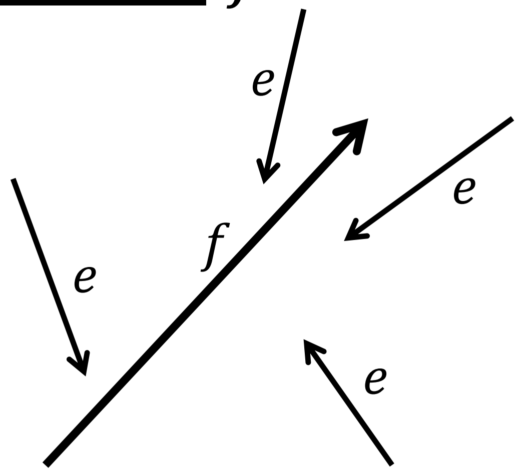
Notation:

Given a (directed) edge $f = (a, b)$ we refer to b as the head of f.

Given edges e and f we say that e is interior / exterior to f if the head of e is left / right of f.

Given edges e and f we say that e aims at f if:

- (e,f) is CW and e is exterior to f, or
- (e,f) is CCW and e is interior to f.

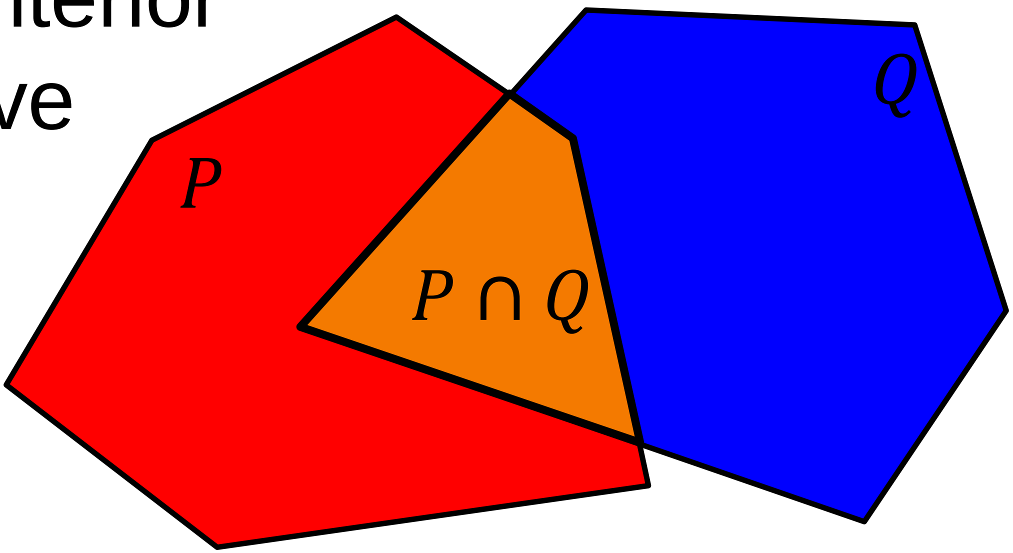


Convex Polygon Intersection

Given convex polygons P and Q, find the (convex) intersection $P \cap Q$.

Approach:

Find intersections between P and Q and track which polygon is interior between successive crossings.



Convex Polygon Intersection

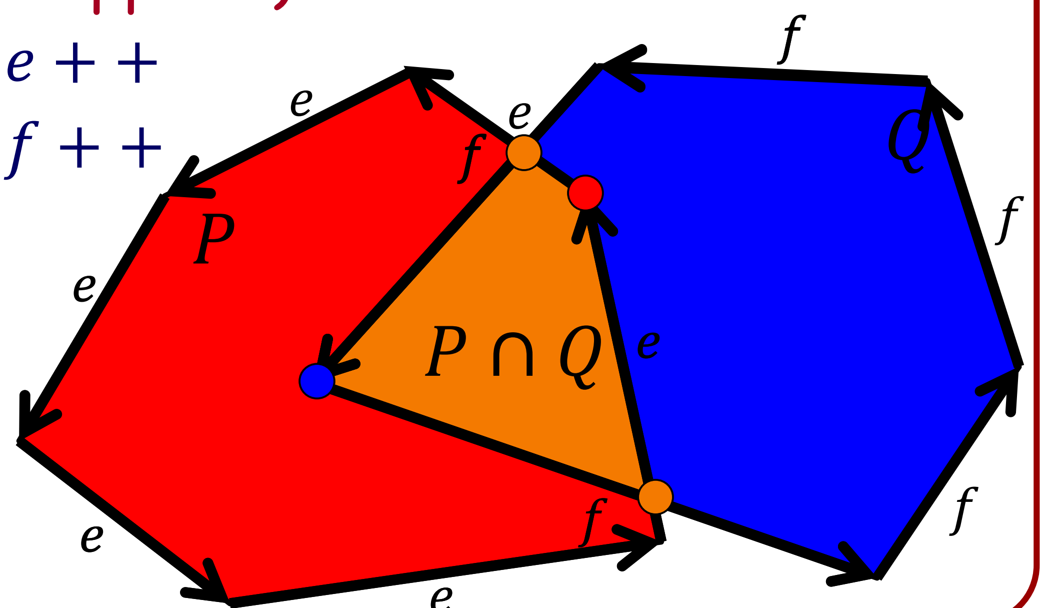
Given convex polygons P and Q, find the (convex) intersection $P \cap Q$.

Greedy Algorithm:

Advance an edge e if it aims at the line through the other edge f.

