Voronoi Diagrams and Delaunay Triangulations

O’Rourke, Chapter 5

Outline

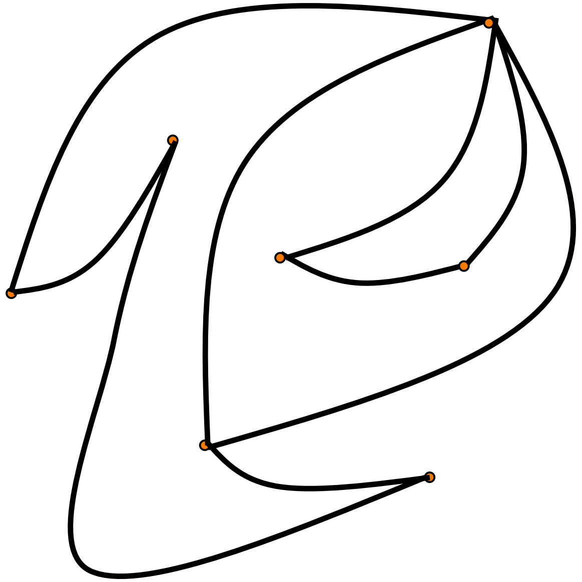
- Preliminaries
- Voronoi Diagrams / Delaunay Triangulations
- Lloyd’s Algorithm

Preliminaries

Claim:

Given a connected planar graph with V vertices, E edges, and F faces*, the graph satisfies:

$$V - E + F = 2$$

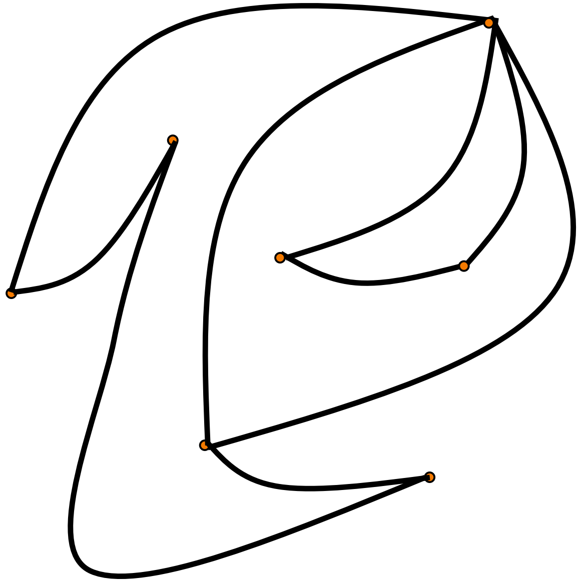


*The "external" face also counts. (Can think of this as a graph on the sphere.)

Preliminaries

Proof:

1. Show that this is true for trees.
2. Show that this is true by induction.



Preliminaries

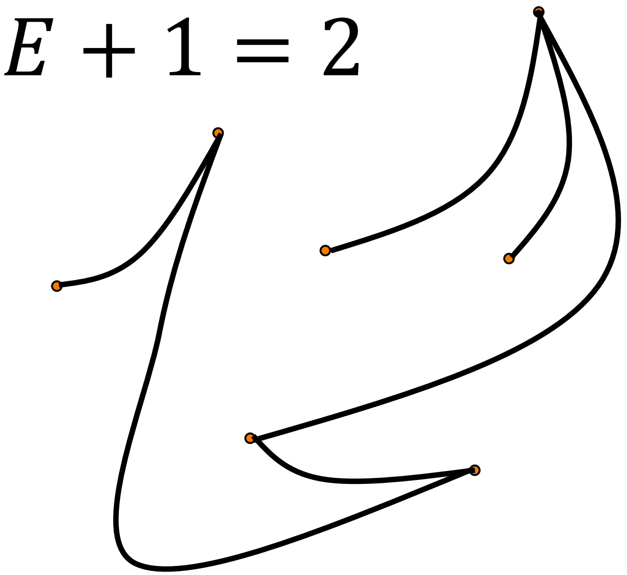
Proof (for Trees):

If a graph is a connected tree, it satisfies:

$$V = E + 1.$$

Since there is only one (external) face:

$$V - E + F = (E + 1) - E + 1 = 2$$



Preliminaries

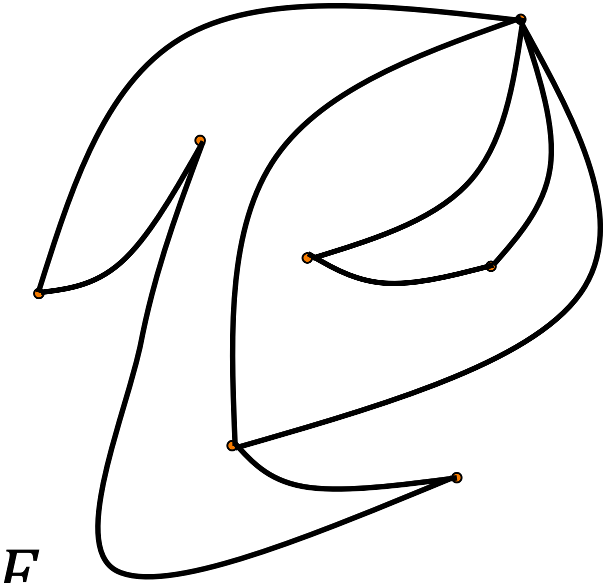
Proof (by Induction):

Suppose that we are given a graph G.

- If it’s a tree, we are done.
- Otherwise, it has a cycle.

Removing an edge on the cycle gives a graph G' with:

- The same vertex set ($V' = V$)
- One less edge ($E' = E - 1$)
- One less face ($F' = F - 1$)



By induction:

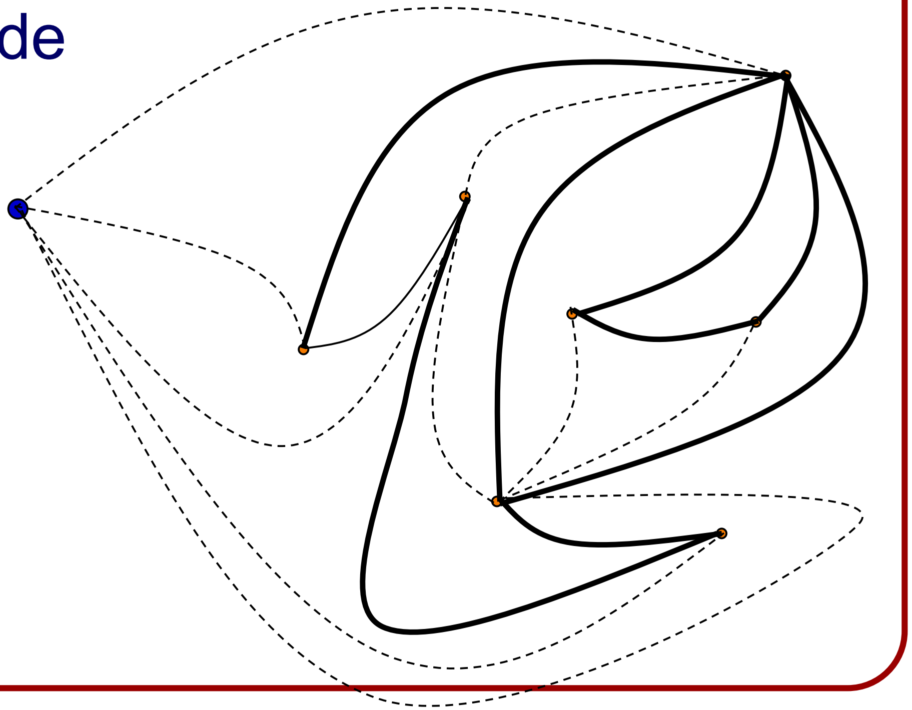
$$2 = V' - E' + F' = V - E + F$$

Preliminaries

Note:

Given a planar graph G, we can get a planar graph G' with triangle faces:

- Triangulate the interior polygons
- Add a "virtual point" outside and triangulate the exterior polygon.

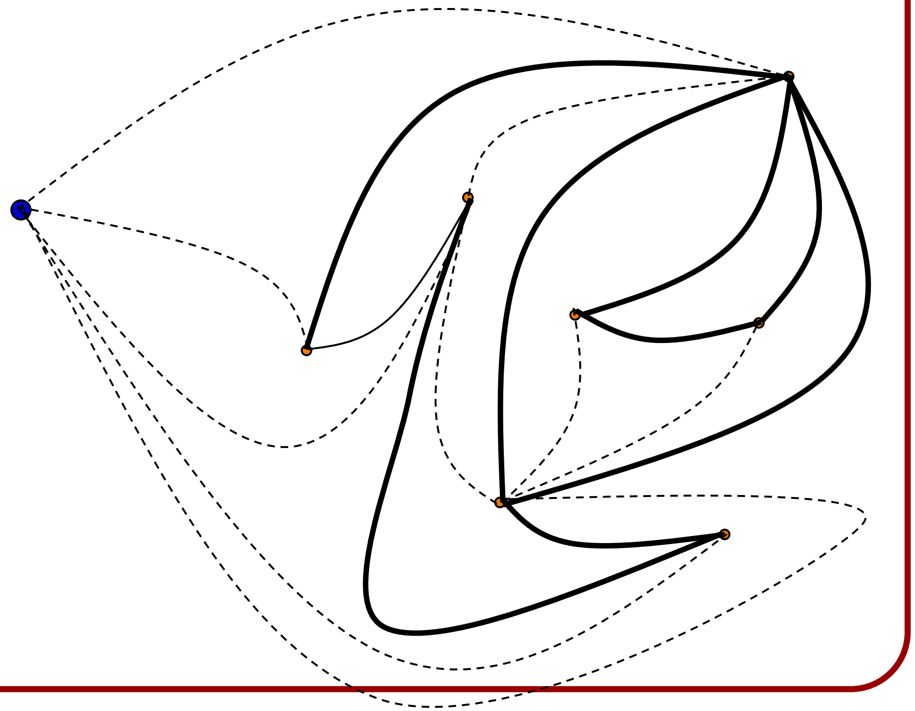


Preliminaries

Note:

The new graph has:

- $V' = V + 1,\ E' \geq E,\ F' \geq F$
- $V' - E' + F' = 2$
- $3E' = 2F'$



Preliminaries

Note:

The new graph has:

- $V' = V + 1,\ E' \geq E,\ F' \geq F$
- $V' - E' + F' = 2$
- $3E' = 2F'$

This gives:

$$\begin{array}{ccc} E' = 3V' - 6 & \qquad & F' = 2V' - 4 \\ \Downarrow & & \Downarrow \\ E \leq 3V - 3 & & F \leq 2V - 2 \end{array}$$

The number of edges/faces of a planar graph is linear in the number of vertices.

