Convex Hulls (3D)

O'Rourke, Chapter 4

Announcements

- Assignment 2 has been posted

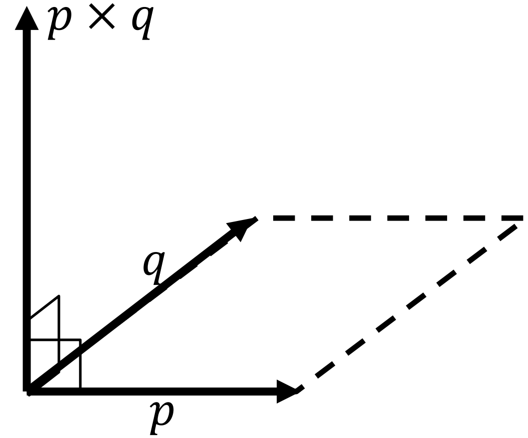
Outline

- Review
- Gift-Wrapping
- Divide-and-Conquer

Recall

Given points $p, q \in \mathbb{R}^3$, the *cross-product* $p \times q \in \mathbb{R}^3$ is the vector:

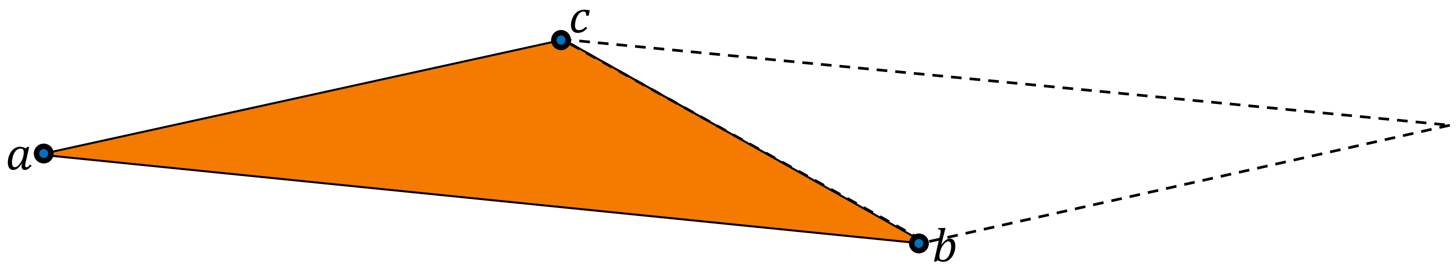
- perpendicular to both p and q,
- oriented according to the right-hand-rule,
- with length equal to the area of the parallelogram defined by p and q.



Recall

Given a triangle T with vertices $(a, b, c) \in \mathbb{R}^3$, the area of the triangle is:

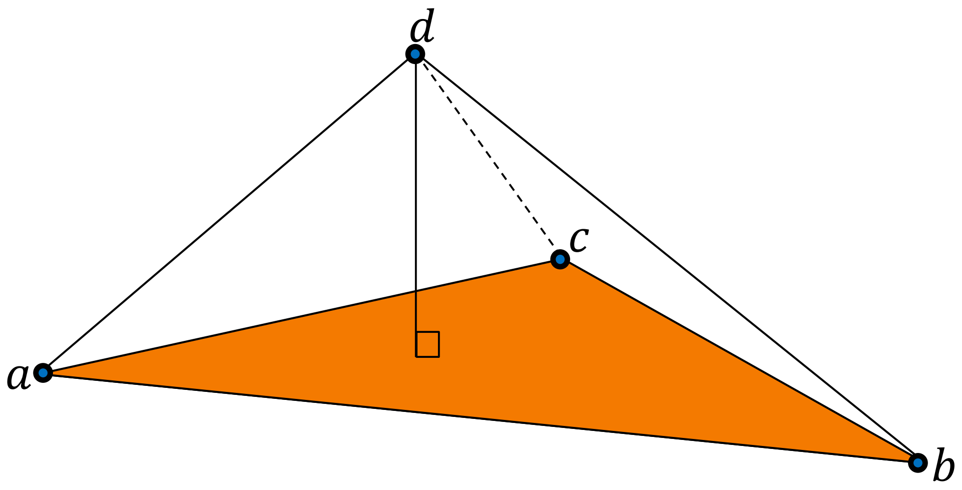
$$\text{Area}(T) = \frac{1}{2} \times \|(b - a) \times (c - a)\|$$



Recall

Given a tetrahedron T with vertices $(a, b, c, d) \in \mathbb{R}^3$, the volume of the tetrahedron is:

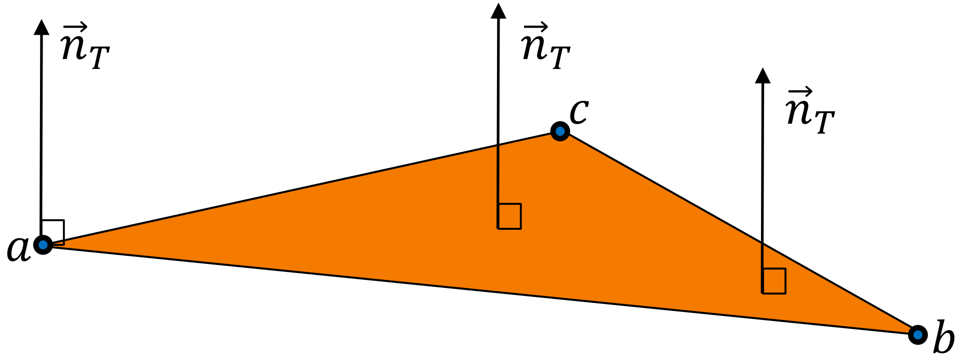
$$\text{Volume}(T) = \frac{1}{3} \times \text{base} \times \text{height}$$



Recall

Given a triangle T with vertices $(a, b, c) \in \mathbb{R}^3$, the triangle normal is:

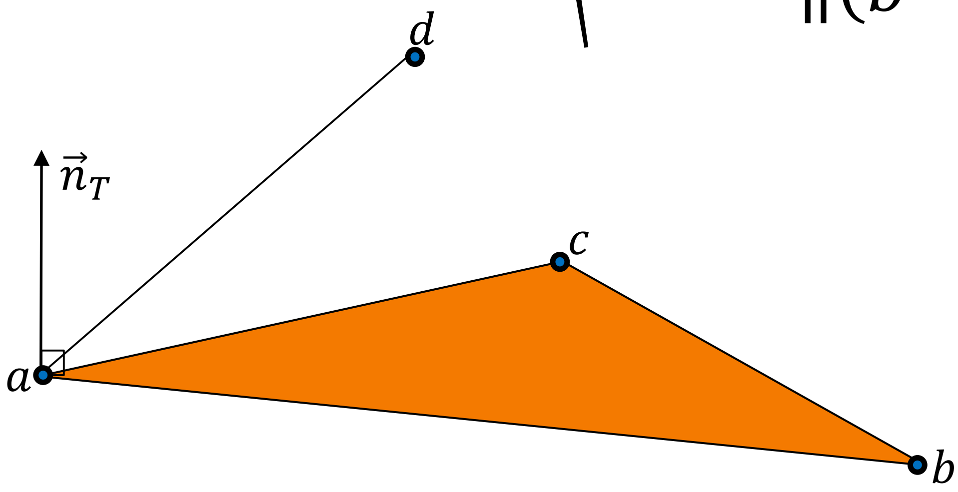
$$\vec{n}_T = \frac{(b-a)\times(c-a)}{\|(b-a)\times(c-a)\|}$$



Recall

Given a triangle T with vertices $(a, b, c) \in \mathbb{R}^3$ and given a point $d \in \mathbb{R}^3$, the signed perpendicular height of d from the plane containing (a, b, c) is:

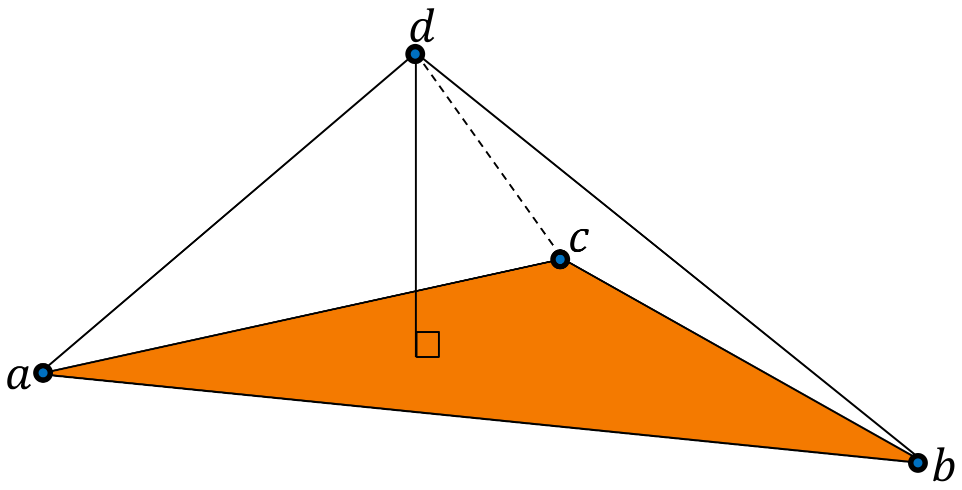
$$\text{Height}(T, d) = \langle d - a, \vec{n}_T \rangle$$
$$= \left\langle d - a, \frac{(b-a) \times (c-a)}{\|(b-a) \times (c-a)\|} \right\rangle$$



Recall

Given a tetrahedron T with vertices $(a, b, c, d) \in \mathbb{R}^3$, the signed volume of the tetrahedron is:

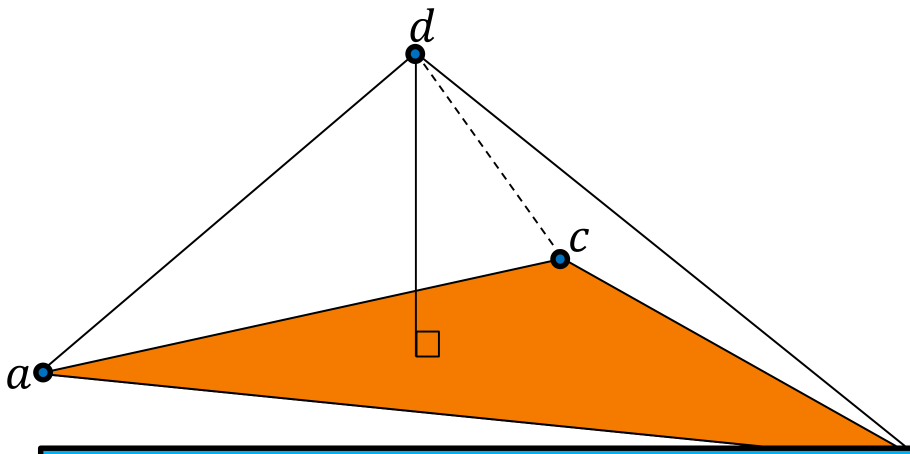
$$\text{Volume}(T) = \frac{1}{3} \times \text{base} \times \text{height}$$
$$= \frac{1}{6} \times \langle d - a, (b - a) \times (c - a) \rangle$$



Recall

Given a tetrahedron T with vertices $(a, b, c, d) \in \mathbb{R}^3$, the signed volume of the tetrahedron is:

$$\text{Volume}(T) = \frac{1}{3} \times \text{base} \times \text{height}$$
$$= \frac{1}{6} \times \langle d - a, (b - a) \times (c - a) \rangle$$



The volume is positive if d is to the left of the plane defined by the triangle (a, b, c).

Recall

If we have a graph G with bounded degree, we can identify the connected component containing a node v by performing a flood-fill.