Convex Polygon Intersection

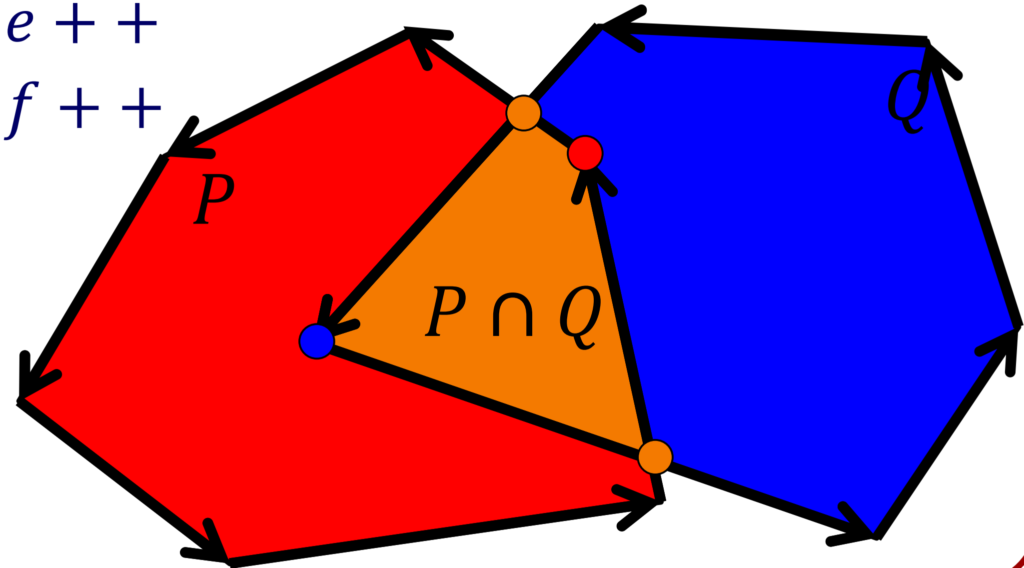
- Choose edges $e \in P$ and $f \in Q$.
- While not done:
  - If neither/both edge aim at each other:
    - » If f interior to e: $e++$
    - » Else if e interior to f: $f++$
    - » Else: exit("Can't happen")
  - Else if e aims at f: $e++$
  - Else if f aims at e: $f++$



Convex Polygon

	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

- Choose edges $e \in P$
- While not done:
  - If neither/both edge aim at each other:
    - » If f interior to e: $e++$
    - » Else if e interior to f: $f++$
    - » Else: exit("Can't happen")
  - Else if e aims at f: $e++$
  - Else if f aims at e: $f++$



Convex Polygon

	$\boldsymbol{aim(e,f)}$	$\boldsymbol{\neg aim(e,f)}$
$\boldsymbol{aim(f,e)}$	**exterior**	$\boldsymbol{f}$
$\boldsymbol{\neg aim(f,e)}$	$\boldsymbol{e}$	**exterior**

<u>Claim</u>:

This algorithm outputs the correct solution and iterates at most $2(|P| + |Q|)$ times.

Convex Polygon

	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

Sub-Claim 1:

The algorithm finds at least one intersection point.*

*Assume P and Q intersect non-degenerately.
(i.e. At most one point of intersection in the interior of an edge.)

Convex Polygon

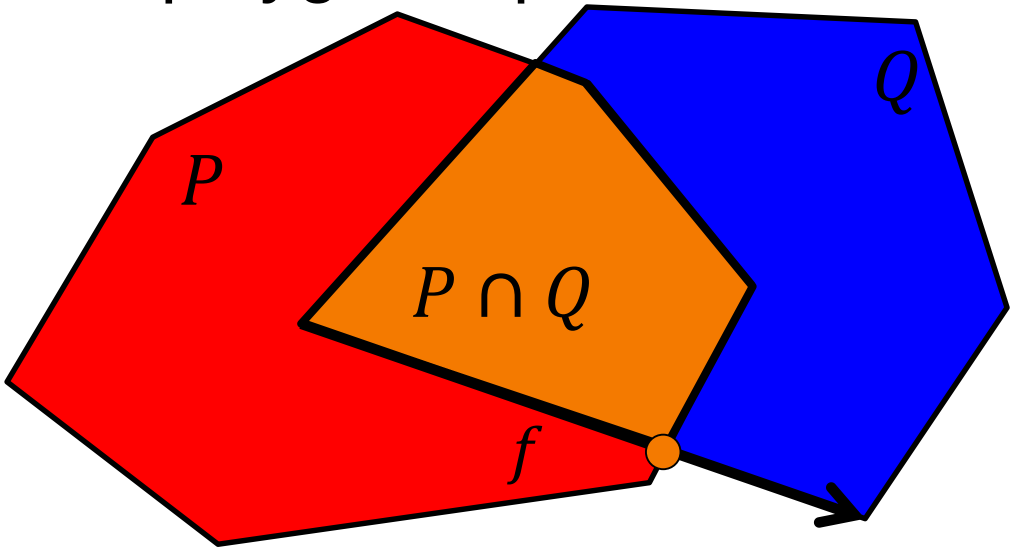
	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

Proof (Sub-Claim 1

Assume to the contrary.

After $|P| + |Q|$ iterations we will have completed a cycle of either P or Q, w.l.o.g. assume Q.

$\Rightarrow$ At some edge $f \in Q$ the polygon P passes from outside Q to inside.



