Search and Intersection

O’Rourke, Chapter 7

Outline

- Review
  - Barycentric Coordinates
- Primitive Intersection

Segment-Triangle Intersection

Barycentric Coordinates:

Given points $v_1, \ldots, v_n \in \mathbb{R}^d$ (with $n \leq d$) a point p in the $(n-1)$-dimensional hyperplane passing through the $\{v_i\}$ can be expressed as:

$$p = \sum_{i=1}^{n} \lambda_i \cdot v_i \quad \text{with} \quad 1 = \sum_{i=1}^{n} \lambda_i \,.$$

p is in the convex hull of the points if $\lambda_i \geq 0$ for all $1 \leq i \leq n$.

Segment-Triangle Intersection

Barycentric Coordinates:

If a point p is on the plane passing through the $\{v_i\}$ we can get the barycentric coordinates of p by solving the (over-constrained) $n \times d$ linear system:

$$A(\lambda) = p \quad \Leftrightarrow \quad \begin{pmatrix} v_1^1 & \cdots & v_n^1 \\ \vdots & \ddots & \vdots \\ v_1^d & \cdots & v_n^d \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} p^1 \\ \vdots \\ p^d \end{pmatrix}$$

Segment-Triangle Intersection

Barycentric Coordinates:

If a point p is on the plane passing through the $\{v_i\}$ we can get the barycentric coordinates of p by solving the (over-constrained) $n \times d$ linear system:

$$A(\lambda) = p$$

In general, the least-squares solution is given by the solution to the normal equation:

$$(A^t \cdot A)\lambda = A^t p$$

Since p is on the plane passing through the $\{v_i\}$, the least-squares solution is the exact solution.

Segment-Triangle Intersection

<u>Barycentric Coordinates</u>:

If a point p is on the plane passing through the $\{v_i\}$ we can get the barycentric coordinates of p by solving the (over-constrained) $n \times d$ linear system:

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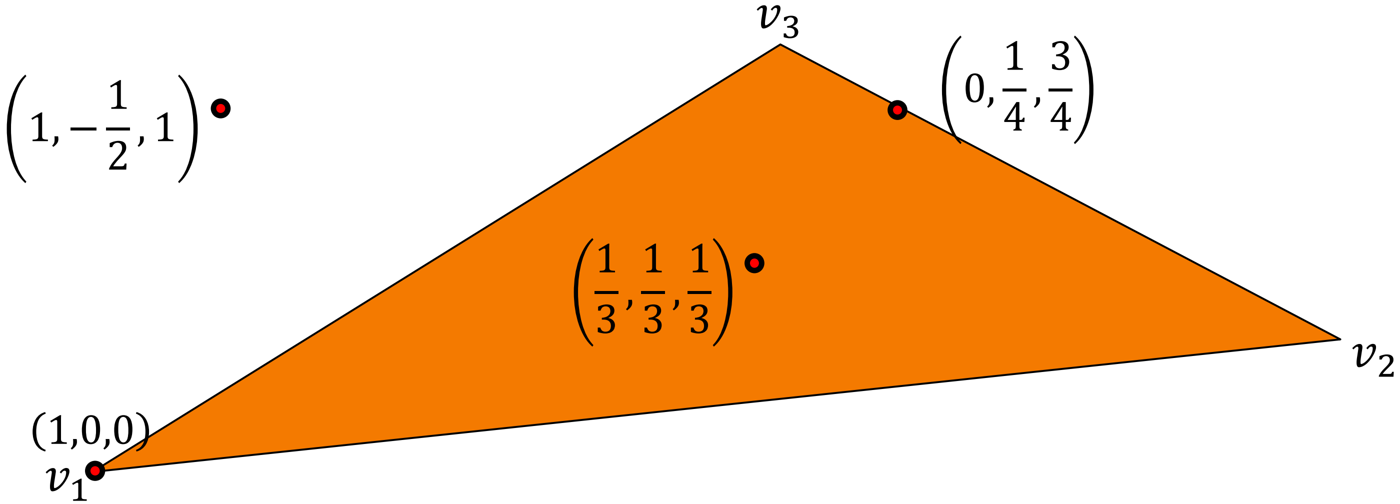
The matrix $A^t \cdot A$ will be non-singular if and only if the $\{v_i\}$ are linearly independent.

Point-Triangle Intersection

Barycentric Coordinates ($n = 3$):



For a point p with barycentric coordinates (α, β, γ):

- p is outside the triangle if $\alpha, \beta, \gamma < 0$ or $1 < \alpha, \beta, \gamma$
- p is on an edge if $0 \leq \alpha, \beta, \gamma \leq 1$ and one of α, β, γ is 0
- p is on a vertex if $0 \leq \alpha, \beta, \gamma \leq 1$ and two of α, β, γ are 0
- p is inside if $0 < \alpha, \beta, \gamma < 1$.

Outline

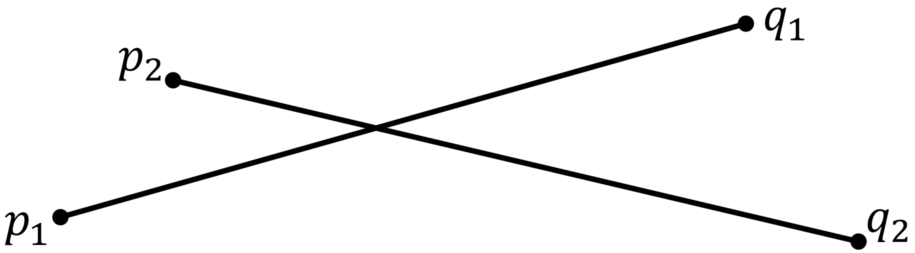
- Review
- Primitive Intersection
  - Segment-Segment (2D)
  - Segment-Triangle (3D)
  - Point-Polygon/Polyhedron (2D/3D)

Segment-Segment Intersection

Given segments (p_1, q_1) and (p_2, q_2) in $\mathbb{R}^2$, we would like to determine where and how they intersect.

Where: Requires floating point precision.

How: Can be done with integer precision.