FloodFill(v , G)

- if(NotMarked(v))
  - Mark(v)
  - for $w \in$ Neighbors(v)
    - FloodFill(w , G)

Complexity: $O(|G|)$

Recall

If we have a graph G with bounded degree, we can identify the connected component containing a node v by performing a flood-fill.

In particular given a winged-edge representation of a triangle mesh and given a face in the mesh, we can compute the connected component of the face in linear time.

Outline

- Review
- Gift-Wrapping
- Divide-and-Conquer

Gift-Wrapping

Initialization:

Find a triangle on the hull.

Iteratively:

Until the hull closes, pivot around a boundary edge.

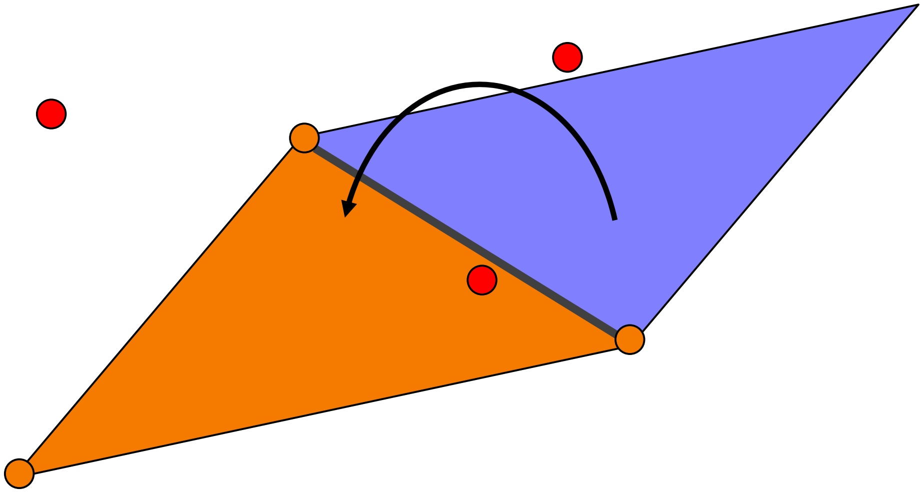
Gift-Wrapping

Initialization:

Find a triangle on the hull.

Iteratively:

Until the hull closes, pivot around a boundary edge.



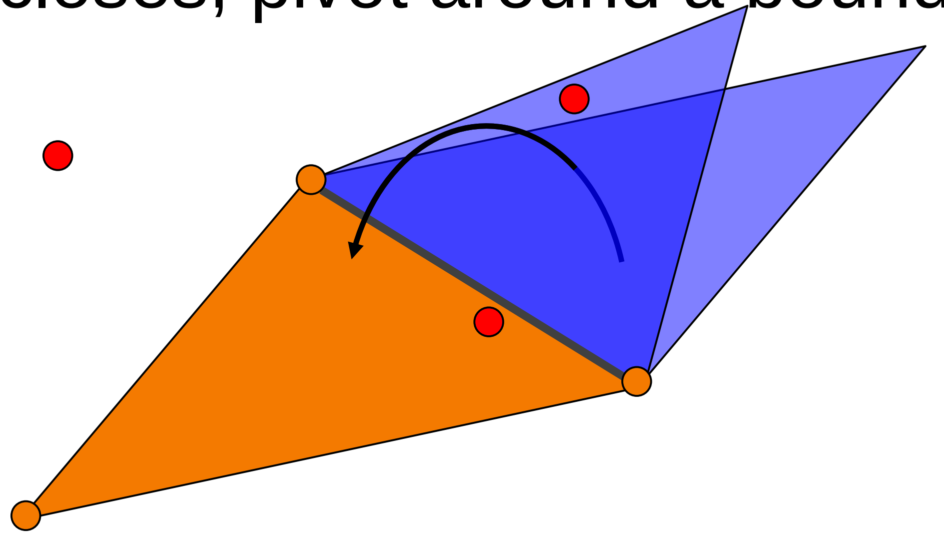
Gift-Wrapping

Initialization:

Find a triangle on the hull.

Iteratively:

Until the hull closes, pivot around a boundary edge.



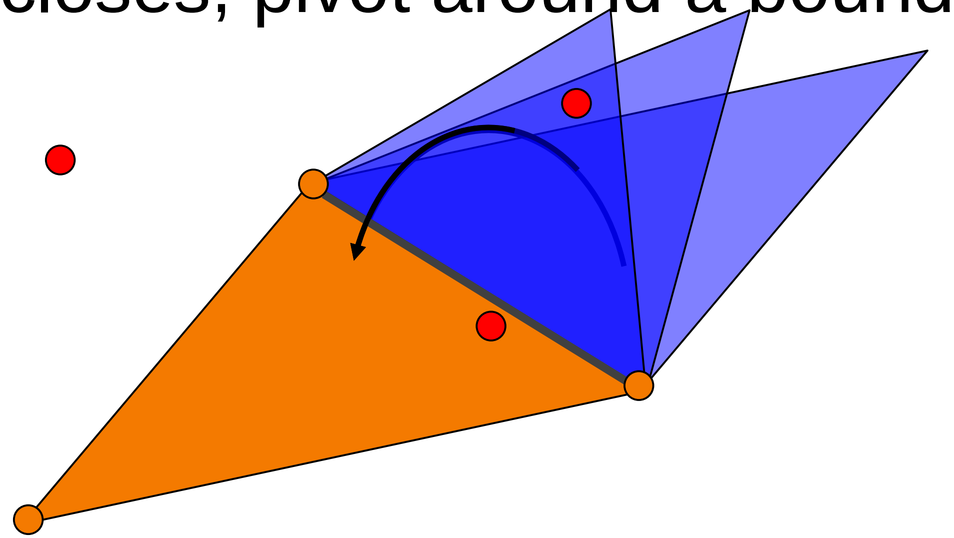
Gift-Wrapping

Initialization:

Find a triangle on the hull.