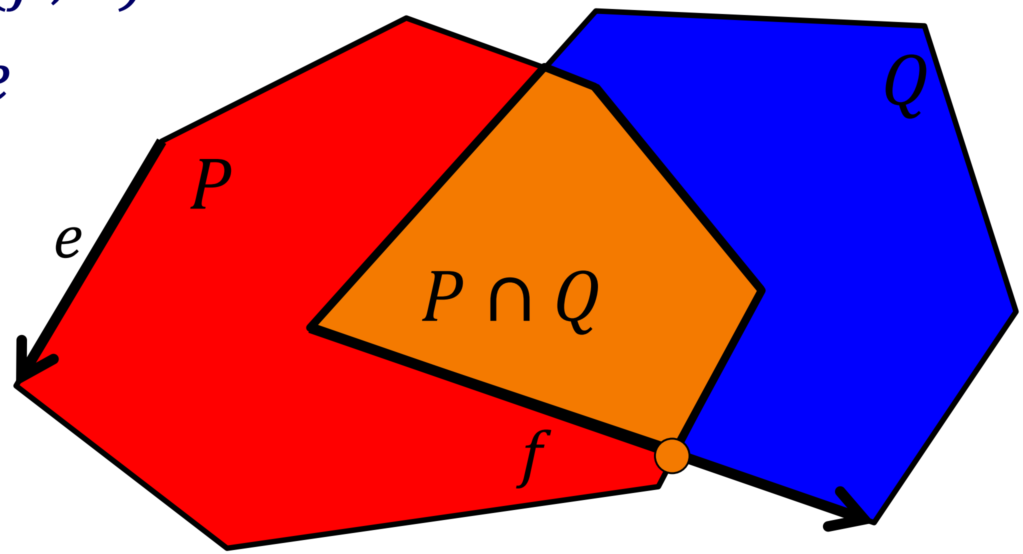
Convex Polygon

	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

P passes from outside Q to inside at f.

<u>Case 1 (e is exterior)</u>:

- If e does not aim at f:
  - $\Rightarrow f$ is interior and (f, e) is CW
  - $\Rightarrow f$ cannot aim at e
  - $\Rightarrow$ Advance e



Convex Polygon

	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

P passes from outside Q to inside at f.

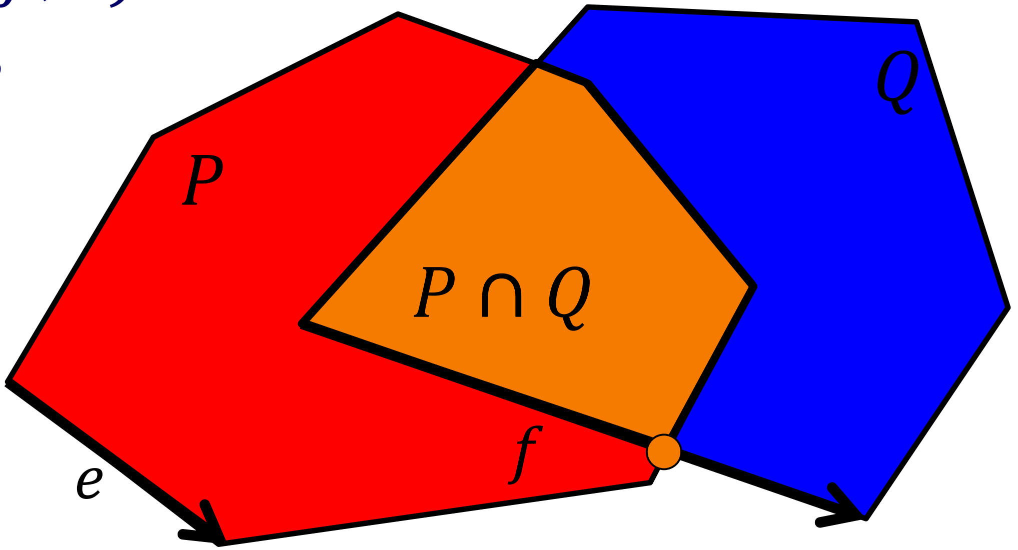
Case 1 (e is exterior):

- If e does not aim at f:

  $\Rightarrow$ f is interior and (f,e) is CW

  $\Rightarrow$ f cannot aim at e

  $\Rightarrow$ Advance e



Convex Polygon

	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

P passes from outside Q to inside at f.

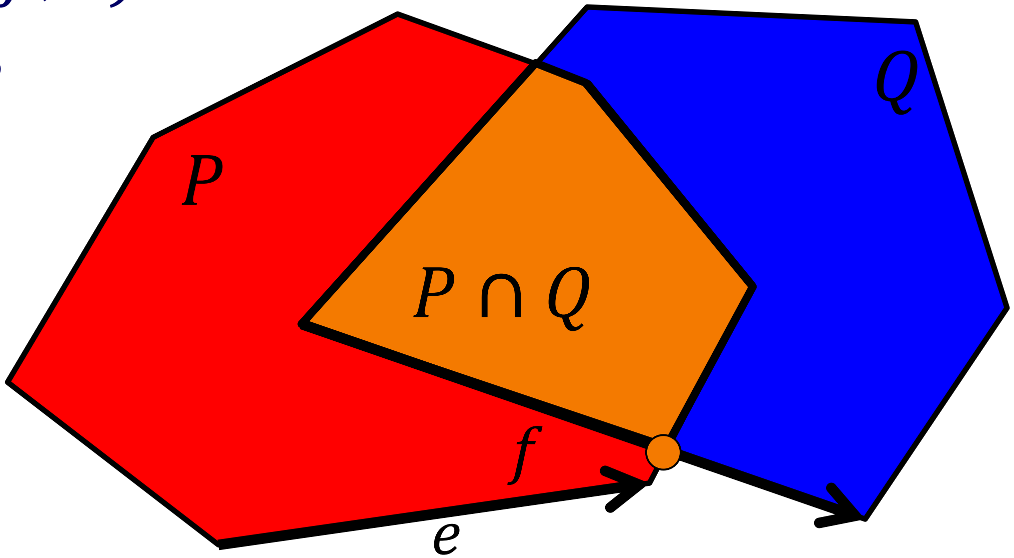
Case 1 (e is exterior):

- If e does not aim at f:

  $\Rightarrow$ f is interior and (f,e) is CW

  $\Rightarrow$ f cannot aim at e

  $\Rightarrow$ Advance e



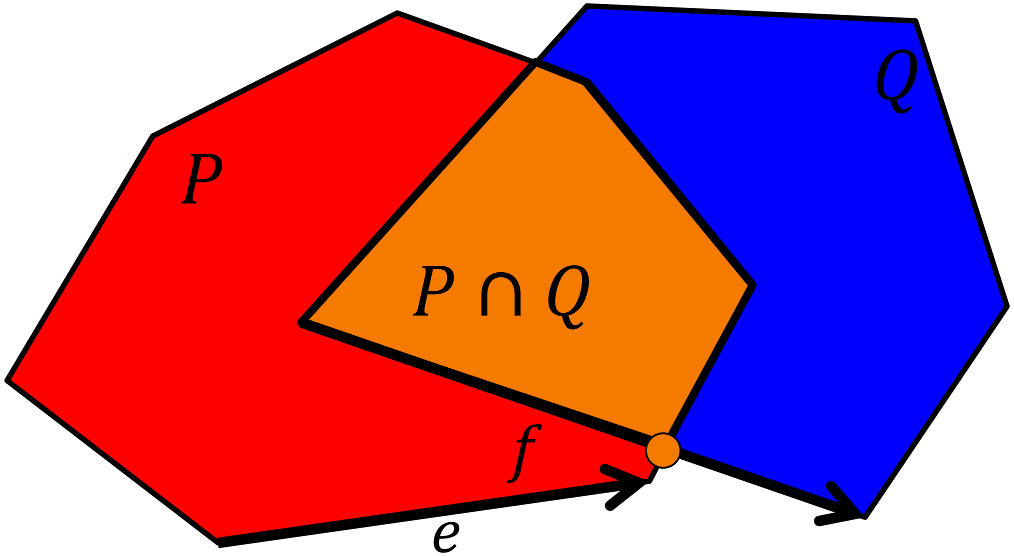
Convex Polygon

	$\boldsymbol{aim(e,f)}$	$\boldsymbol{\neg aim(e,f)}$
$\boldsymbol{aim(f,e)}$	**exterior**	$\boldsymbol{f}$
$\boldsymbol{\neg aim(f,e)}$	$\boldsymbol{e}$	**exterior**

P passes from outside Q to inside at f.

<u>Case 1 (e is exterior)</u>:

- If e does not aim at f: advance e
- If e aims at f: advance e



Convex Polygon

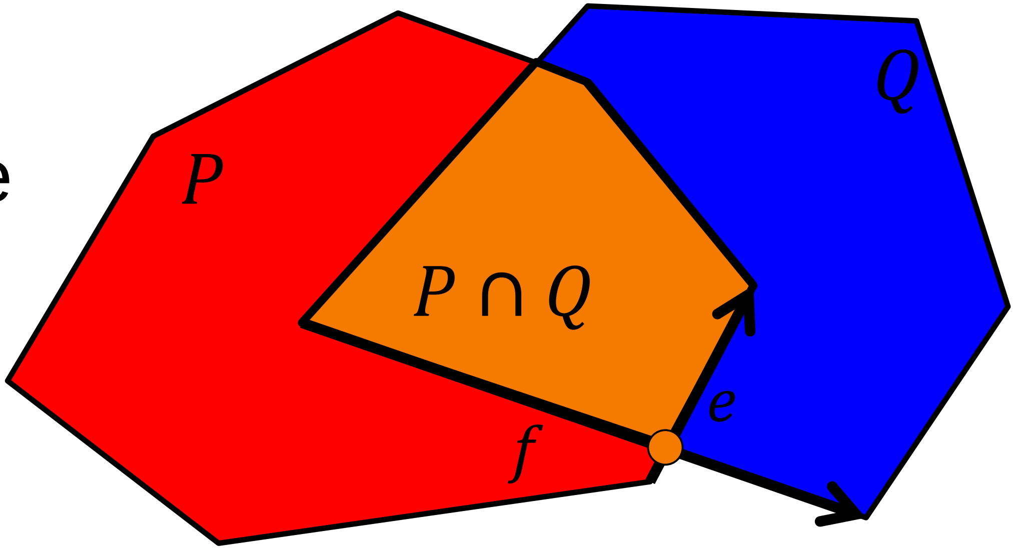
	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

P passes from outside Q to inside at f.

Case 1 (e is exterior):

- If e does not aim at f: advance e
- If e aims at f: advance e

$\Rightarrow$ Until e crosses f we advance e.



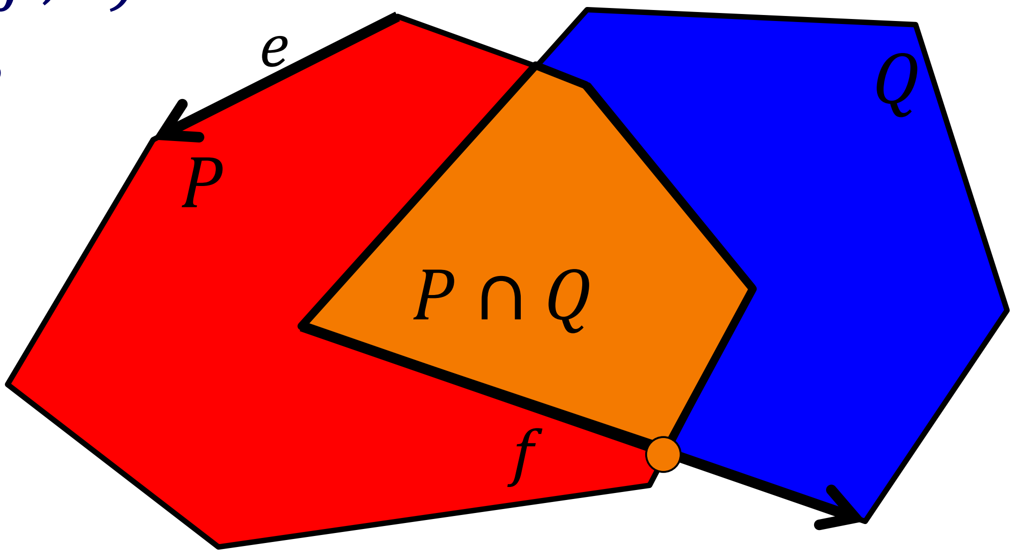
Convex Polygon

	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

P passes from outside Q to inside at f.