Segment-Segment Intersection

Points on $\overline{pq}$ can be expressed as:

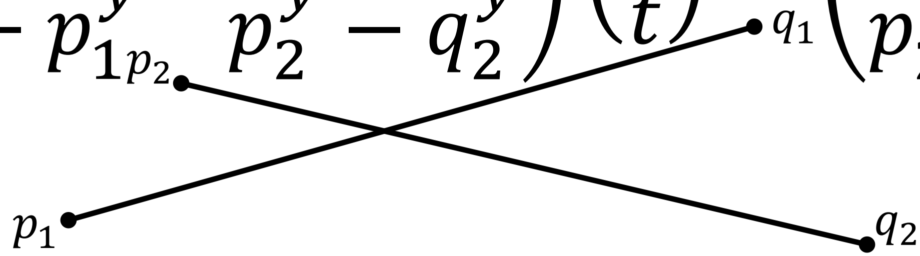
$$\Phi(s) = p + t \cdot (q - p), \qquad t \in [0,1].$$

The intersection can be computed by first intersecting the lines, solving:

$$p_1 + s(q_1 - p_1) = p_2 + t(q_2 - p_2)$$

Rewriting, we get:

$$\begin{pmatrix} q_1^x - p_1^x & p_2^x - q_2^x \\ q_1^y - p_1^y & p_2^y - q_2^y \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} p_2^x - p_1^x \\ p_2^y - p_1^y \end{pmatrix}$$



Segment-Segment Intersection

$$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} q_1^x - p_1^x & p_2^x - q_2^x \\ q_1^y - p_1^y & p_2^y - q_2^y \end{pmatrix}^{-1} \begin{pmatrix} p_2^x - p_1^x \\ p_2^y - p_1^y \end{pmatrix}$$

The matrix is not invertible if the vectors $q_1 - p_1$ and $q_2 - p_2$ are linearly dependent (i.e. the segment directions are parallel).

Otherwise, if $s, t \in [0,1]$ they intersect.

$p = p_1 + s \cdot (q_1 - p_1)$ gives the where.

Segment-Segment Intersection

How:

$$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} e \\ f \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}$$

- Parallel $\Leftrightarrow ad - bc = 0.$
- $s = 0 \Leftrightarrow de - cf = 0.$
- $t = 0 \Leftrightarrow -be + af = 0.$
- $s = 1 \Leftrightarrow de - cf = ad - bc.$
- $t = 1 \Leftrightarrow -be + af = ad - bc.$

Segment-Segment Intersection

<u>How</u>:

$$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} e \\ f \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}$$

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- $s = 1 \Leftrightarrow de - cf = ad - bc$
-

Assuming integer coordinates, we can identify the intersection type using only integer arithmetic, using <u>twice</u> the number of bits of precision.

Segment-Segment Intersection

Parallel Intersection:

We have a parallel intersection if:

- The lines are parallel ($ad - bc = 0$)

And

- Points p_1, q_1, and p_2 are collinear

And

- Point p_1 is between q_1 and q_2, or
- Point p_2 is between q_1 and q_2, or
- Point q_1 is between p_1 and p_2, or
- Point q_2 is between p_1 and p_2.

Segment-Segment Intersection

Parallel Intersection:

We have a parallel intersection if:

- The lines are parallel ($ad - bc = 0$)

And

- Points p_1, q_1, and p_2 are collinear

And

- Point p_1 is between q_1 and q_2, or

We defined these predicates, when performing triangulation.

In the case of parallel segments, we can identify if there is an intersection. And, if there is, we can compute the interval of intersection.

Outline

- Review
- Primitive Intersection
  - Segment-Segment (2D)
  - Segment-Triangle (3D)
  - Point-Polygon/Polyhedron (2D/3D)

Segment-Triangle Intersection

Given a segment $\overline{pq}$ and a triangle $\Delta v_1 v_2 v_3$ in $\mathbb{R}^3$, an intersection can be computed first intersecting the line with the plane containing the triangle.

The plane containing the triangle is:

$$\pi = \{p \in \mathbb{R}^3 | \langle p, n \rangle - d = 0\}$$

with normal $n \in \mathbb{R}^3$ and distance $d \in \mathbb{R}$ to the origin.

Segment-Triangle Intersection

Where:

The point of intersection is the solution to:

$$\langle p + s(q-p), n\rangle - d = 0,$$

or equivalently:

$$s = \frac{d - \langle p, n\rangle}{\langle q-p, n\rangle}$$

There is a solution if $q-p$ and n are not orthogonal – $\overline{pq}$ is not parallel to π.

Otherwise, if $s \in [0,1]$ they intersect.

Segment-Triangle Intersection

How (Segment):

$$s = \frac{d - \langle p, n \rangle}{\langle q - p, n \rangle}$$

- Parallel $\Leftrightarrow \langle q - p, n \rangle = 0.$
  - In plane $\Leftrightarrow d - \langle p, n \rangle = 0.$
- $s = 0 \Leftrightarrow d - \langle p, n \rangle = 0.$
- $s = 1 \Leftrightarrow d - \langle p, n \rangle = \langle q - p, n \rangle.$

Assuming integer coordinates, we can identify intersection type using only integer arithmetic, using three times the number of bits of precision.

Segment-Triangle Intersection

How (Point-Triangle):

Given the point of intersection of $\overline{pq}$ with the plane π, we can use barycenctric coordinates to test if the point of intersection is:

- On a triangle edge
- On a triangle vertex
- Inside the triangle
- Outside the triangle

Segment-Triangle Intersection

<u>How (Point-Triangle)</u>:

Given the point of intersection of $\overline{pq}$ with the plane π, we can use barycenctric coordinates to test if the point of intersection is:

- On a triangle edge
- On a triangle vertex
- Inside the triangle

<u>Challenge</u>:
This would entail making a discrete decision using floating point arithmetic (both for computing the point of intersection and the barycentric coordinates).

Segment-Triangle Intersection

How (Segment End-Point):

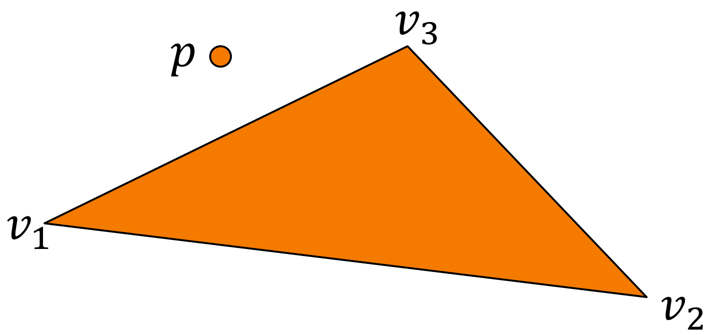
Given a triangle $\Delta v_1 v_2 v_3$ in $\mathbb{R}^3$ and a point p in the plane through the $\{v_i\}$, the point p is inside the triangle if and only if for any projection to 2D.

If we project onto one of the axis, we can get the 2D position while still keeping integer precision.

Point-Triangle Intersection (2D)

Barycentric Coordinates (Geometric):

Compute the ratio of the signed areas of the $\Delta p v_i v_{i+1}$ with the signed area of $\Delta v_1 v_2 v_3$.