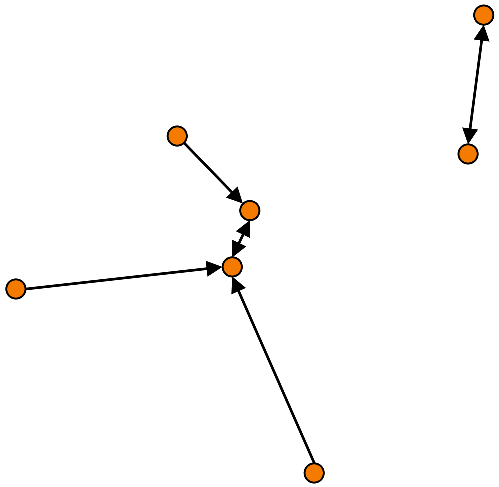
Preliminaries

Definition:

Given a set of points $\{p_1, \ldots, p_n\} \subset \mathbb{R}^d$, the *nearest-neighbor graph* is the directed graph with an edge from p_i to p_j, whenever:

$$\|p_k - p_i\| \geq \|p_j - p_i\| \quad \forall 1 \leq k \leq n.$$

Naively, the nearest-neighbor can be computed in $O(n^2)$ time by testing all possible neighbors.



Outline

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- Voronoi Diagrams / Delaunay Triangulations
- Lloyd’s Algorithm

Voronoi Diagrams

Definition:

Given points $P = \{p_1, \ldots, p_n\}$, the *Voronoi region* of point p_i, $V(p_i)$ is the set of points at least as close to p_i as to any other point in P:

$$V(p_i) = \left\{x \middle| |p_i - x| \leq \left|p_j - x\right| \forall 1 \leq j \leq n\right\}$$

Voronoi Diagrams

Definition:

The set of points with more than one nearest neighbor in P is the *Voronoi Diagram* of P:

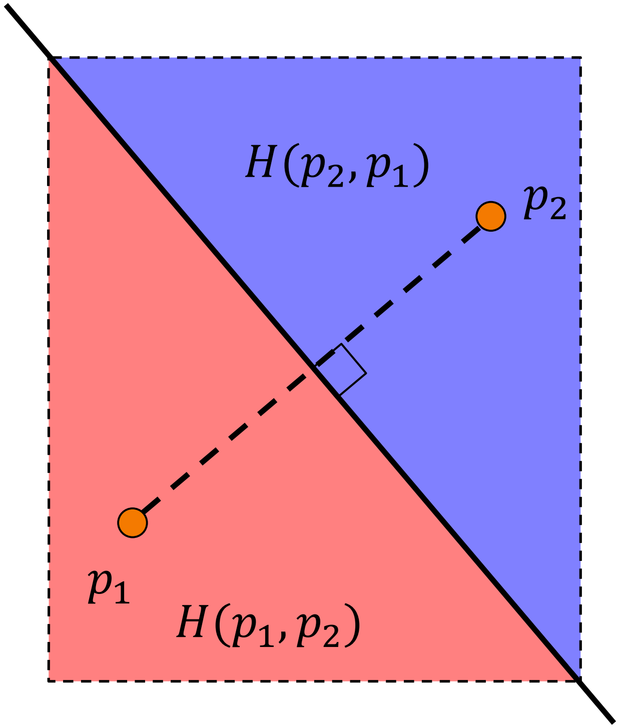
- The set with two nearest neighbors make up the *edges* of the diagram.
- The set with three or more nearest neighbors make up the *vertices* of the diagram.

The points P are called the *sites* of the Voronoi diagram.

Voronoi Diagrams

2 Points:

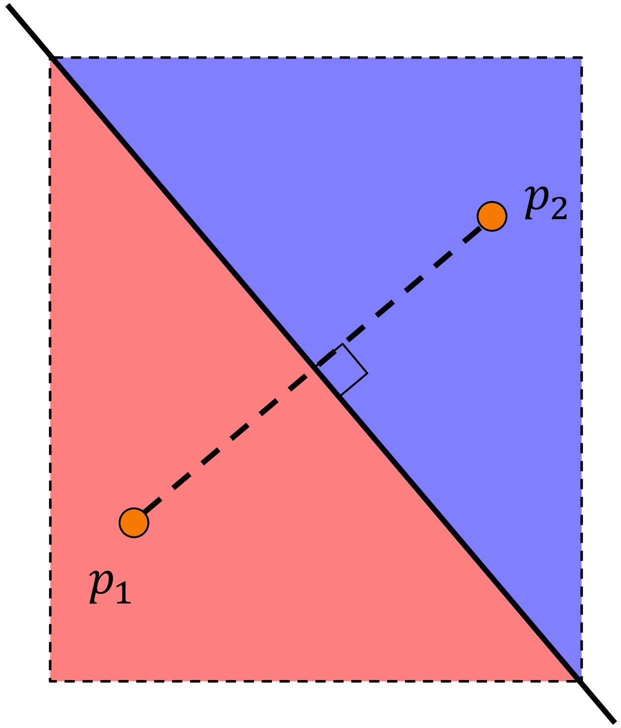
When $P = \{p_1, p_2\}$, the regions are defined by the perpendicular bisector:

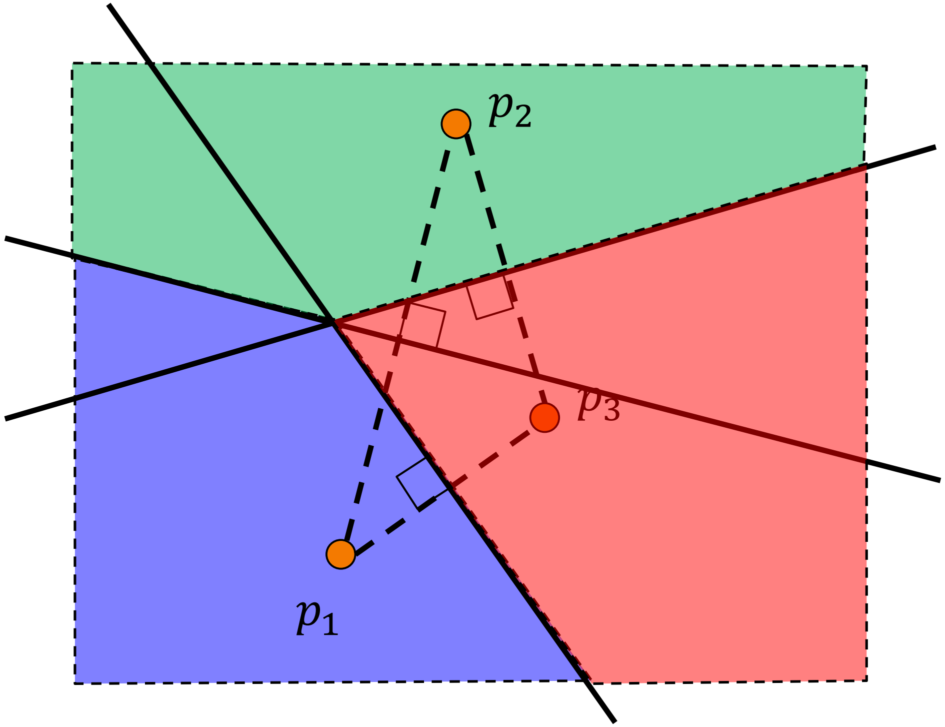


Voronoi Diagrams

3 Points:

When $P = \{p_1, p_2, p_3\}$, the regions are defined by the three perpendicular bisectors:





Voronoi Diagrams

3 Points:

The three bisectors intersect at a point

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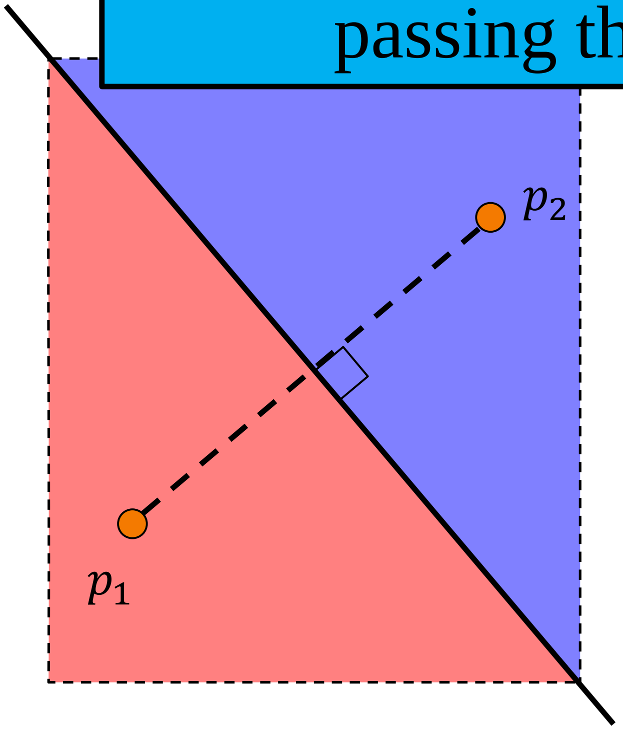
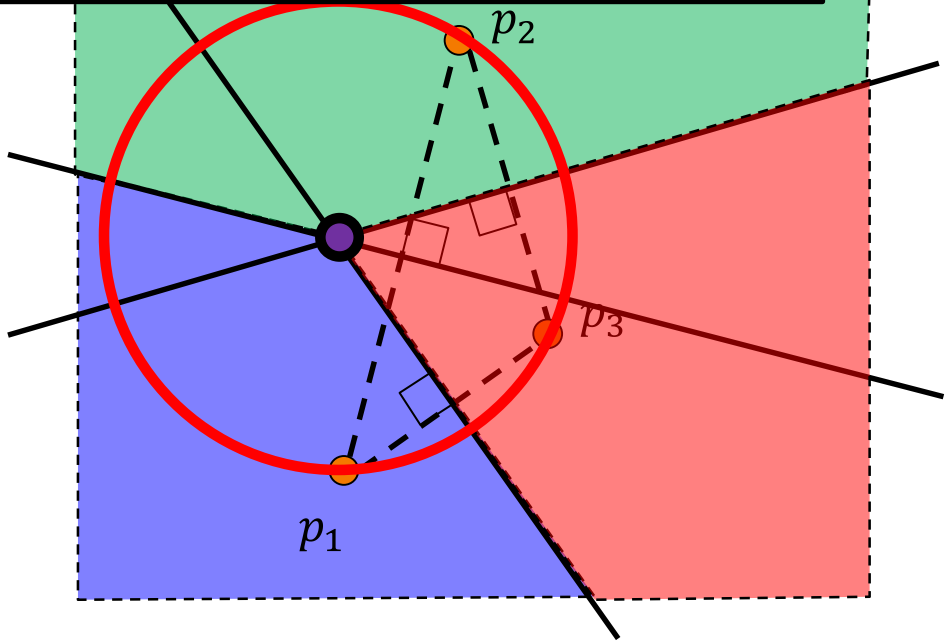
The intersection can be outside the triangle.

The point of intersection is center of the circle passing through the three points.



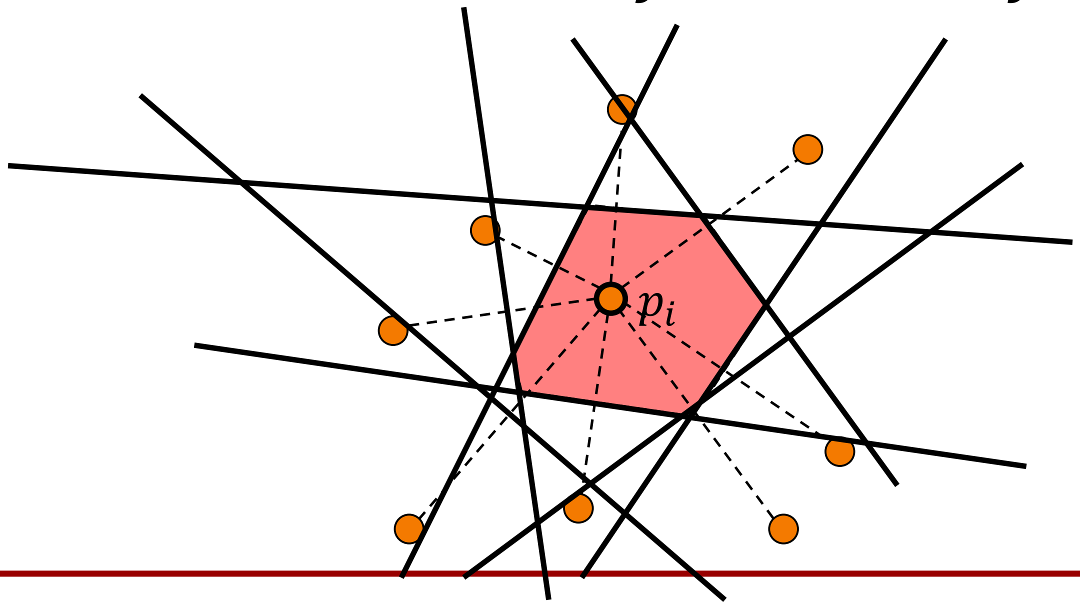


Voronoi Diagrams

More Generally:

The Voronoi region associated to point p_i is the intersection of the half-spaces defined by the perpendicular bisectors:

$$V(p_i) = \cap_{j \neq i} H(p_i, p_j)$$

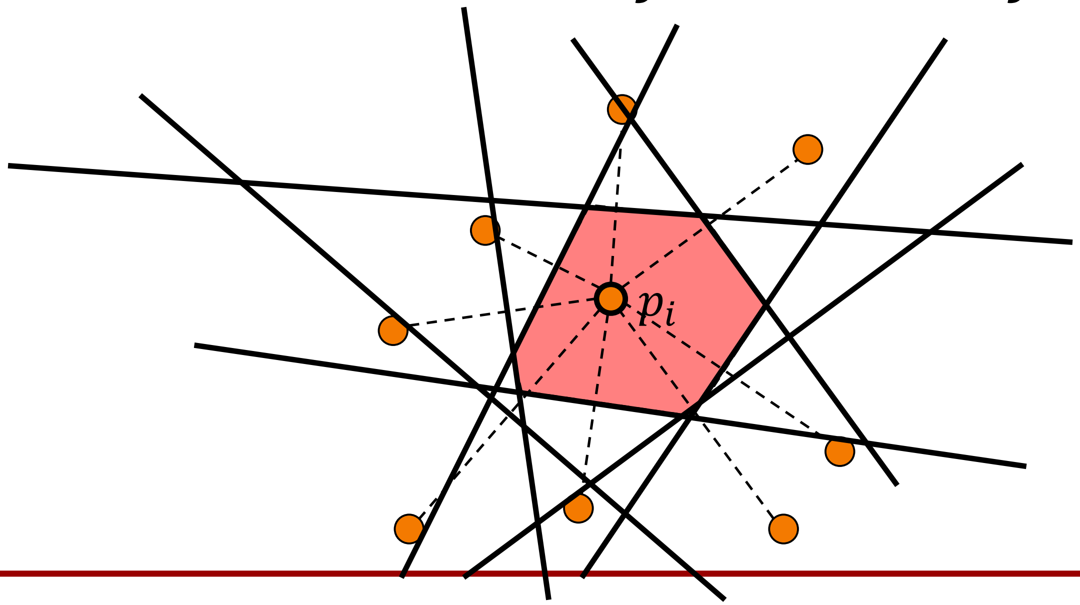


Voronoi Diagrams

More Generally:

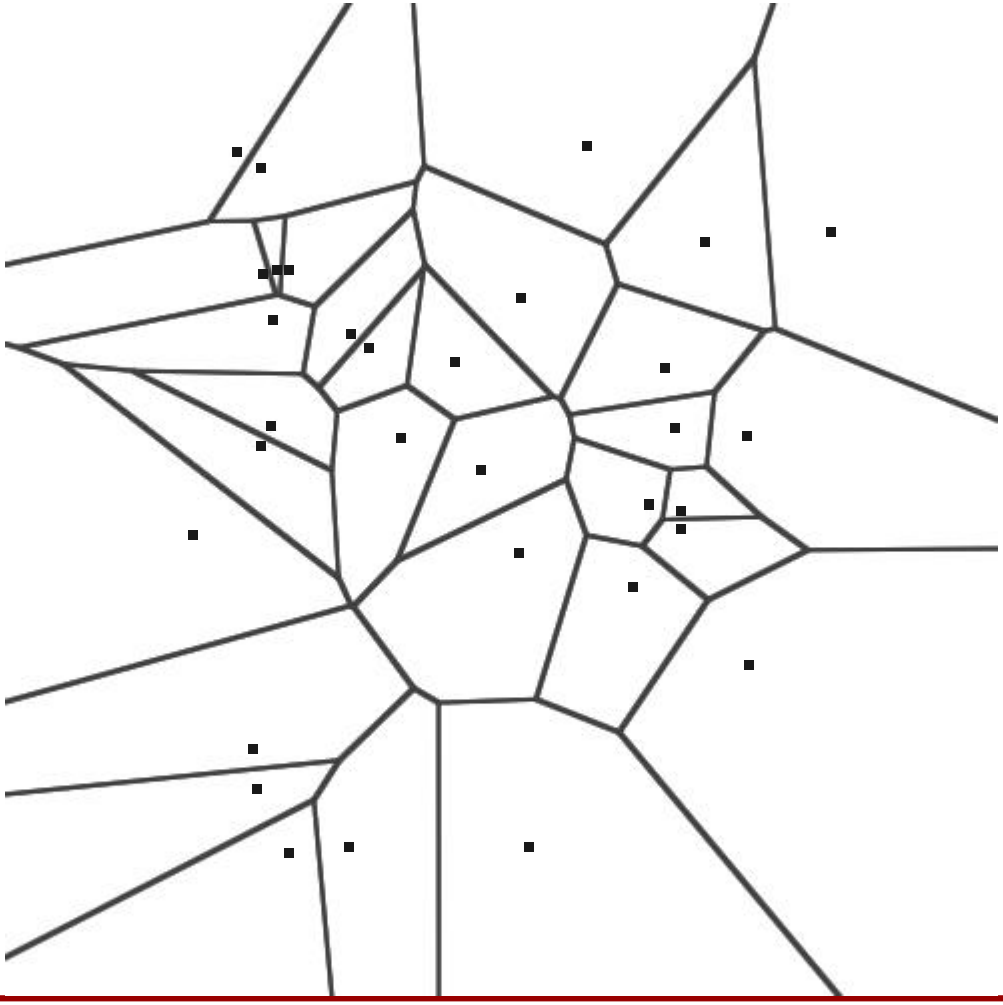
The Voronoi region associated to point p_i is the intersection of the half-spaces defined by the perpendicular bisectors:

⇒ Voronoi regions are convex polygons.