<u>Case 2 (e is interior)</u>:

- If e aims at f:
  - $\Rightarrow$ f is interior and (f,e) is CCW
  - $\Rightarrow$ f cannot aim at e
  - $\Rightarrow$ Advance e



$\Rightarrow$ Until e is exterior, advance e.

Convex Polygon

	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

P passes from outside Q to inside at f.

<u>Case 2 (e is interior)</u>:

- If e aims at f:

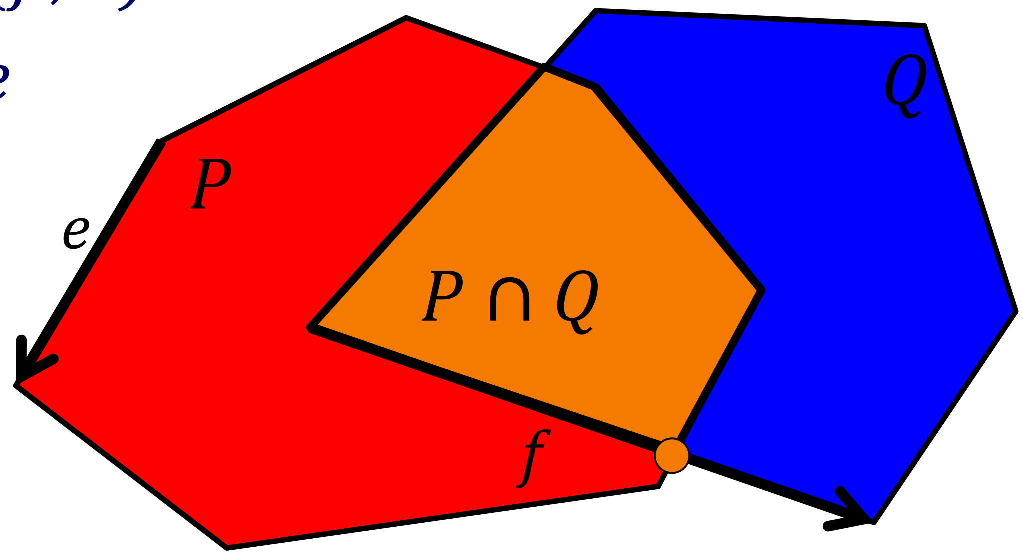
  $\Rightarrow$ f is interior and (f,e) is CCW

  $\Rightarrow$ f cannot aim at e

  $\Rightarrow$ Advance e

$\Rightarrow$ Until e is exterior, advance e.

$\Rightarrow$ Back to case 1.



Convex Polygon

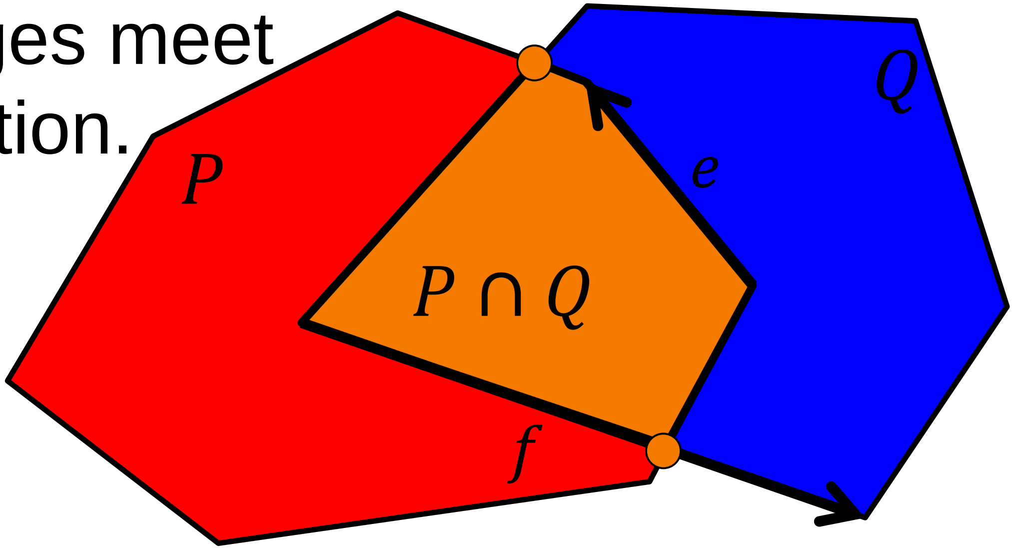
	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	exterior	f
$\neg aim(f,e)$	e	exterior

P passes from outside Q to inside at f.

Case 2 (e is interior):

- If e does not aim at f:
  Claim:
  In this case the edges meet at the next intersection.

  Whichever edge gets to the next intersection first waits for the other.