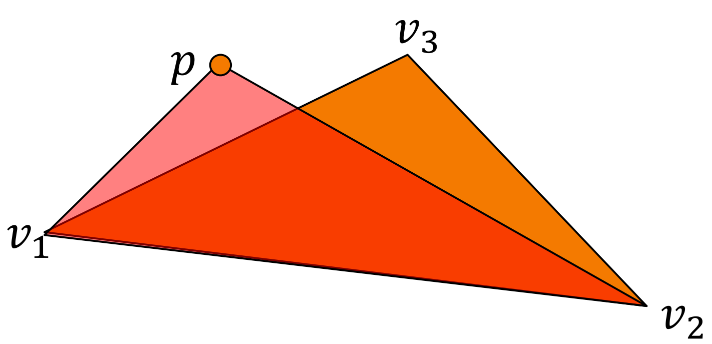


Point-Triangle Intersection (2D)

Barycentric Coordinates (Geometric):

Compute the ratio of the signed areas of the $\Delta p v_i v_{i+1}$ with the signed area of $\Delta v_1 v_2 v_3$.

$$\alpha = \frac{\text{Area}(p, v_1, v_2)}{\text{Area}(v_1, v_2, v_3)} > 0$$



Point-Triangle Intersection (2D)

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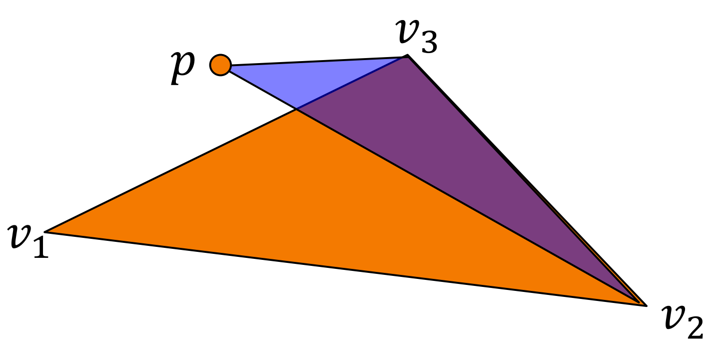
$$\alpha = \frac{\text{Area}(p, v_1, v_2)}{\text{Area}(v_1, v_2, v_3)} > 0$$

$$\beta = \frac{\text{Area}(p, v_2, v_3)}{\text{Area}(v_1, v_2, v_3)} > 0$$



Point-Triangle Intersection (2D)

Barycentric Coordinates (Geometric):

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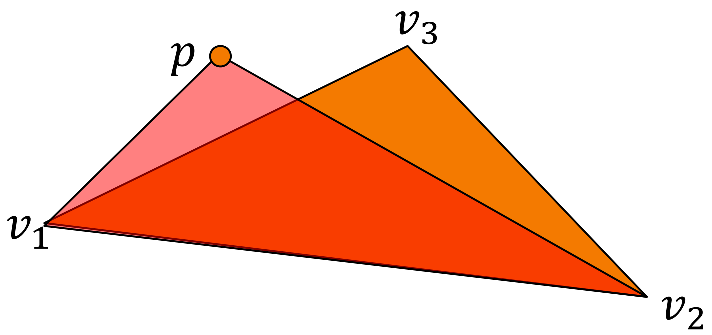
$$\alpha = \frac{\text{Area}(p, v_1, v_2)}{\text{Area}(v_1, v_2, v_3)} > 0$$

$$\beta = \frac{\text{Area}(p, v_2, v_3)}{\text{Area}(v_1, v_2, v_3)} > 0$$

$$\gamma = \frac{\text{Area}(p, v_3, v_1)}{\text{Area}(v_1, v_2, v_3)} < 0$$



Point-Triangle Intersection (2D)

Barycentric Coordinates (Geometric):

Compute the ratio of the signed areas of the $\Delta p v_i v_{i+1}$ with the signed area of $\Delta v_1 v_2 v_3$.

Note:

$$\text{Area}(p, v_1, v_2) + \text{Area}(p, v_2, v_3) + \text{Area}(p, v_3, v_1) = \text{Area}(v_1, v_2, v_3)$$
$$\Updownarrow$$
$$\alpha + \beta + \gamma = 1$$

$$\alpha = \frac{\text{Area}(p, v_1, v_2)}{\text{Area}(v_1, v_2, v_3)} > 0$$
$$\beta = \frac{\text{Area}(p, v_2, v_3)}{\text{Area}(v_1, v_2, v_3)} > 0$$
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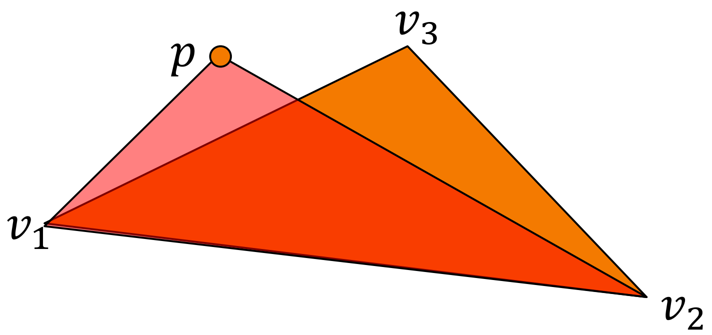
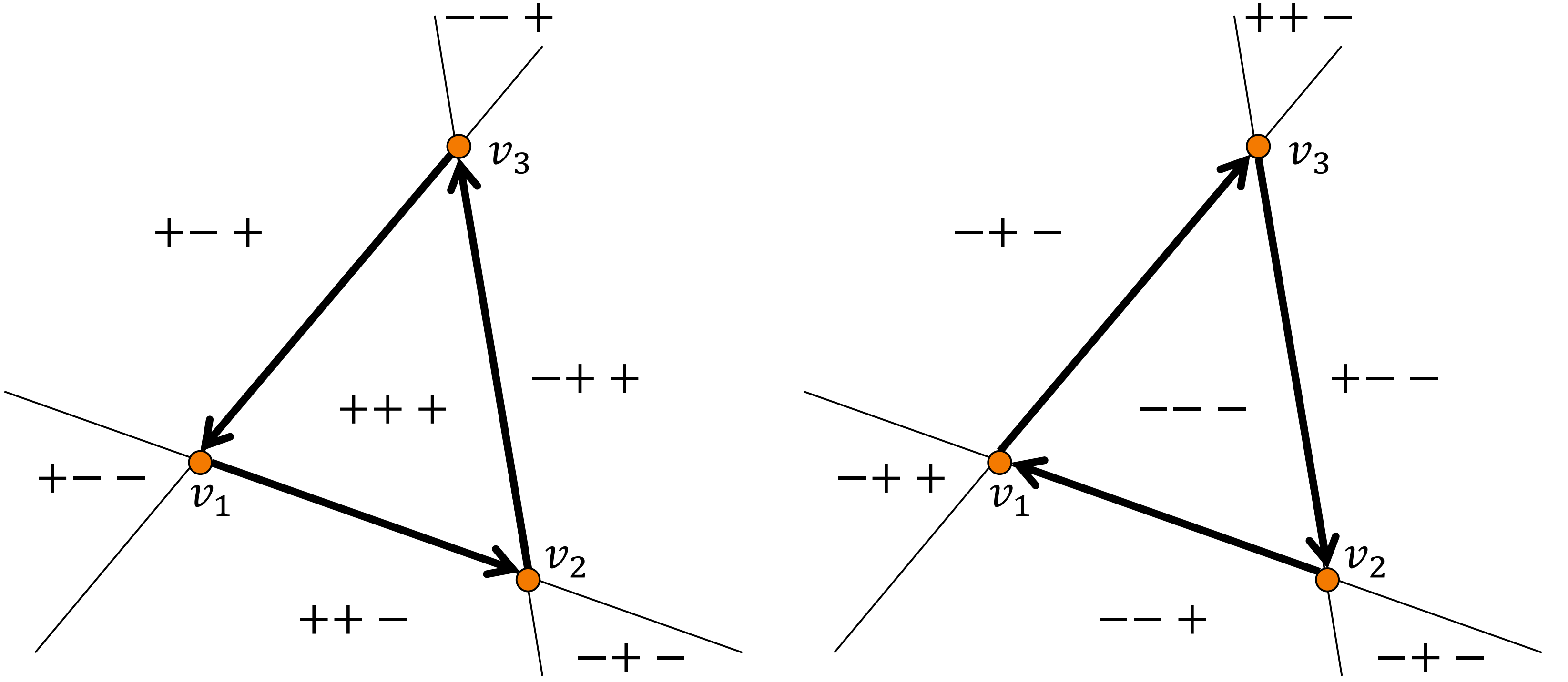