Iteratively:

Until the hull closes, pivot around a boundary edge.



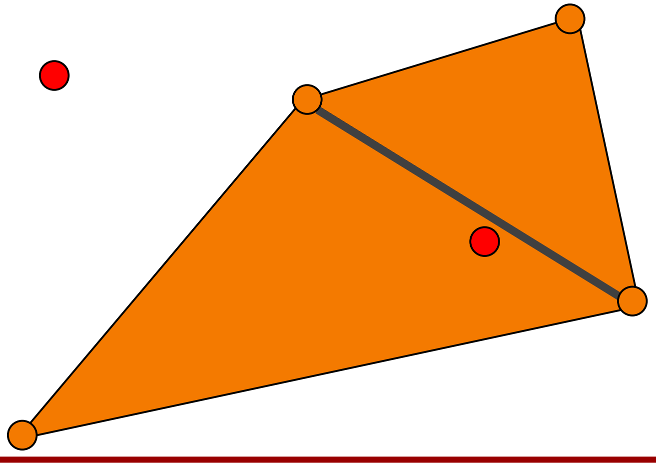
Gift-Wrapping

Initialization:

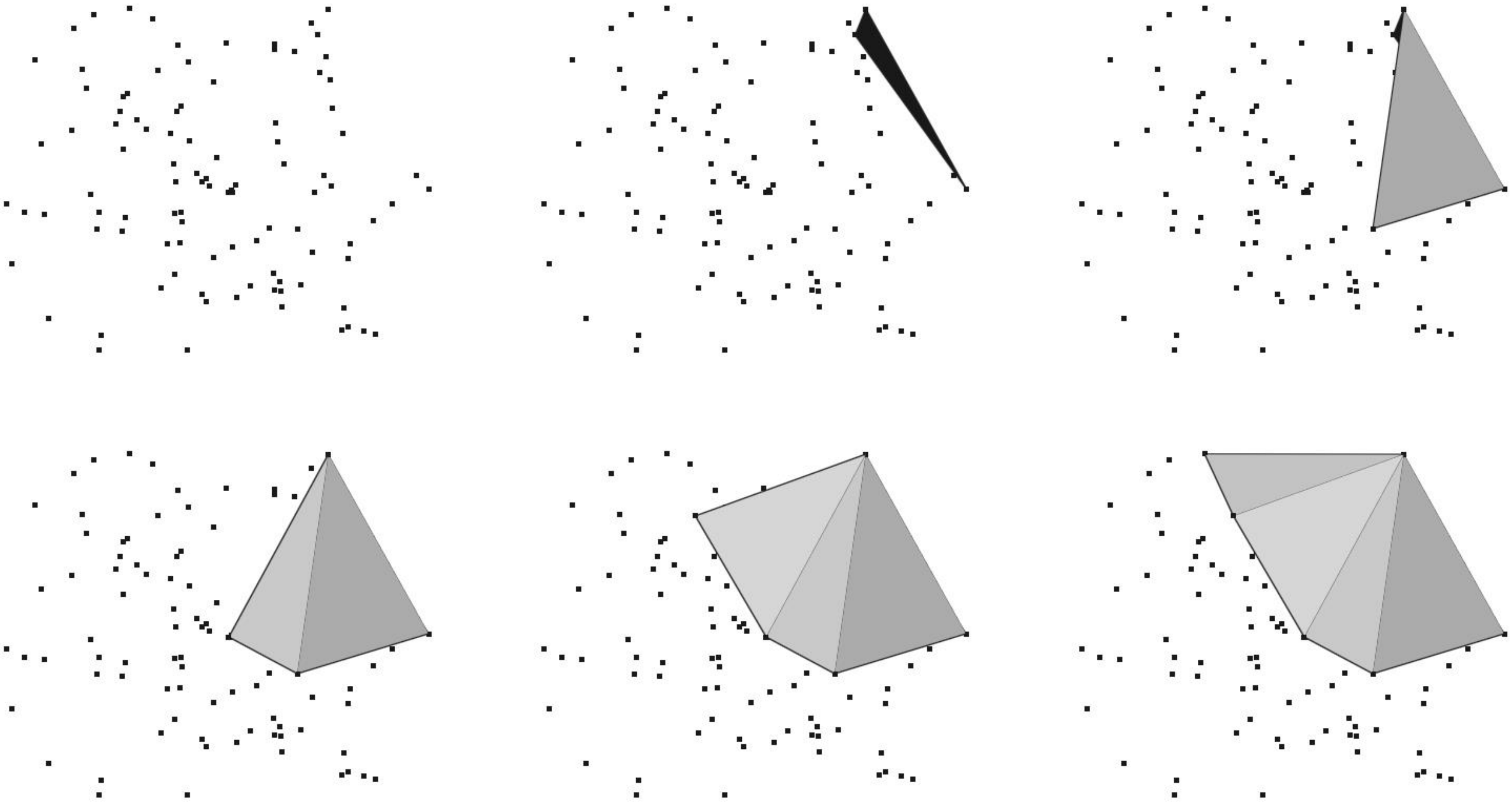
Find a triangle on the hull.