Convex Polygon

	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

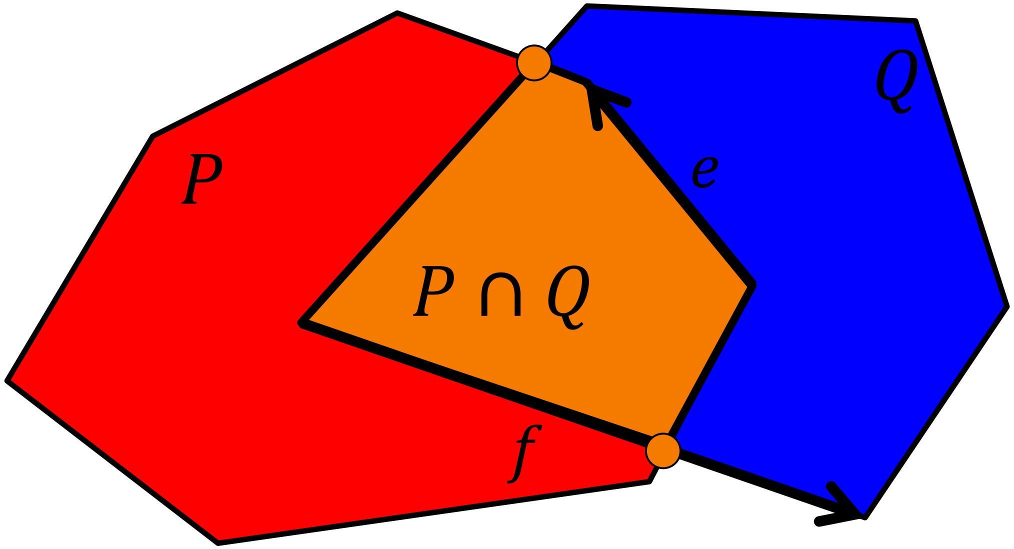
P passes from outside Q to inside at f.

<u>Case 2 (e is interior)</u>:

- If e does not aim at f:
  - If f is interior:

    $\Rightarrow$ f aims at e

    $\Rightarrow$ Advance f



Convex Polygon

	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

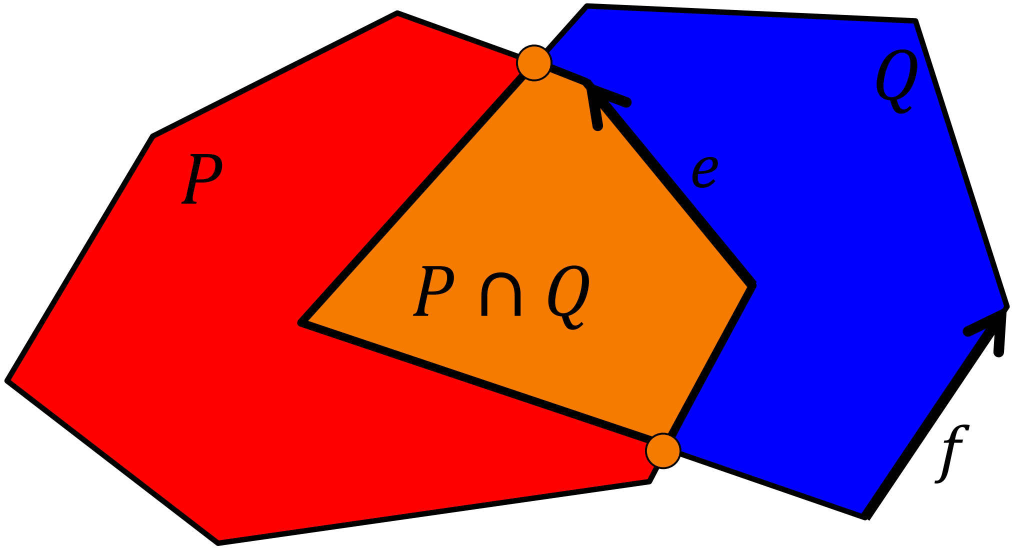
P passes from outside Q to inside at f.

Case 2 (e is interior):

- If e does not aim at f:
  - If f is interior:

    $\Rightarrow$ f aims at e

    $\Rightarrow$ Advance f



$\Rightarrow$ Until f is exterior, advance f.

$\Rightarrow$ At that point, e and f aim away from each other

Convex Polygon

	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

P passes from outside Q to inside at f.

Case 2 (e is interior):

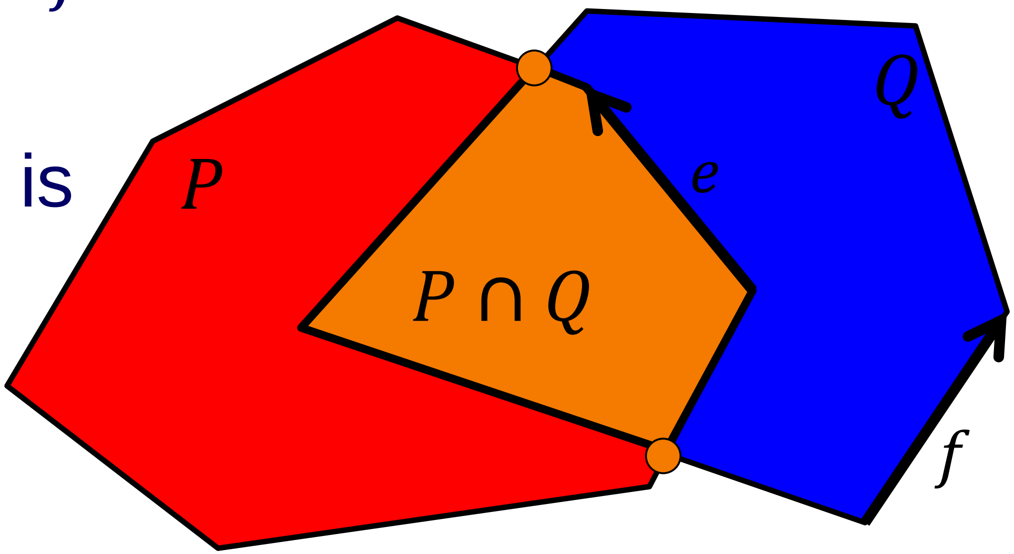
- Until the next intersection:

  $\Rightarrow$ e only advances if f is interior

  Note:

  If e advances and f is exterior then e aims at f and f does not aim at e.