Point-Triangle Intersection (2D)

<u>Areas</u>:

We can test if there is an intersection by looking at the signs of the areas of the triangles made between the point and the edges.



Point-Triangle Intersection (2D)

Classification:

- If all area are positive/negative, the point is interior.
- If two are positive/negative and one is zero, the point is on an edge.
- If two signs are zero, the point is at a vertex.

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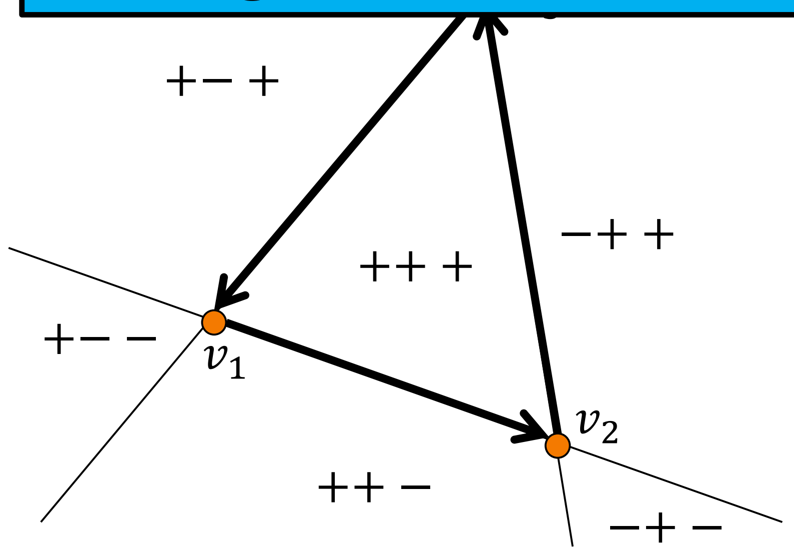
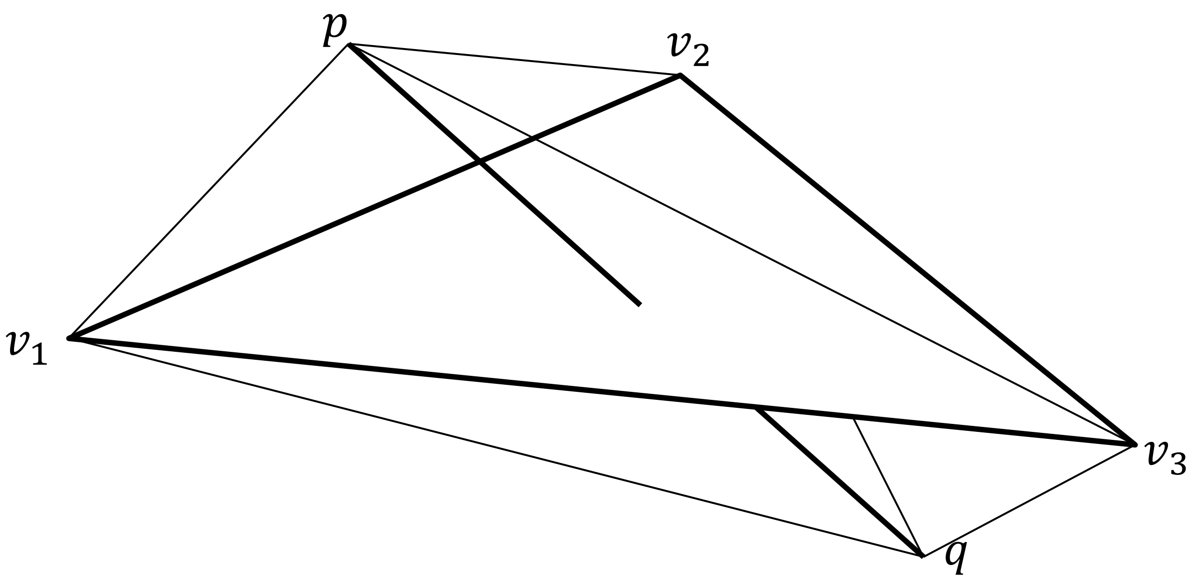
Assuming integer coordinates, we can identify intersection type using only integer arithmetic, using two times the number of bits of precision.