Iteratively:

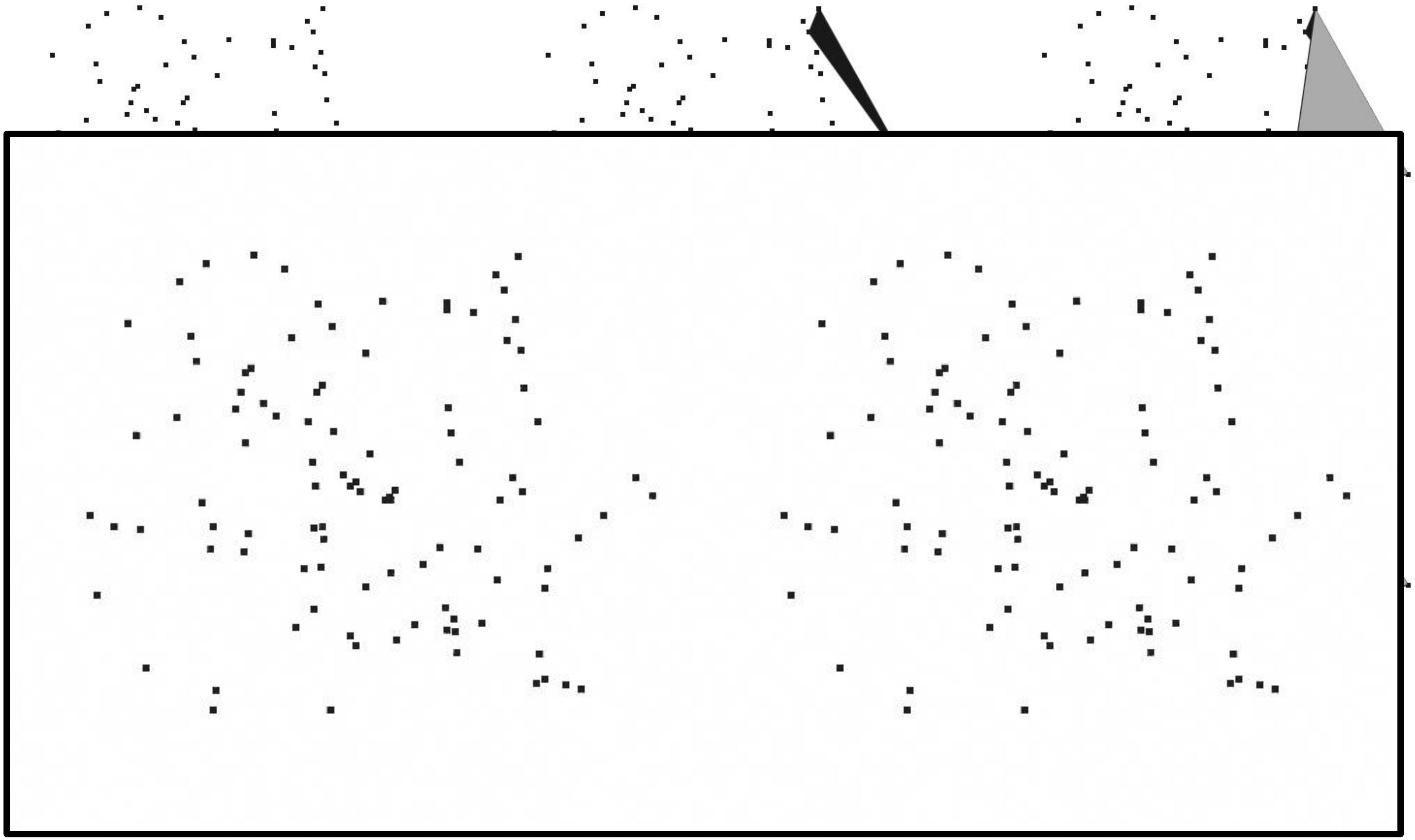
Until the hull closes, pivot around a boundary edge.



Gift-Wrapping



Gift-Wrapping



Gift-Wrapping

PivotAroundEdge($e = \{q_0, q_1\}$, $P = \{p_0, \dots, p_{n-1}\}$)

- $p \leftarrow p_0$
- area2 ← SquaredArea(q_0 , q_1 , p)
- for $p' \in \{p_1, \dots, p_{n-1}\}$:
  - » volume ← SignedVolume(q_0 , q_1 , p , p')
  - » if(volume<0)
    - – $p \leftarrow p'$
  - » else if(volume==0)
    - – _area2 ← SquaredArea(q_0 , q_1 , p')
    - – if(_area2>area2)
      - • $p \leftarrow p'$
      - • area2 ← _area2
- return p

Complexity: $O(n)$

Gift-Wrapping

FindTriangleOnHull($P = \{p_0, \ldots, p_{n-1}\}$)

- $\{p, q\} \leftarrow$ FindEdgeOnHull(P)
- $r \leftarrow$ PivotAroundEdge($\{p, q\}$, P)
- return $\{p, q, r\}$

Gift-Wrapping

FindEdgeOnHull($P = \{p_0, \ldots, p_{n-1}\}$)

- $p \leftarrow$ BottomMostLeftMostBackMost(P)
- $q \leftarrow p$
- for $r \in P$:
  - » if(q_z==r_z && q_y==r_y && q_x<r_x)
    - – $q \leftarrow r$
- if(q==p)
  - » $q \leftarrow p + (1,0,0)$
- $q \leftarrow$ PivotOnEdge($\{p, q\}$, P)
- return $\{p, q\}$

Gift-Wrapping

GiftWrap(P):

- $t \leftarrow$ FindTriangleOnHull(P)
- $Q \leftarrow \{(t_1, t_0), (t_2, t_1), (t_0, t_2)\}$
- $H \leftarrow \{t\}$
- while($Q \neq \emptyset$)
  - » $e \leftarrow Q$.pop_back()
  - » if(NotProcessed(e))
    - – $q \leftarrow$ PivotOnEdge(e)
    - – $t \leftarrow$ Triangle(e , q)
    - – $H \leftarrow H \cup \{t\}$
    - – $Q \leftarrow Q \cup \{(t_1, t_0), (t_2, t_1), (t_0, t_2)\}$
    - – MarkProcessedEdges(e)

Complexity: $O(n^2)$

Outline

- Review
- Gift-Wrapping
- Divide-and-Conquer

Divide And Conquer

DivideAndConquer(P):