  But this cannot be.



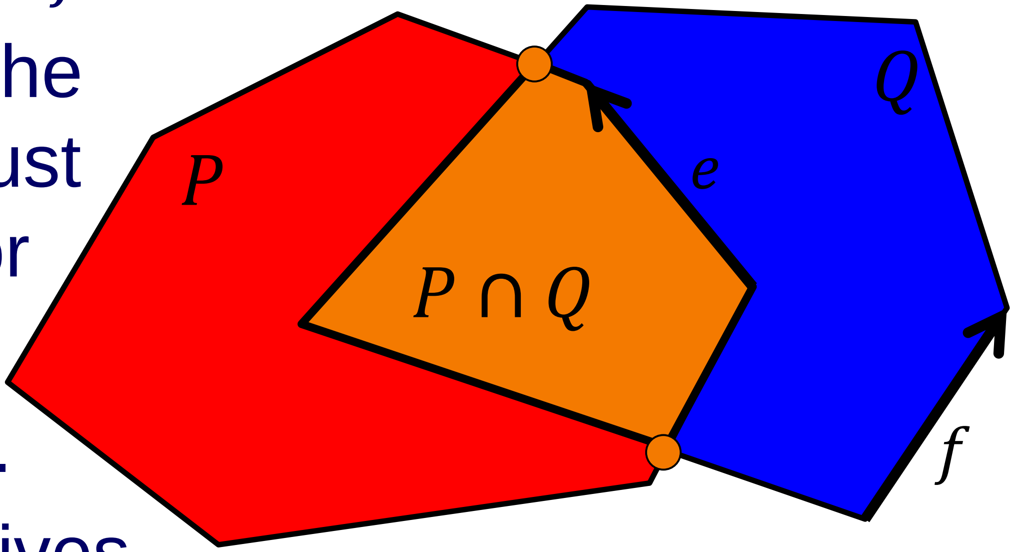
Convex Polygon

	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

P passes from outside Q to inside at f.

Case 2 (e is interior):

- Until the next intersection:
  - $\Rightarrow$ e only advances if f is interior
  - $\Rightarrow$ If e advances to the intersection, f must have been interior before.
  - $\Rightarrow$ f is exterior after.
  - $\Rightarrow$ e waits until f arrives.



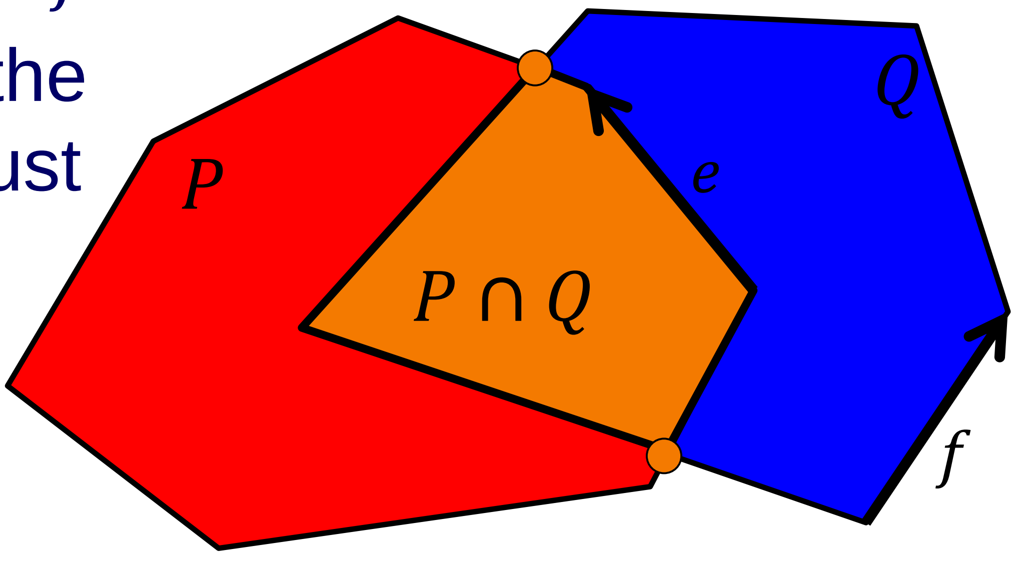
Convex Polygon

	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

P passes from outside Q to inside at f.

Case 2 (e is interior):

- Until the next intersection:
  - $\Rightarrow$ e only advances if f is interior
  - $\Rightarrow$ If f advances to the intersection, f must be interior.
  - $\Rightarrow$ f waits until e arrives.