Segment-Triangle Intersection

How (Segment Interior):

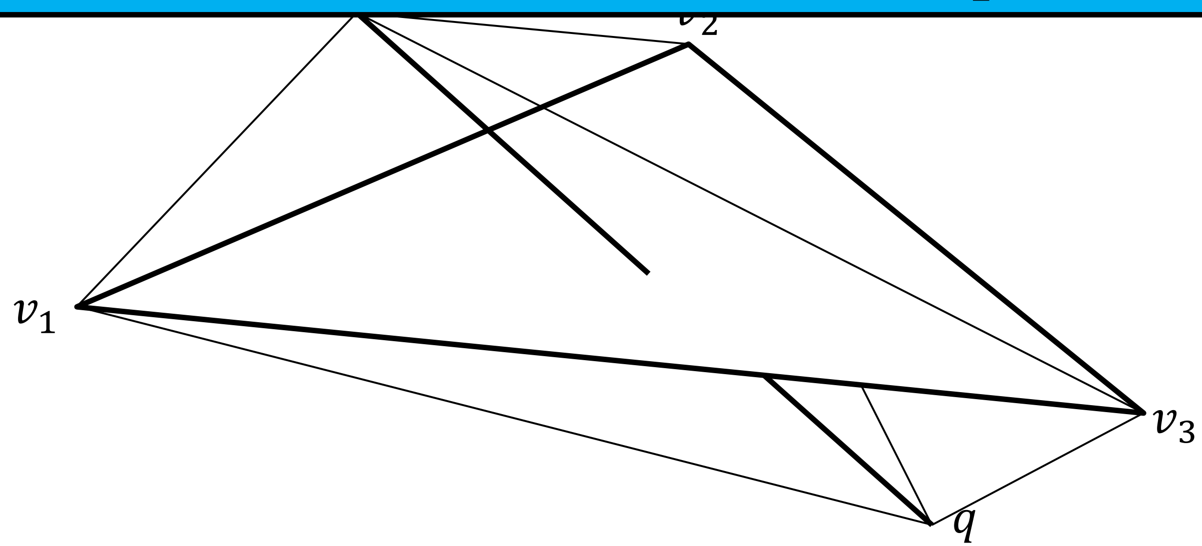
If the $\overline{pq}$ intersects π properly, we can classify the point of intersection by considering the volumes of the tetrahedra.



Segment-Triangle Intersection (3D)

<u>Classification</u>:
- If all volumes are positive/negative, the point is interior.
- If two are positive/negative and one is zero, the point is on an edge.
- If two signs are zero, the point is at a vertex.

Assuming integer coordinates, we can identify intersection type using only integer arithmetic, using three times the number of bits of precision.



Segment-Triangle Intersection

How:

- If end-points are on the plane:
  - » Do segment-triangle (2D) intersection
- If interior of the edge crosses:
  - » Do segment-triangle (3D) intersection
- If segment and plane are parallel
  - » …
- Otherwise
  - » No intersection

Note:

- All the predicates can be performed using integer arithmetic.
- Implementation requires three times the number of bits of precision.

Outline

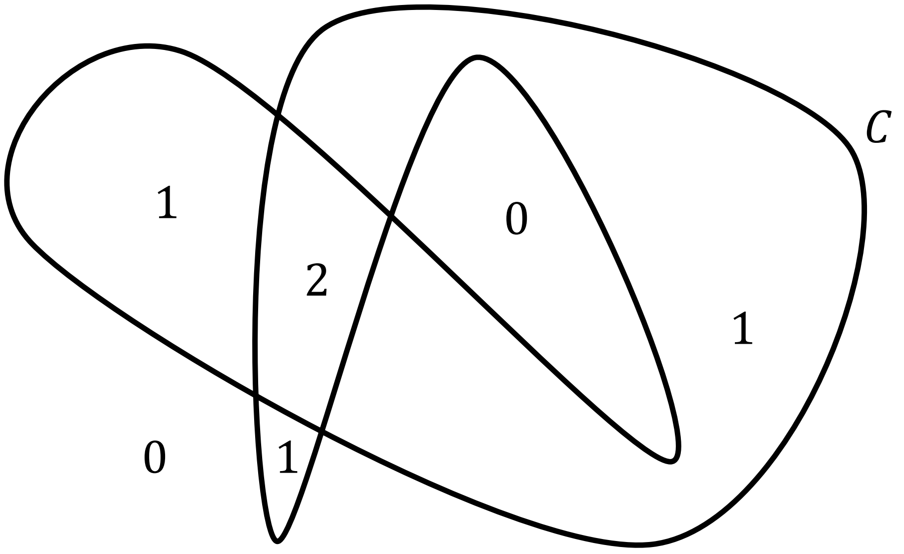
- Review
- Primitive Intersection
  - Segment-Segment (2D)
  - Segment-Triangle (3D)
  - Point-Polygon/Polyhedron (2D/3D)
    - » Winding Number
    - » Parity Test

Point-Polygon/Polyhedron

Winding Number:

Given a point p and a curve C in the plane, we can compute the number of times the curve winds around p.

A point is interior if its winding number is odd/one.



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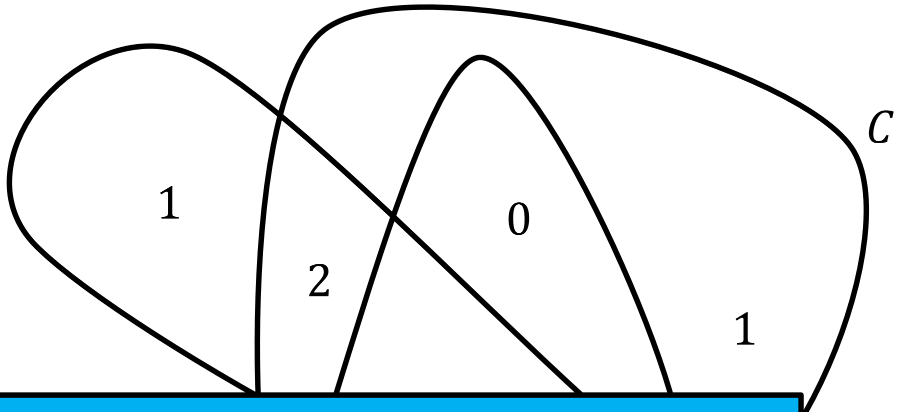