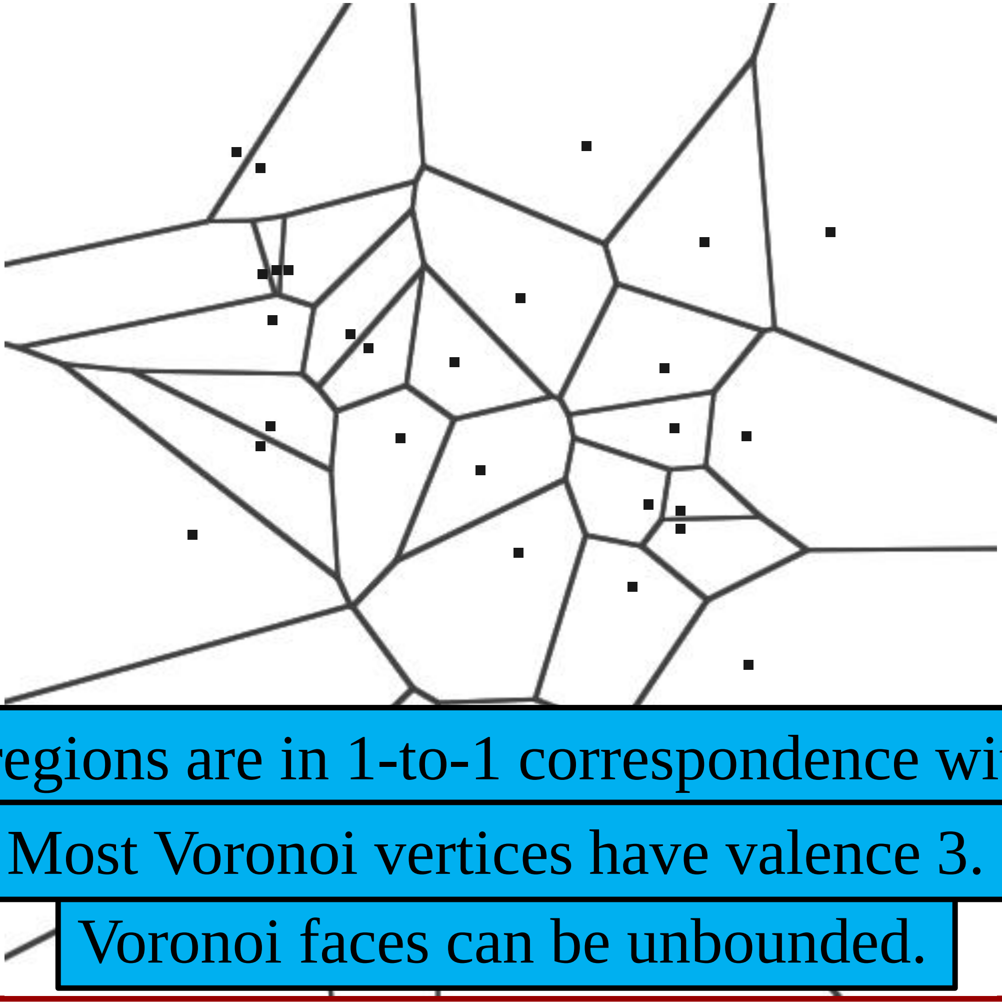
Voronoi Diagrams

More Generally:



Voronoi Diagrams

More Generally:



Voronoi regions are in 1-to-1 correspondence with points.

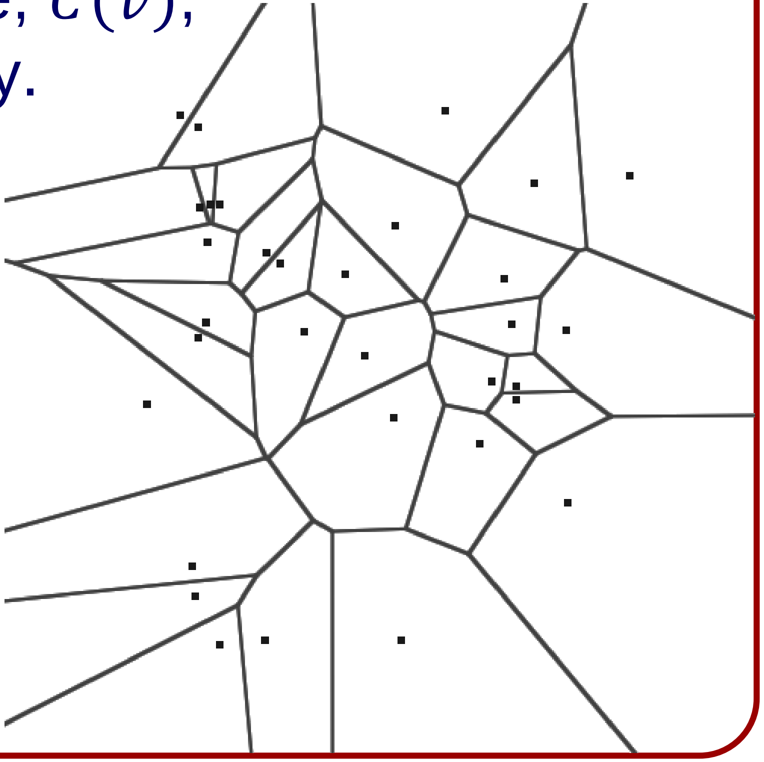
Most Voronoi vertices have valence 3.

Voronoi faces can be unbounded.

Voronoi Diagrams

Properties:

- Each Voronoi region is convex.
- $V(p_i)$ is unbounded $\Leftrightarrow$ p_i is on the convex hull of P.
- If v is a at the junction of $V(p_1),\ldots,V(p_k)$,with $k \geq 3$, then v is the center of a circle, $C(v)$, with $p_1,\ldots,p_k$ on the boundary.
- The interior of $C(v)$ contains no points.



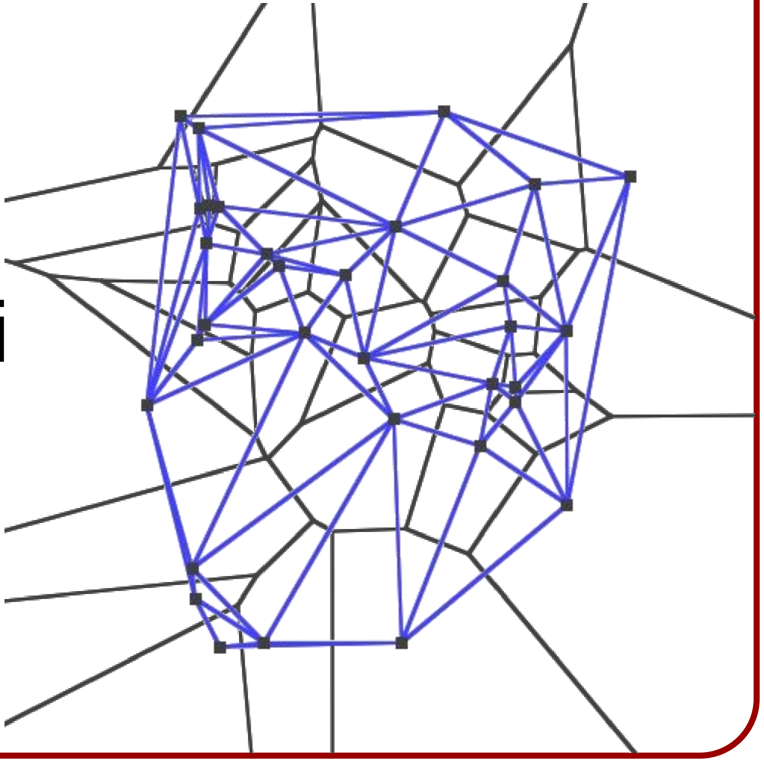
Delaunay Triangulation

Definition:

The *Delaunay triangulation* is the straight-line dual of the Voronoi Diagram.