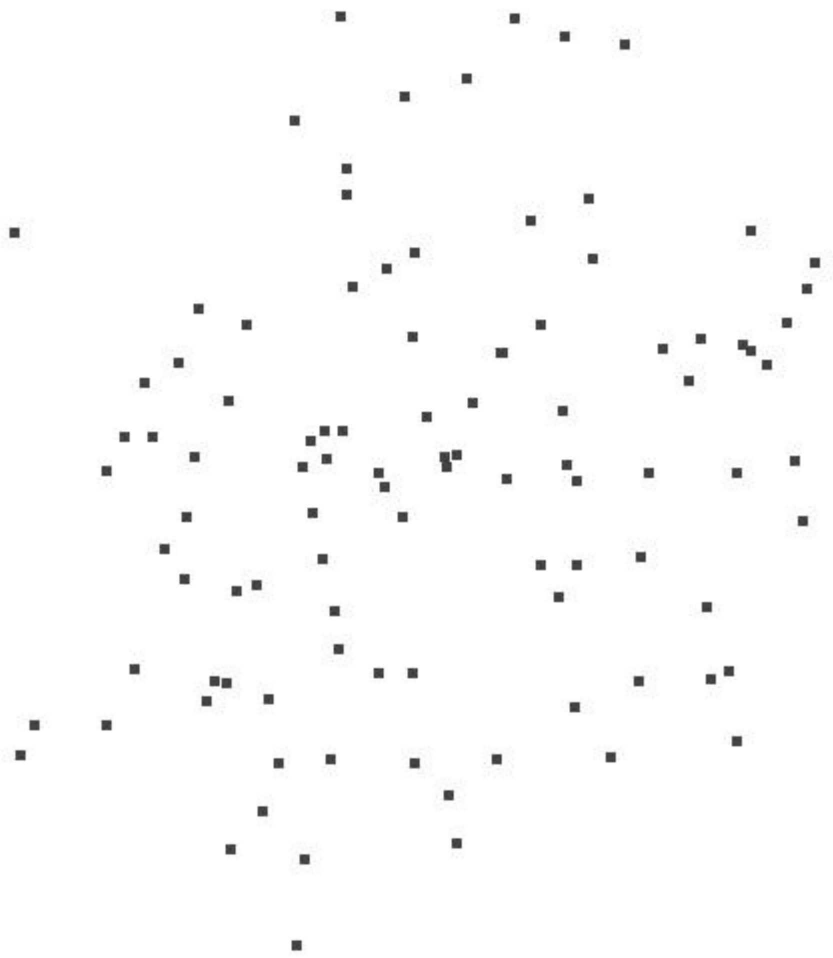
- $P \leftarrow$ SortByX(P)
- return _DivideAndConquer(P)

_DivideAndConquer(P)

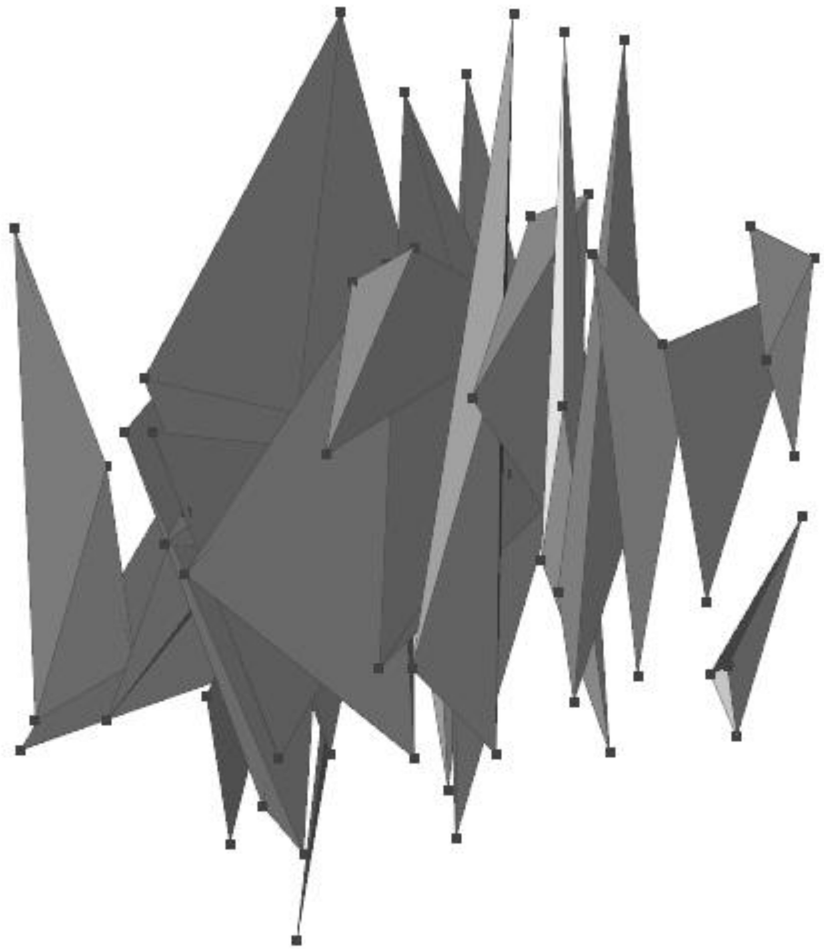
- if($|P| < 8$) return Incremental(P)
- $(P_1, P_2) \leftarrow$ SplitInHalf(P)
- $H_1 \leftarrow$ _DivideAndConquer(P_1)
- $H_2 \leftarrow$ _DivideAndConquer(P_2)
- return Merge(H_1 , H_2)