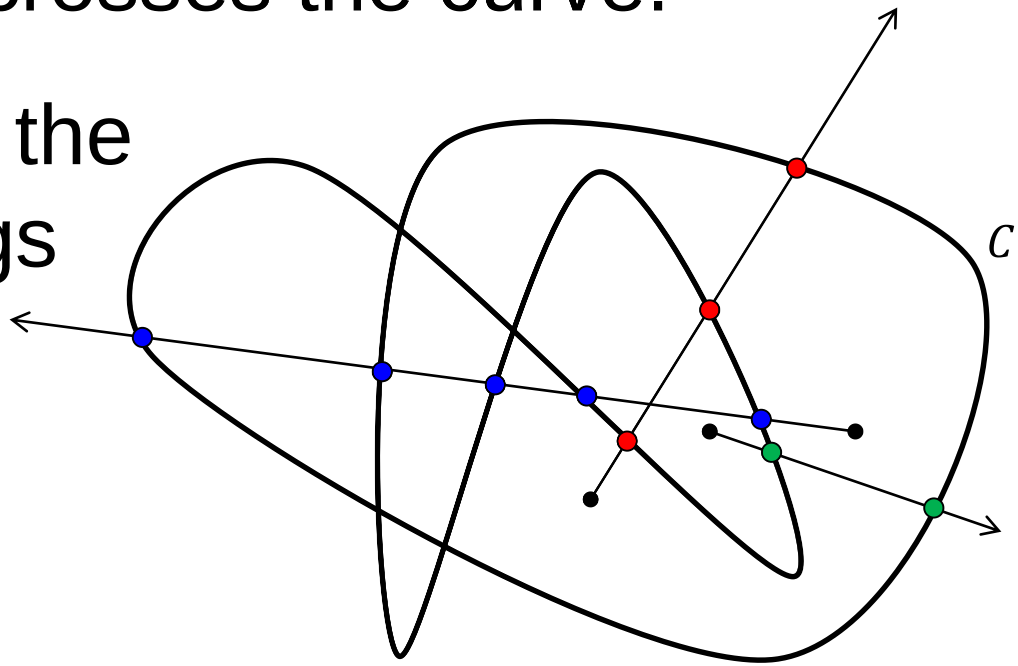
A similar approach (measuring steridians instead of angles) can be used to test for points in polyhedra.

Point-Polygon/Polyhedron

Parity Test:

Given a point p and a curve C in the plane, we can compute the number of times a ray emanating from p crosses the curve.

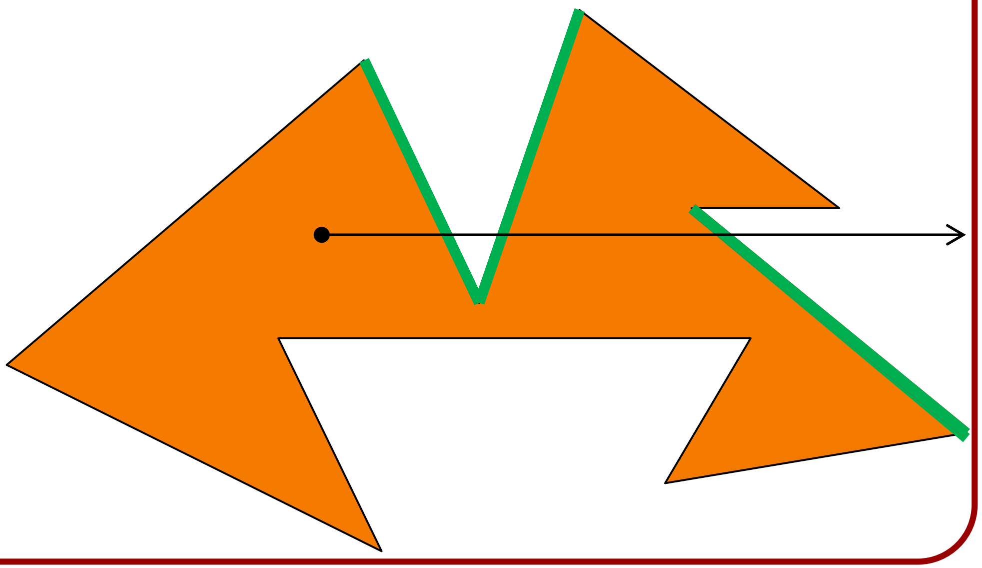
A point is interior if the number of crossings is odd.



Point-Polygon/Polyhedron

Parity Test:

When the curve is a polygon, we can test using, e.g. a ray directed along the positive x-axis, and test for intersection with edges.

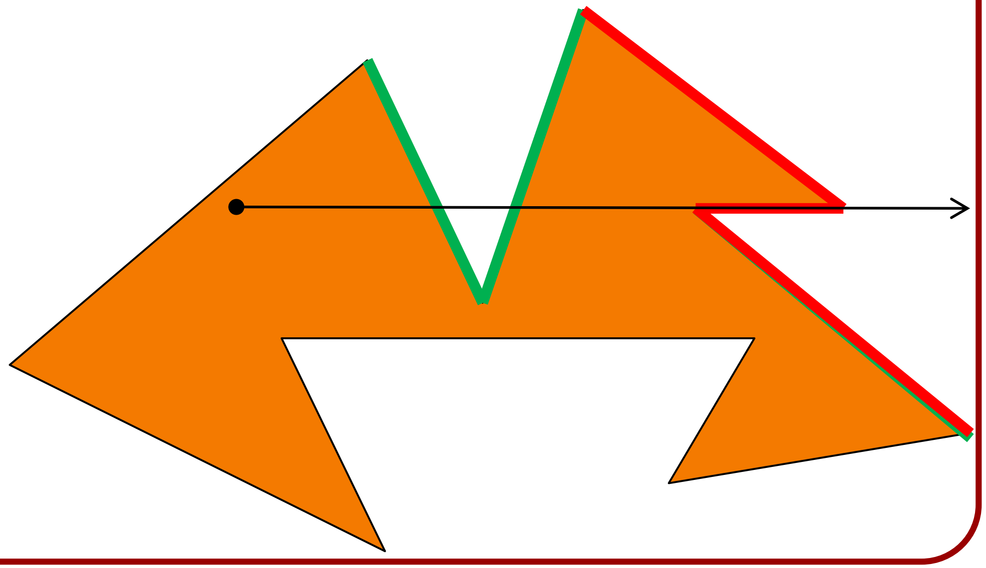


Point-Polygon/Polyhedron

Parity Test:

When the curve is a polygon, we can test using, e.g. a ray directed along the positive x-axis, and test for intersection with edges.

What happens if the intersection is degenerate?