Note:

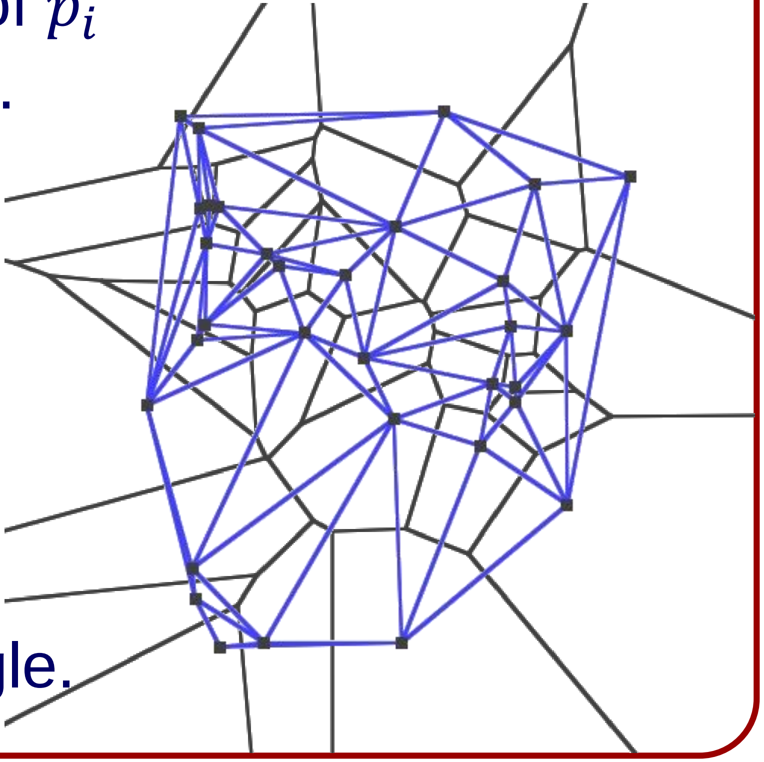
The Delaunay edges don't have to cross their Voronoi duals.



Delaunay Triangulation

Properties:

- The edges of $D(P)$ don't intersect.
- $D(P)$ is a triangulation if no 4 points are co-circular.
- The boundary of $D(P)$ is the convex hull of P.
- If p_j is the nearest neighbor of p_i then $\overline{p_i p_j}$ is a Delaunay edge.
- There is a circle through p_i and p_j that does not contain any other points $\Leftrightarrow$ $\overline{p_i p_j}$ is a Delaunay edge.
- The circumcircle of p_i, p_j, and p_k is empty $\Leftrightarrow$ $\Delta p_i p_j p_k$ is Delaunay triangle.



Delaunay Triangulation

Note:

Assuming that the edges of $D(P)$ do not cross, we get a planar graph.

⇒ The number of edges/faces in a Delaunay Triangulation is linear in the number of vertices.

⇒ The number of edges/vertices in a Voronoi Diagram is linear in the number of faces.

⇒ The number of vertices/edges/faces in a Voronoi Diagram is linear in the number of sites.

Delaunay Triangulation

Properties:

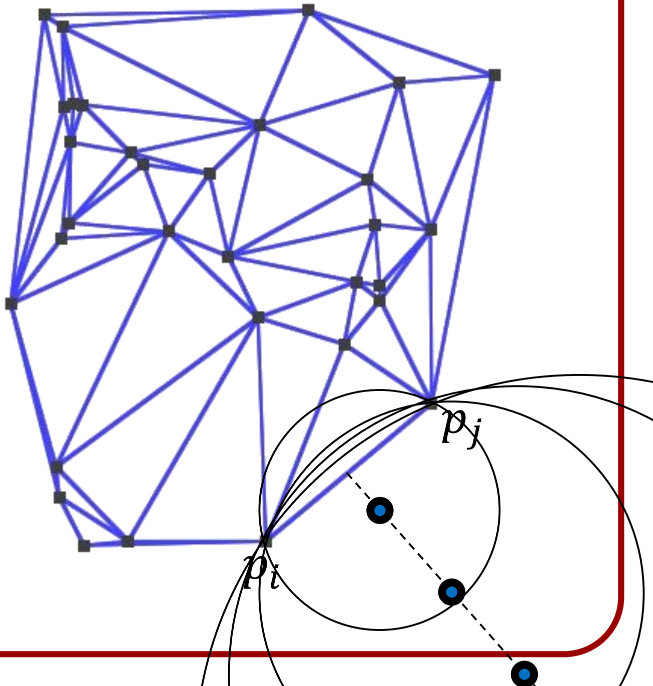
- The boundary of $D(P)$ is the convex hull of P.

Proof:

Suppose that $\overrightarrow{p_i p_j}$ is an edge of the hull of P.

Consider circles with center on the bisector that intersect p_i and p_j.

As we move out along the bisector the circle converges to the half-space to the right of $\overrightarrow{p_i p_j}$.



Delaunay Triangulation

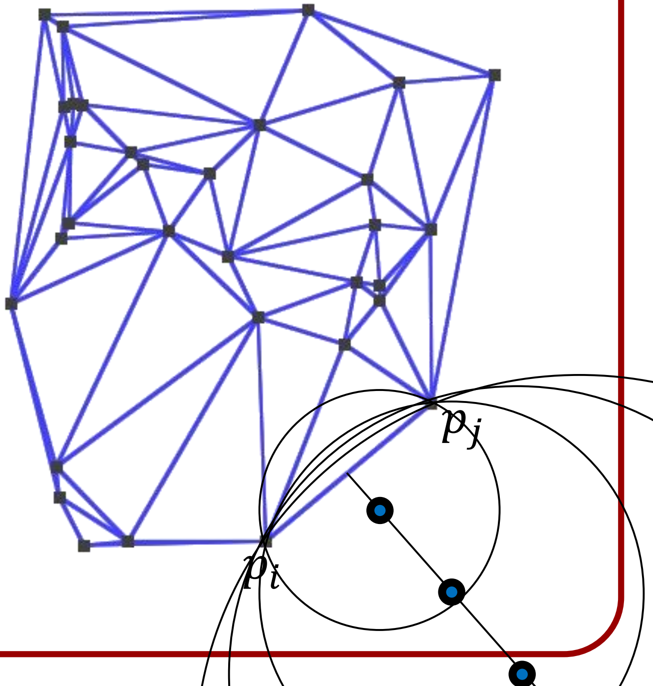
Properties:

- The boundary of $D(P)$ is the convex hull of P.

Proof:

Suppose that $\overrightarrow{p_i p_j}$ is an edge of the hull of P.

⇒ There is an (infinite) region on the bisector that is closer to p_i and p_j than to any other points.



⇒ There is a Voronoi edge between p_i and p_j.

⇒ The dual edge is in $D(P)$.

Delaunay Triangulation

Properties:

- If p_j is the nearest neighbor of p_i then $\overline{p_i p_j}$ is a Delaunay edge.

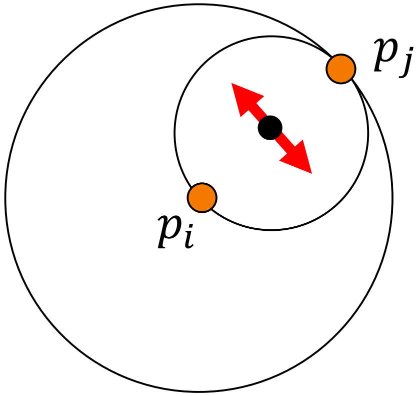
Proof:

p_j is the nearest neighbor of p_i iff. the circle around p_i with radius $|p_i - p_j|$ is empty of other points.

$\Rightarrow$ The circle through $(p_i + p_j)/2$ with radius $|p_i - p_j|/2$ is empty of other points.

$\Rightarrow$ $(p_i + p_j)/2$ is on the Voronoi diagram.

$\Rightarrow$ $(p_i + p_j)/2$ is on a Voronoi edge.



Delaunay Triangulation

Properties:

- If p_j is the nearest neighbor of p_i then $\overline{p_i p_j}$ is a Delaunay edge.

Implications:

The nearest neighbor graph is a subset of the Delaunay triangulation.

We will show that the Delaunay triangulation can be computed in $O(n\log n)$ time.

⇒ We can compute the nearest-neighbor graph in $O(n\log n)$.

Delaunay Triangulation

Properties:

- There is a circle through p_i and p_j that does not contain any other points ⇔ $\overline{p_i p_j}$ is a Delaunay edge.

Proof (⇐):

If $\overline{p_i p_j}$ is a Delaunay edge, then the Voronoi regions $V(p_i)$ and $V(p_j)$ intersect at an edge.

Set v to be some point on the interior of the edge.

$|v - p_i| = \left|v - p_j\right| = r$ and $|v - p_k| > r \ \forall k \neq i, j$.

The circle at v with radius r is empty of other points.

Delaunay Triangulation

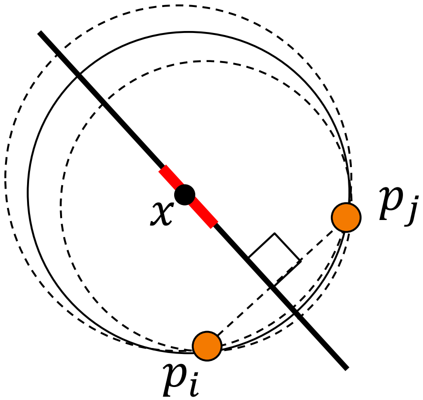
Properties:

- There is a circle through p_i and p_j that does not contain any other points ⇔ $\overline{p_i p_j}$ is a Delaunay edge.

Proof (⇒):

If there is a circle through p_i and p_j, empty of other points, with center x, then $x \in V(p_i) \cap V(p_j)$.

Since no other point is in or on the circle there is a neighborhood of centers around x on the bisector with circles through p_i and p_j empty of other points.



x is on a Voronoi edge.