Complexity: $O(n \log n)$

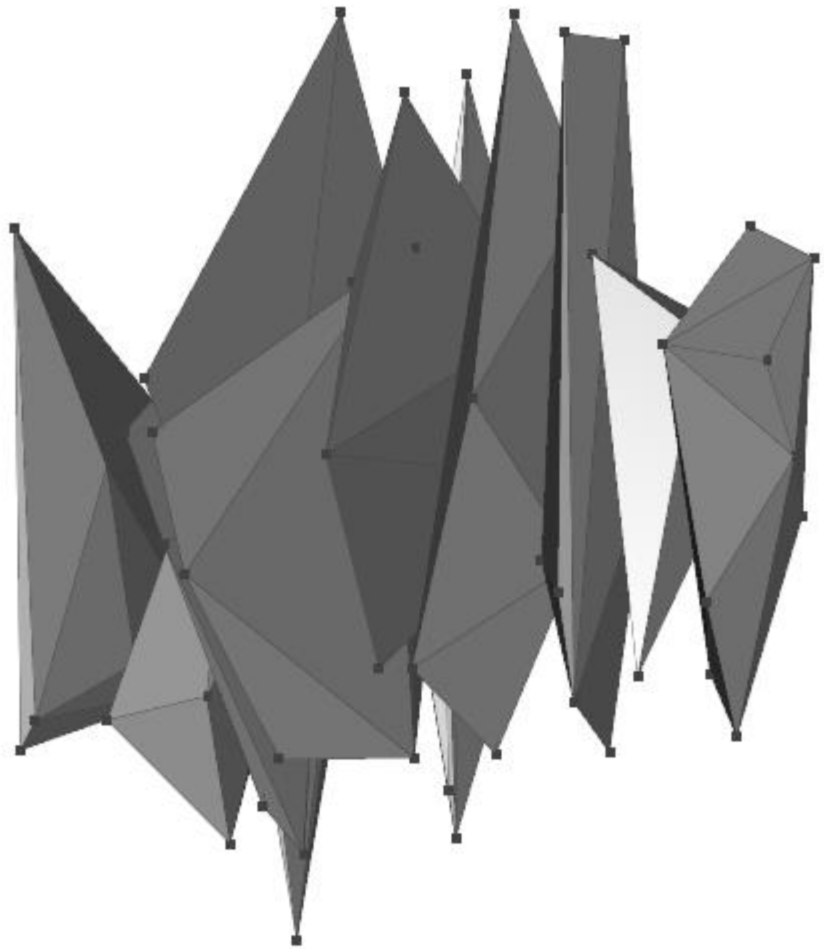
Divide And Conquer



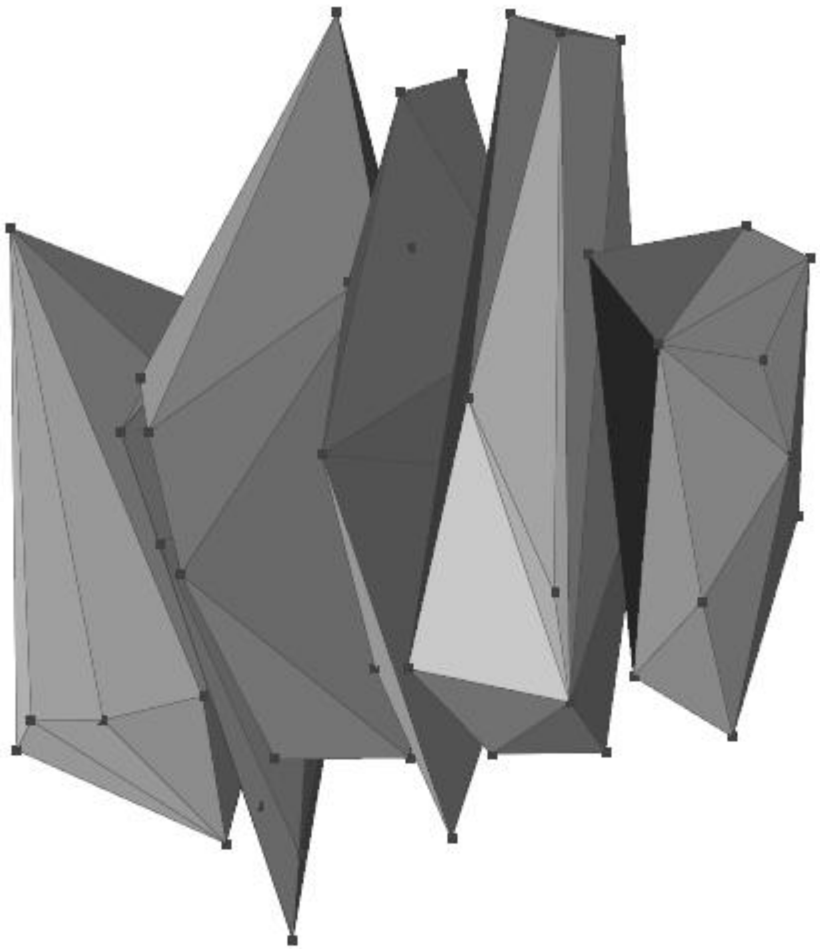
Divide And Conquer



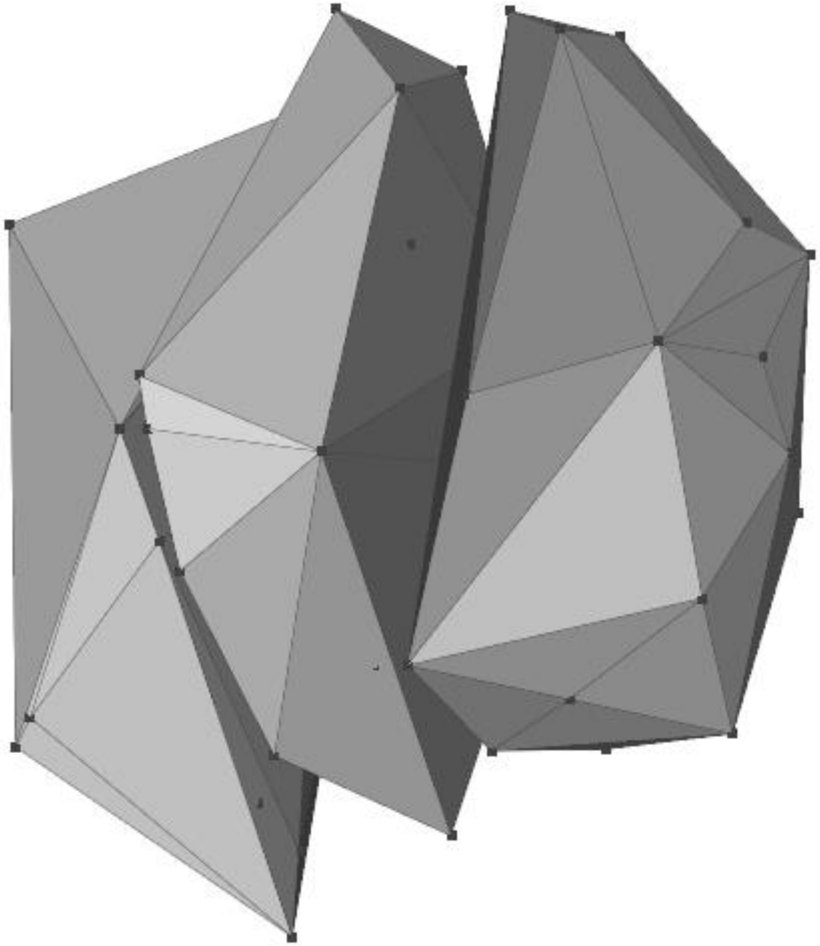
Divide And Conquer



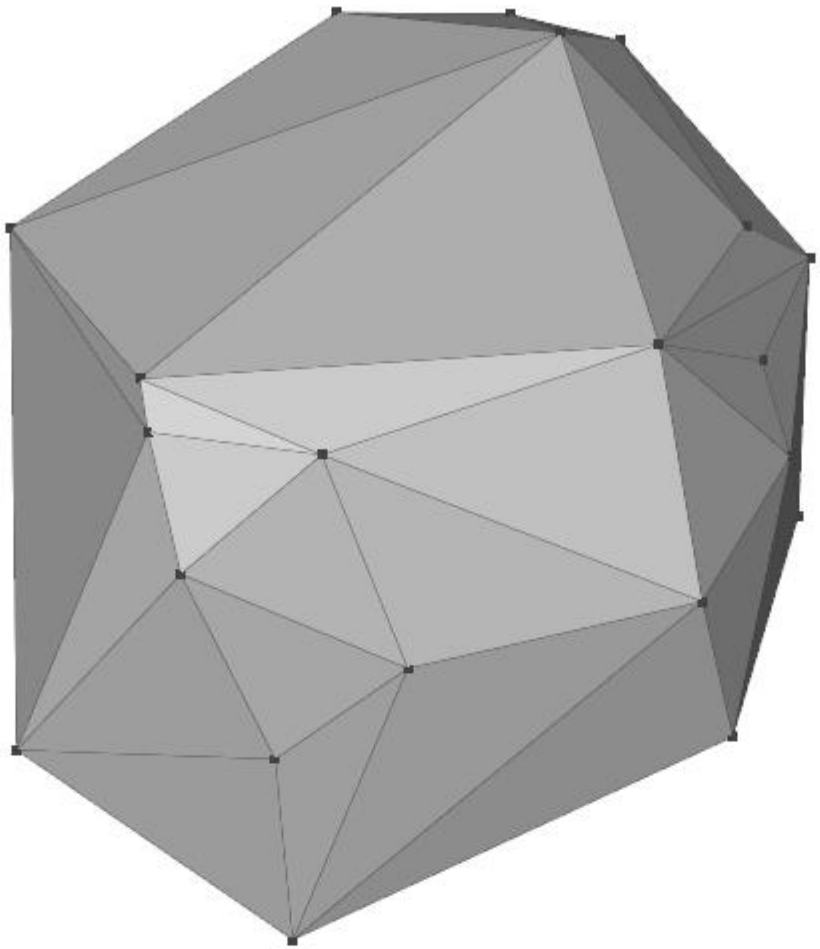