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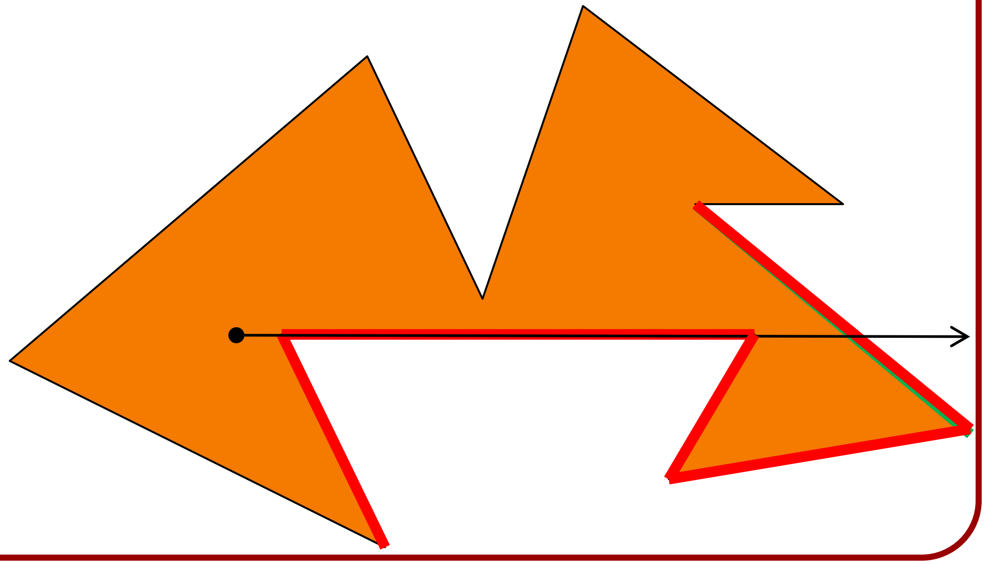

Point-Polygon/Polyhedron

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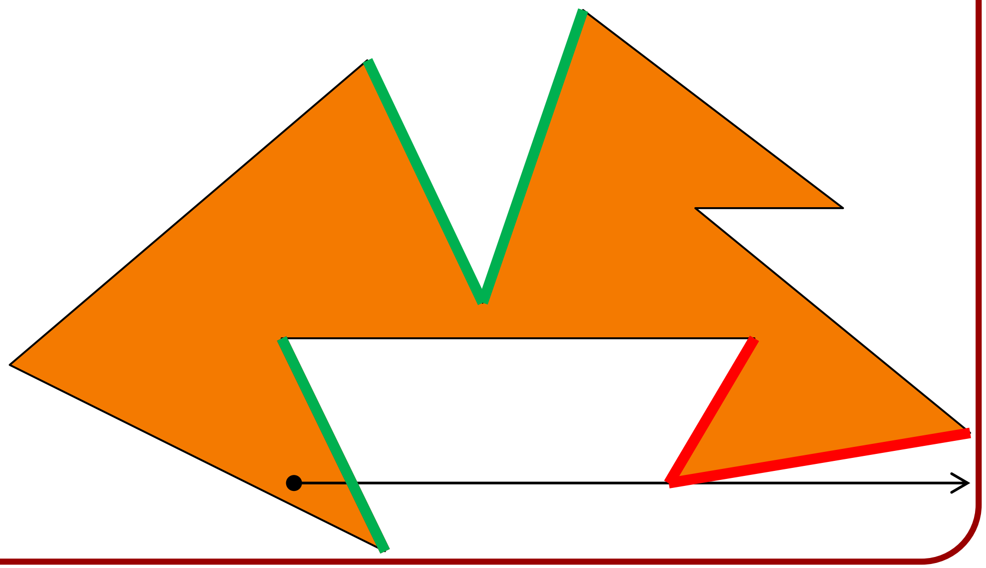
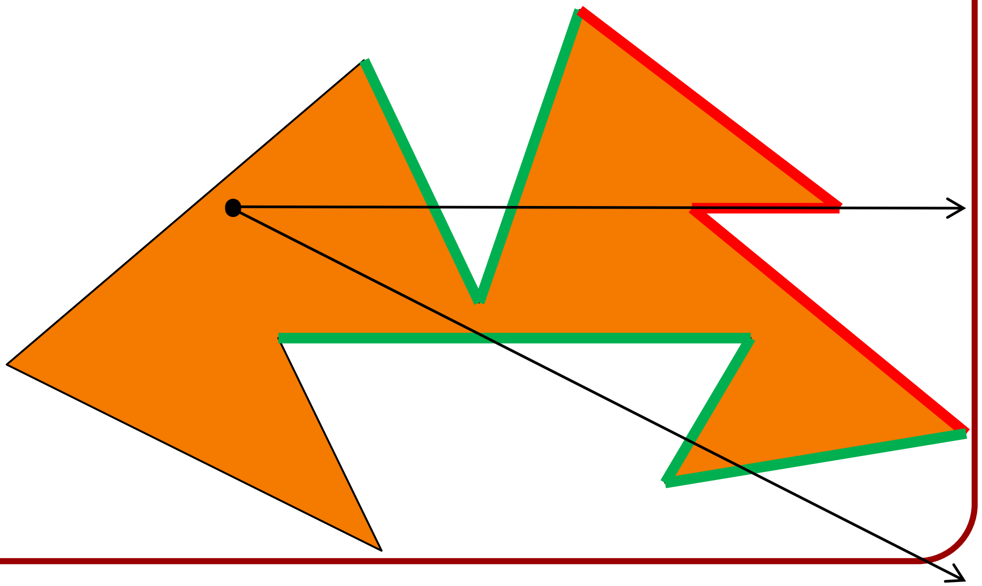


Point-Polygon/Polyhedron

Parity Test (Avoid Degeneracies):

- Test for degeneracies, and if encountered, cast a different ray in some other (random) direction.
- With high likelihood, that ray won't be degenerate.
- Otherwise, cast again.

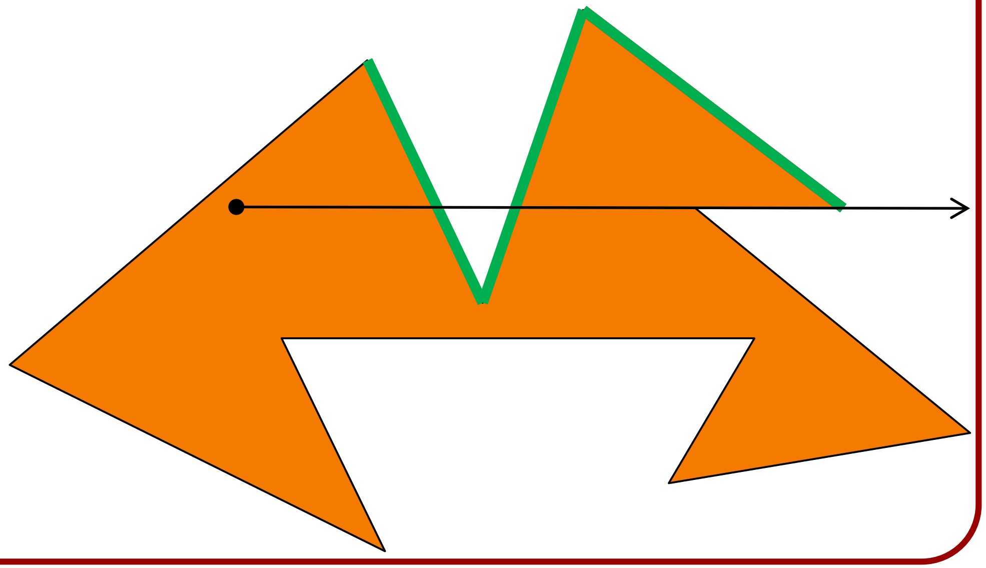


Point-Polygon/Polyhedron

<u>Parity Test (Handle Degeneracies)</u>:

- Define a ray-edge intersection if the ray intersects and one of the end-points is above.