Delaunay Triangulation

Properties:

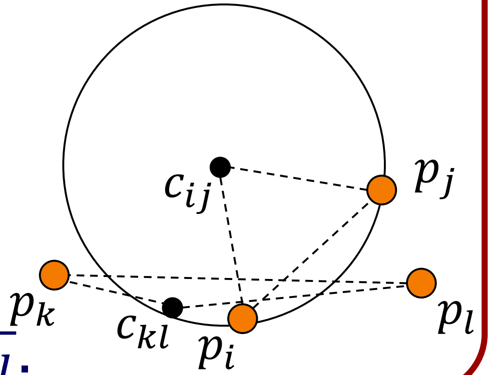
- The edges of $D(P)$ don’t intersect.

Proof:

Given an edge $\overline{p_i p_j}$ in $D(P)$, there is a circle with p_i and p_j on its boundary and empty of other points.

Let be $\overline{p_k p_l}$ be an edge in $D(p)$ that intersect $\overline{p_i p_j}$:

p_k and p_l cannot be in the circle.
$\Rightarrow$ p_k and p_l are not in the triangle $\Delta c_{ij} p_i p_j$
$\Rightarrow$ $\overline{p_k p_l}$ intersects either $\overline{c_{ij} p_i}$ or $\overline{c_{ij} p_j}$.
$\Rightarrow$ $\overline{p_i p_j}$ intersects either $\overline{c_{kl} p_k}$ or $\overline{c_{kl} p_l}$.
$\Rightarrow$ One of $\overline{c_{ij} p_i}$ or $\overline{c_{ij} p_j}$ one of $\overline{c_{kl} p_k}$ or $\overline{c_{kl} p_l}$.



Delaunay Triangulation

Properties:

- The edges of $D(P)$ don't intersect.

Proof:

Given an edge $\overline{p_i p_j}$ in $D(P)$, there is a circle with p_i and p_j on its boundary and empty of other points.

But $\overline{c_{ij} p_i}$ is in the Voronoi region of p_i and $\overline{c_{kl} p_k}$ is in the Voronoi region of p_k, so they cannot intersect.

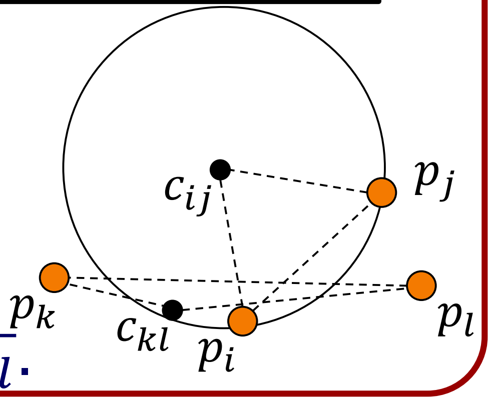
p_k and p_l cannot be in the circle.

$\Rightarrow$ p_k and p_l are not in the triangle $\Delta c_{ij} p_i p_j$

$\Rightarrow$ $\overline{p_k p_l}$ intersects either $\overline{c_{ij} p_i}$ or $\overline{c_{ij} p_j}$.

$\Rightarrow$ $\overline{p_i p_j}$ intersects either $\overline{c_{kl} p_k}$ or $\overline{c_{kl} p_l}$.

$\Rightarrow$ One of $\overline{c_{ij} p_i}$ or $\overline{c_{ij} p_j}$ one of $\overline{c_{kl} p_k}$ or $\overline{c_{kl} p_l}$.



Outline

- Preliminaries
- Voronoi Diagrams / Delaunay Triangulations
  - Naive Algorithm
  - Fortune’s Algorithm
- Lloyd’s Algorithm

Naive Algorithm

```
Delaunay( $\{p_1, \dots, p_n\}$ )
    ◦ for $i \in [1, n]$
        » for $j \in [1, i)$
            – for $k \in [1, j)$
                • $(c, r) \leftarrow$ Circumcircle( $p_i$ , $p_j$ , $p_k$ )
                • isTriangle ← true
                • for $l \in [1, k)$
                    • if( $\|p_l - c\| < r$ ) isTriangle ← false
                • if( isTriangle ) Output( $p_i$ , $p_j$ , $p_k$ )
```

Complexity: $O(n^4)$

Voronoi Diagrams and Cones

Key Idea:

We can think of generating Voronoi regions by expanding circles centered at points of P.

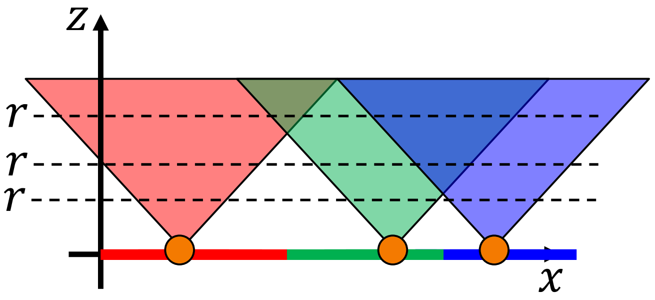
When multiple circles overlap a point, track the one that is closer.

Voronoi Diagrams and Cones

Key Idea:

We can visualize the Voronoi regions by drawing right cones over the points, with axes along the positive z-axis.

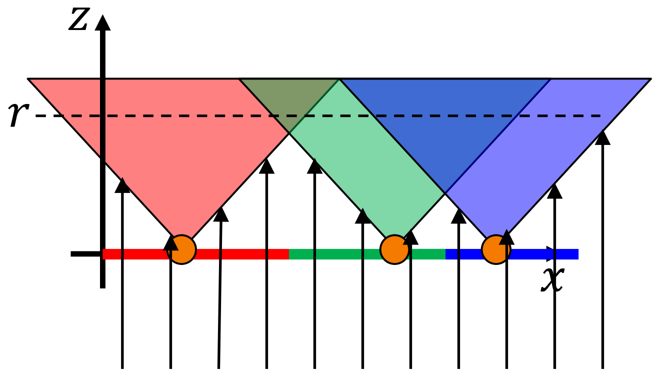
Circles with radius r are the projections of the intersections of the plane $z = r$ plane with the cones, onto the xy-plane.



Voronoi Diagrams and Cones

Key Idea:

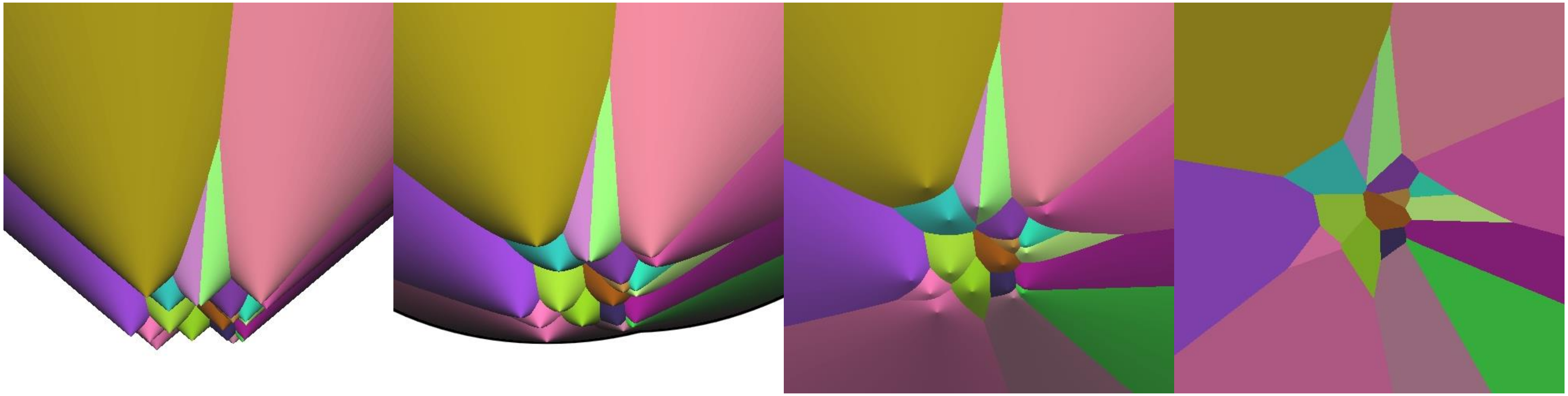
To track the closer circle, we can render the cones with an orthographic camera looking up the z-axis.



Voronoi Diagrams and Cones

Key Idea:

To track the closer circle, we can render the cones with an orthographic camera looking up the z-axis.

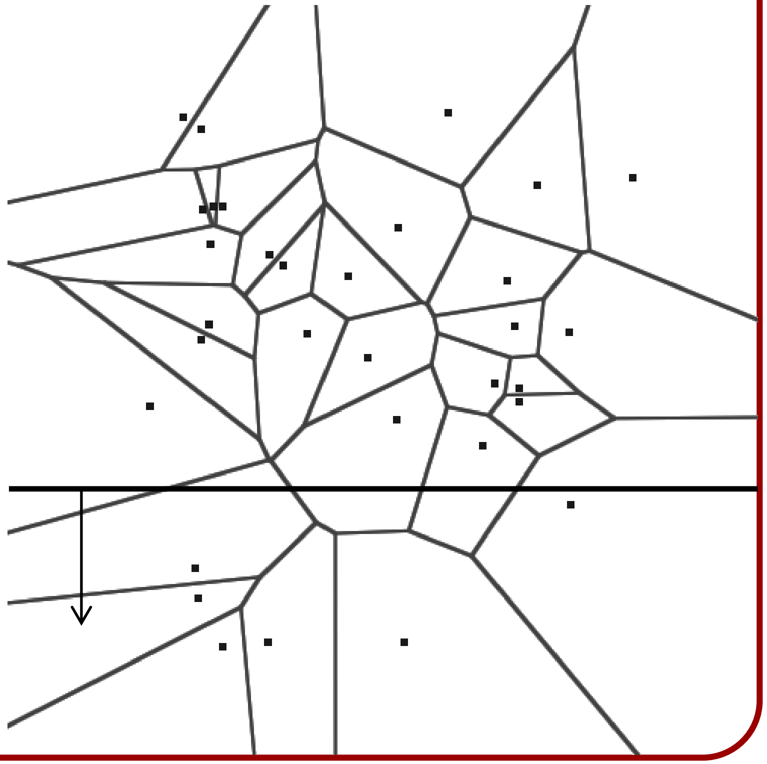


Visualization

Fortune’s Algorithm

<u>Approach</u>:

Sweep a line and maintain the solution for all points behind the line.