Convex Polygon

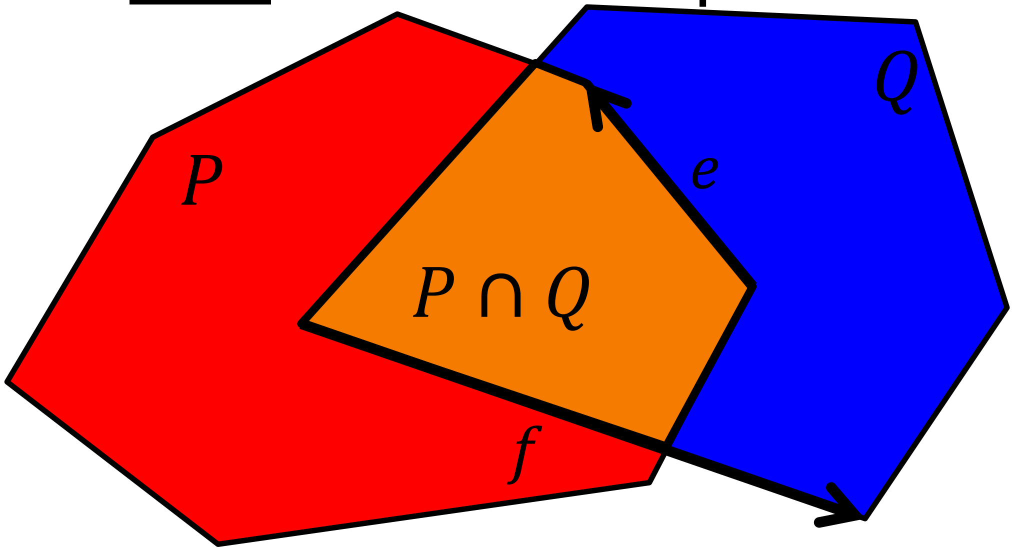
	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

P passes from outside Q to inside at f.

<u>Case 2 (e is interior to f)</u>:

- If e does not aim at f:

$\Rightarrow$ e and f advance to the <u>next</u> intersection point.

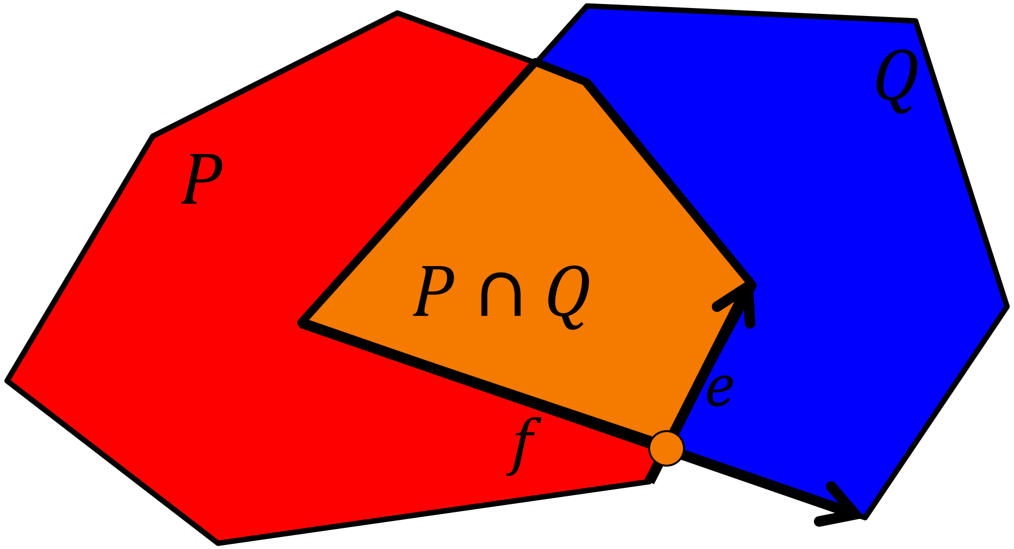


Convex Polygon

	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

<u>Sub-Claim 2</u>:

Once a point of intersection has been found, the next intersection will be found (without skipping).



Convex Polygon

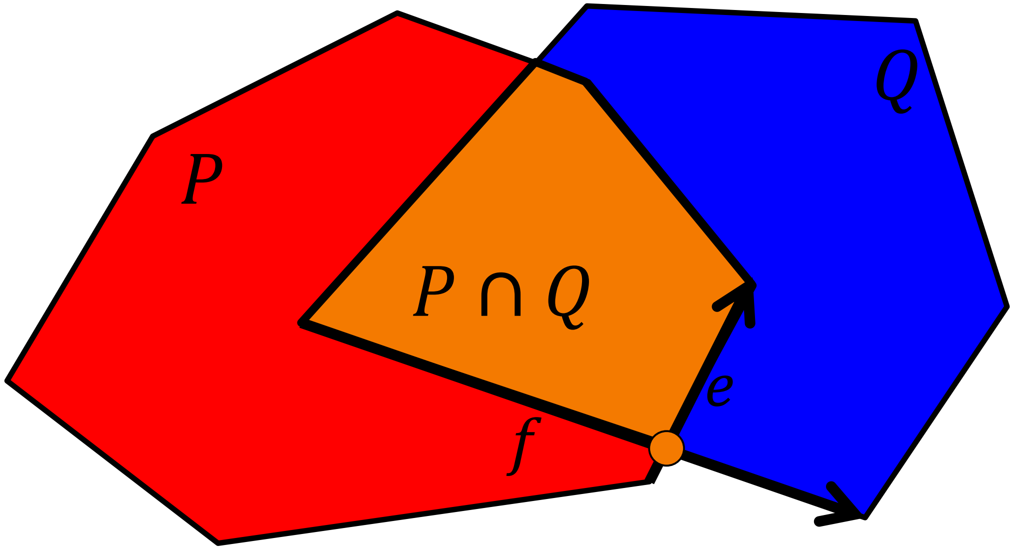
	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

Proof (Sub-Claim 2

W.l.o.g, assume that e is interior.

$\Rightarrow$ e does not aim at f.

$\Rightarrow$ As above, e and f advance to the next intersection.



Convex Polygon

	$aim(e,f)$	$\neg aim(e,f)$
$aim(f,e)$	**exterior**	f
$\neg aim(f,e)$	e	**exterior**

Proof (Sub-Claim 2

W.l.o.g, assume that e is interior to f.

$\Rightarrow$ e does not aim at f.

$\Rightarrow$ As
int

Thus, we find an intersection within the first $|P|+|Q|$ iterations and find the rest within the next $|P|+|Q|$ iterations.