(Equivalent to shifting the ray up by a tiny amount.)



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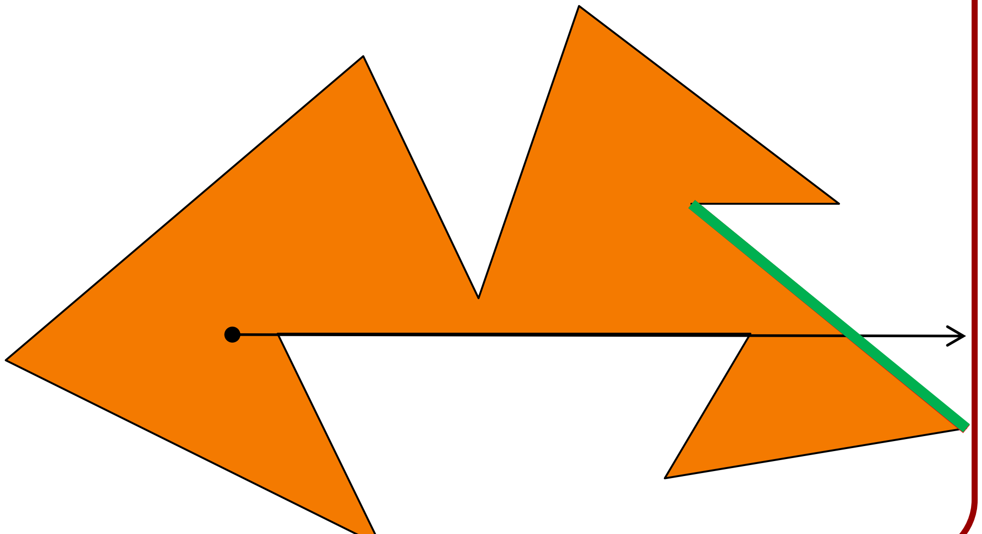

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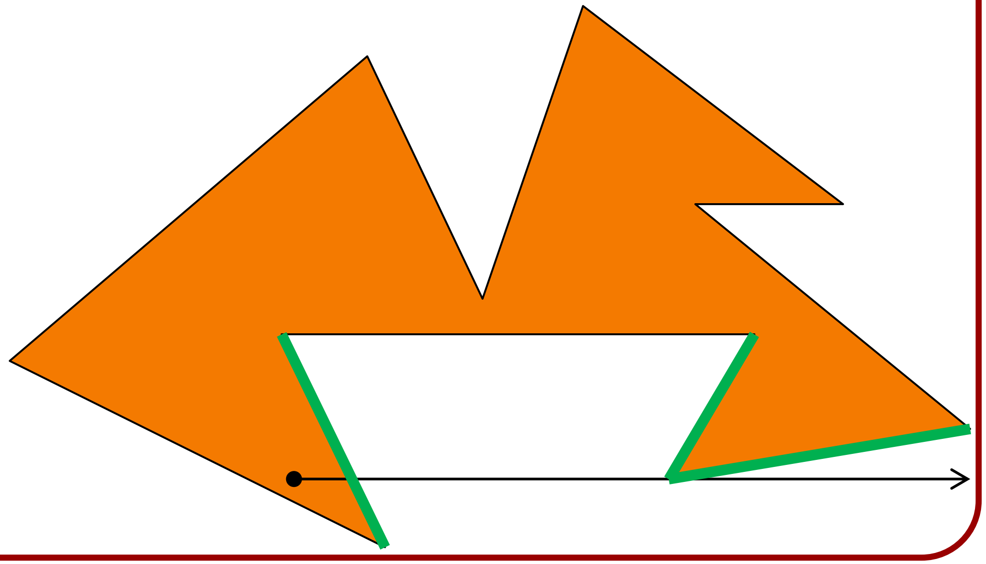


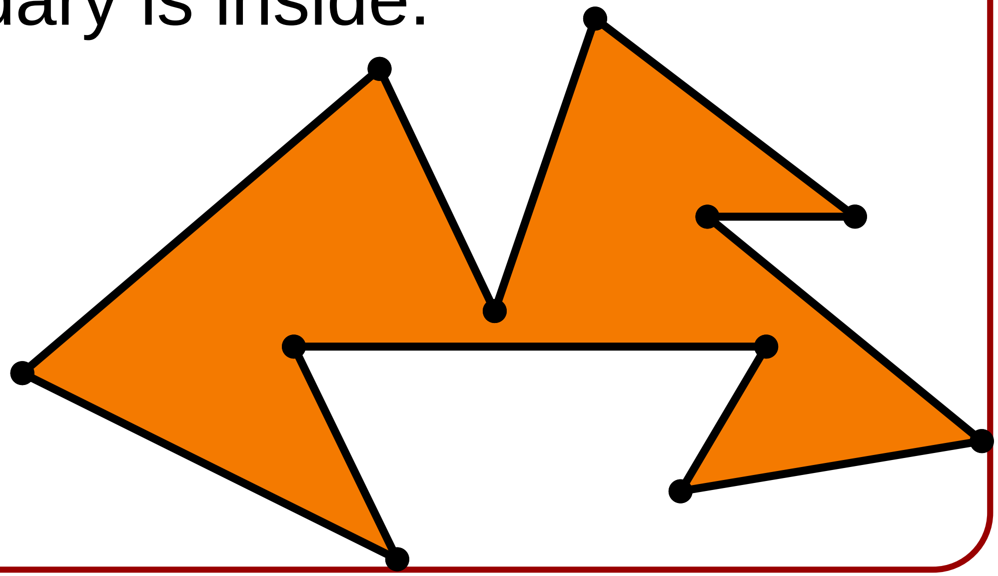
Point-Polygon/Polyhedron

Parity Test (Handle Degeneracies):

- Define a ray-edge intersection if the ray intersects and one of the end-points is above.

Boundary (Closed Polygons):

- A point on the boundary is inside.



Point-Polygon/Polyhedron

Parity Test (Handle Degeneracies):

- Define a ray-edge intersection if the ray intersects and one of the end-points is above.

Boundary (Partitioning Polygons):

- A point on an edge is inside if the points to the right are.

(Equivalent to shifting the ray up and right by a tiny amount.)