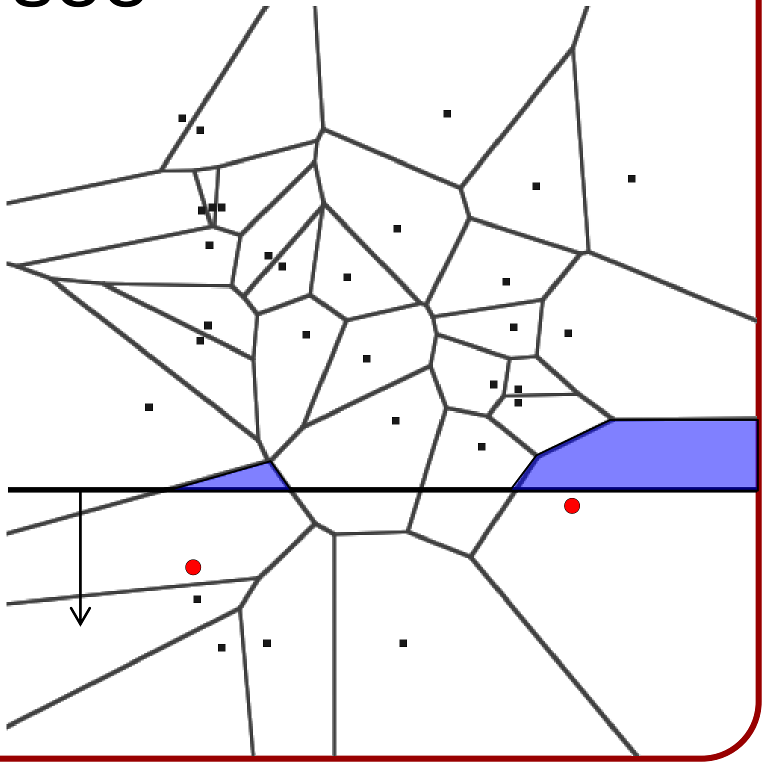
Fortune's Algorithm

Why This Shouldn't Work:

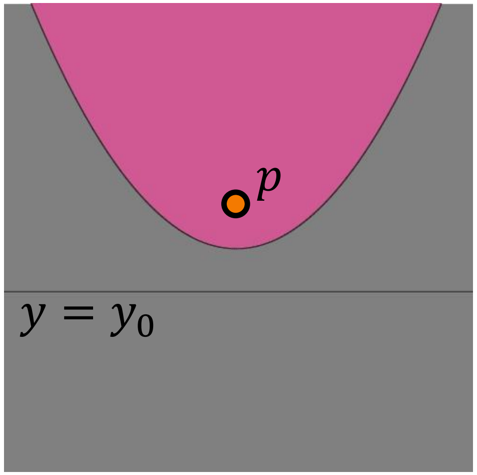
The Voronoi region behind the line can depend on points that are in front of the line! (Looking up the z-axis, we see the cone before the apex.)

Key Idea:

We can finalize points behind the line that are closer to a site than to the line.



Fortune’s Algorithm



Given a site $p \in P$ and the line with height y_0, we can finalize the points satisfying:

$$\{(x, y) | (y - y_0)^2 > \|p - (x, y)\|^2\}$$

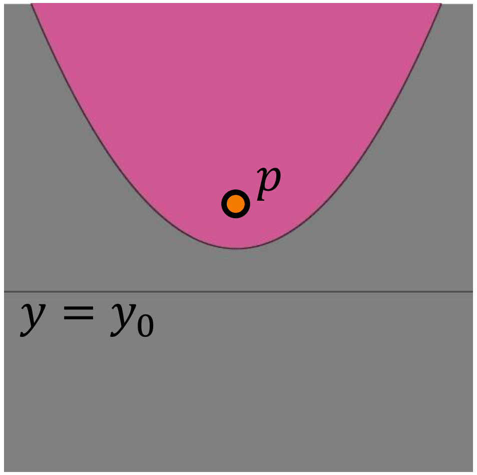
Points on the boundary satisfy:

$$(y - y_0)^2 = \|p - (x, y)\|^2$$

Setting $z = \|p - (x, y)\|$, this gives:

$$z = y - y_0$$

Fortune’s Algorithm



Formally:

⇒ We can describe the points on the boundary as the xy-coordinates of the points in 3D with:

1. $z = \|p - (x,y)\|$ ⟵

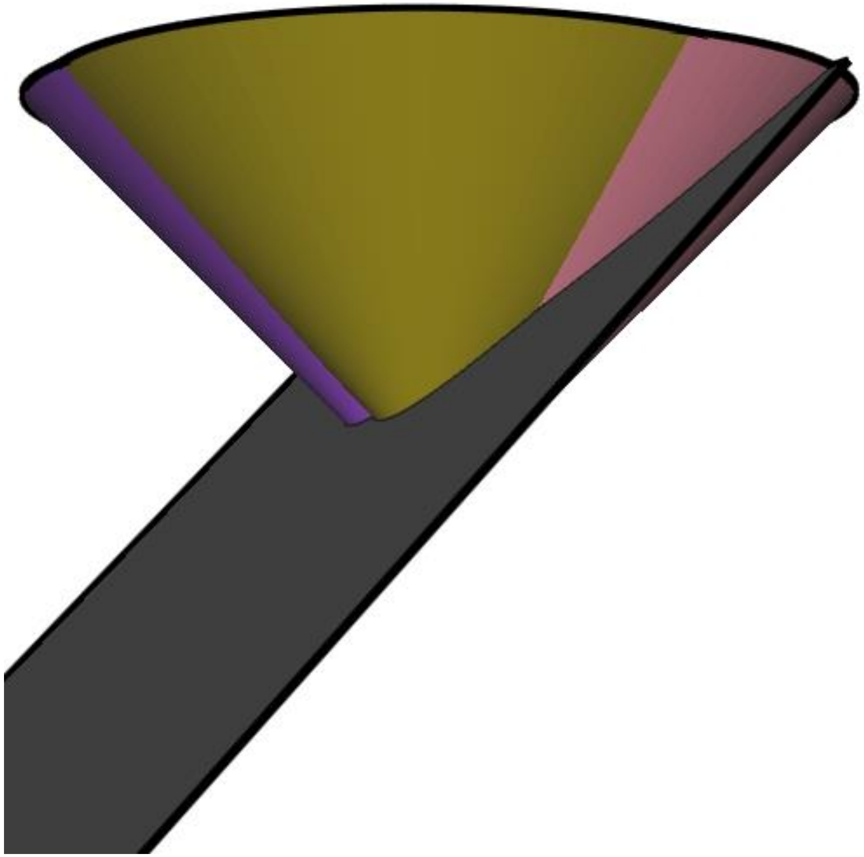
Points on the right cone,
centered at p,
centered around the positive z-axis

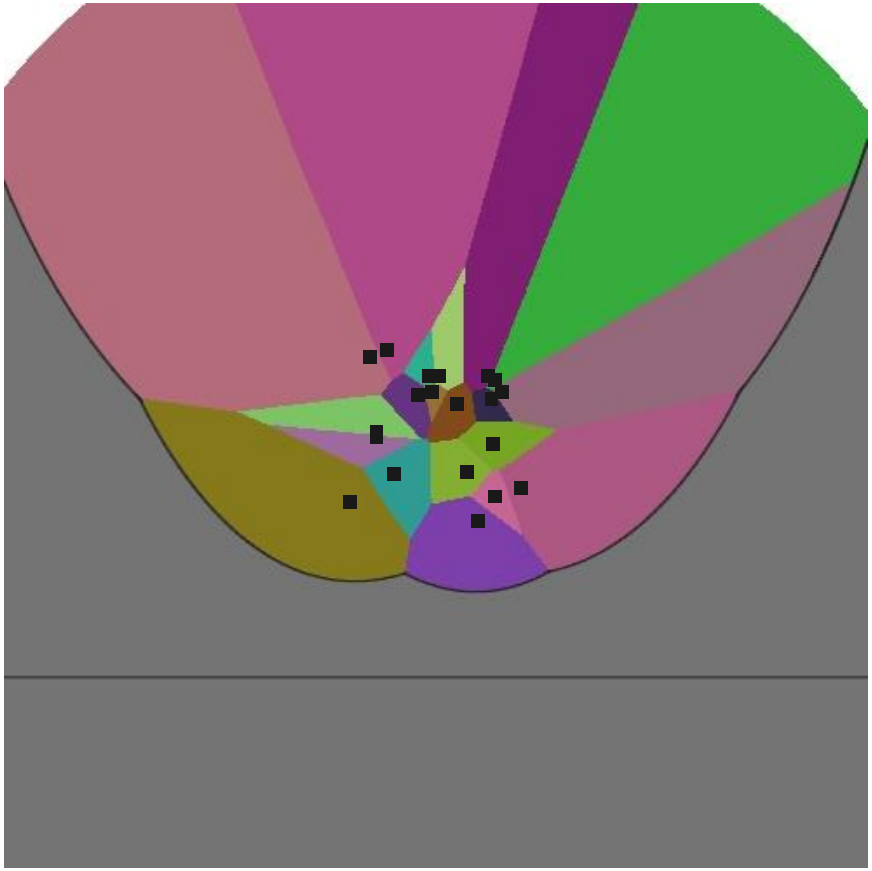
2. $z = y - y_0$

Points on the plane,
making a 45° angle with the xy-plane,
passing through the line $y = y_0$ and $z = 0$

Sweep the cones with a plane parallel to the x-axis making a $45°$ angle with the xy-plane.

Fortune’s Algorithm





Fortune's Algorithm

Sweep with a plane π_y, parallel to the x-axis, making a $45°$ angle with the xy-plane.

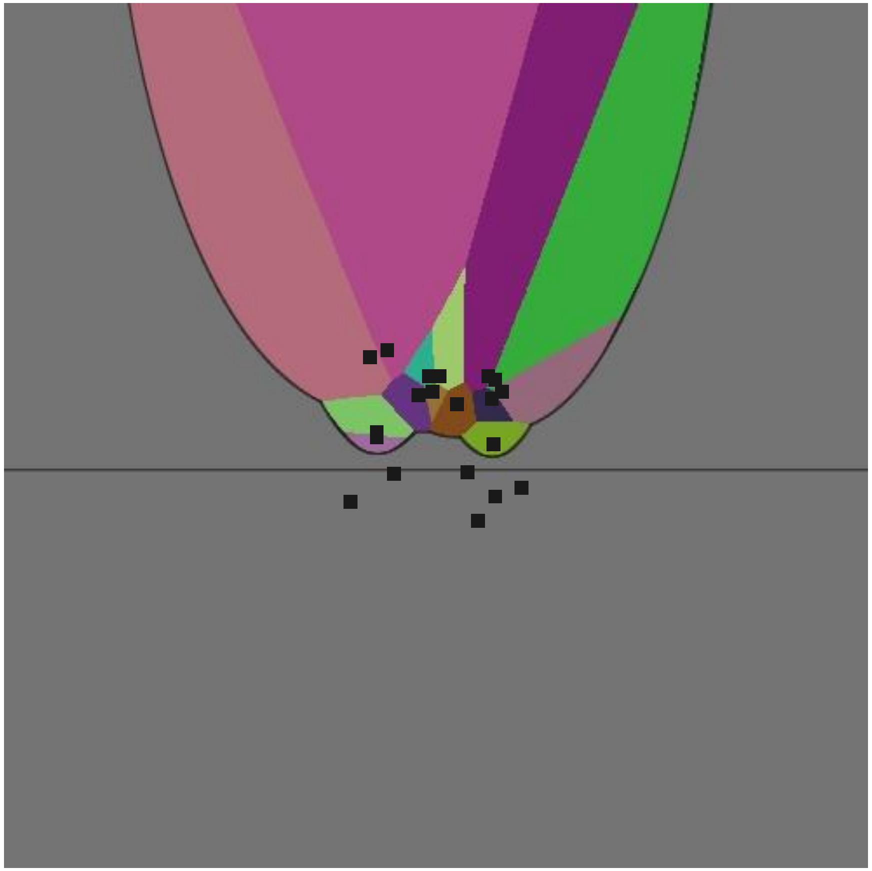
"Render" the cones and the plane with an orthographic camera looking up the z-axis.

At each point, we see:

- The part of π_y that is in front of the line (since it is below the xy-plane and hence below the cones).
- The part of the cones that are behind the line and below π_y.

Fortune’s Algorithm

As y advances, the algorithm maintains a set of parabolic fronts (the projection of the intersections of π_y with the cones).

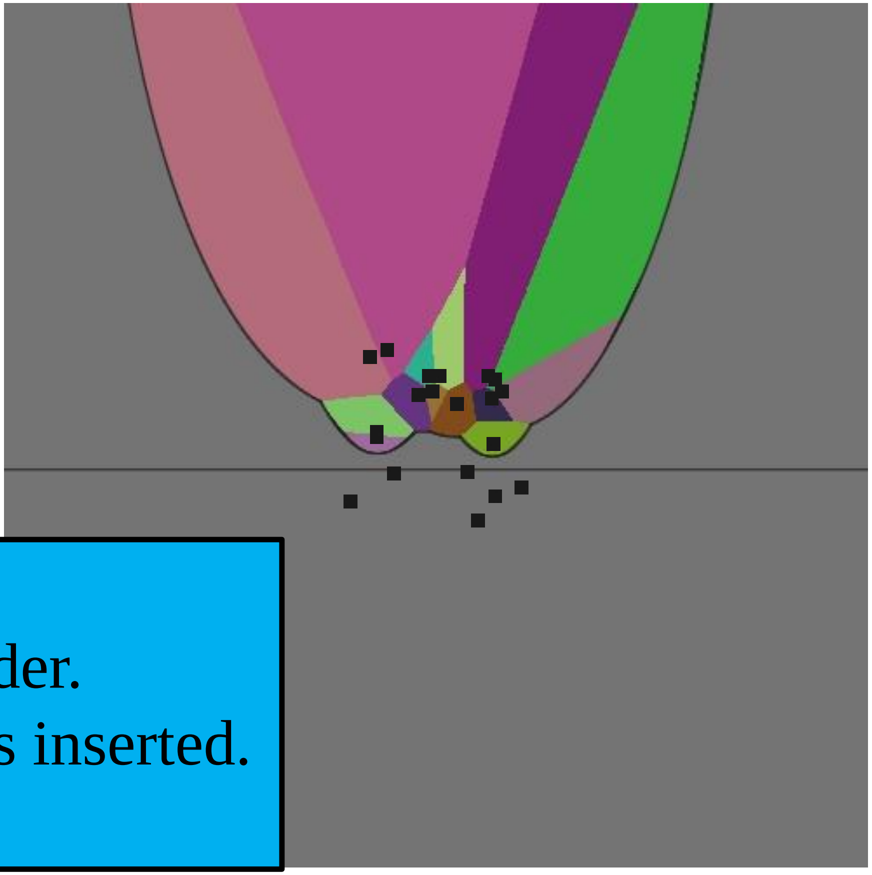


At any point, the Voronoi diagram is finalized behind the parabolic fronts.

Fortune’s Algorithm

As y advances, the algorithm maintains a set of parabolic fronts (the projection of the intersections of π_y with the cones).

At any point, the Voronoi diagram is finalized behind the



Implementation:

- The fronts are maintained in order.
- As y intersects a site, its front is inserted.
- Complexity $O(n \log n)$.

Outline

- Preliminaries
- Voronoi Diagrams / Delaunay Triangulations
- Lloyd’s Algorithm

Lloyd’s Algorithm

Challenge:

Solve for the position of points $P = \{p_1, \dots, p_n\}$ inside the unit square minimizing:

$$E(P) = \int_{[0,1]^2} d^2(q, P)\, dq$$

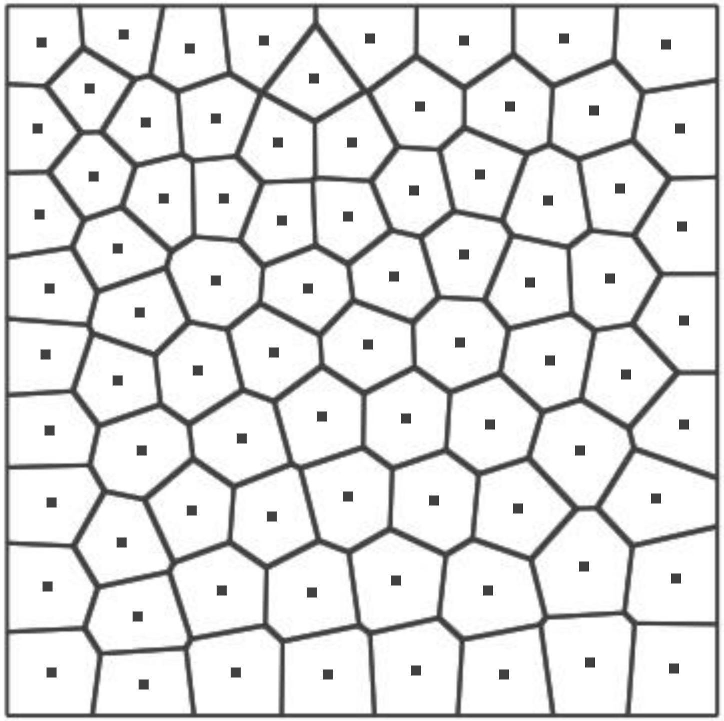
where $d(q, P) = \min_i |p_i - q|$.

Lloyd’s Algorithm

Approach:

1. Initialize the points to random positions.
2. Compute the Voronoi Diagram of the points, clipped to the unit square.
3. Set the positions of the points to the centers of mass of the corresponding Voronoi cells.
4. Go to step 2.

Lloyd’s Algorithm



Lloyd’s Algorithm

2. Compute the Voronoi Diagram of the points, clipped to the unit square.

Since:

$$\int_{[0,1]^2} d^2(q,P)\,dq = \sum_{F_i \in V(P)} \int_{F_i} \|p_i - q\|^2 dq$$

this provides the assignment of points in $[0,1]^2$ to points in P that minimize the energy.

Lloyd’s Algorithm

3. Set the positions of the points to the centers of mass of the corresponding Voronoi cells.

Since:

$$\underset{p\in[0,1]^2}{\arg\min} \int_F \|p-q\|^2 dq = C(F)$$

with $C(F)$ the center of mass of face F, repositioning to the center reduces the energy.