Divide And Conquer



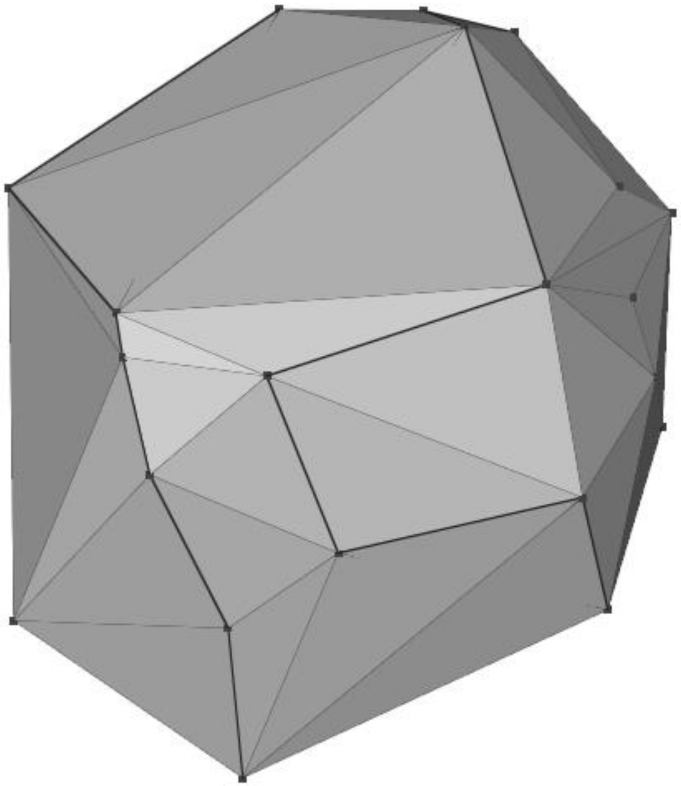
Divide And Conquer



Divide And Conquer



Divide And Conquer



Merge:

- Construct the fillet that merges the two hulls
- Remove the triangles that are no longer visible

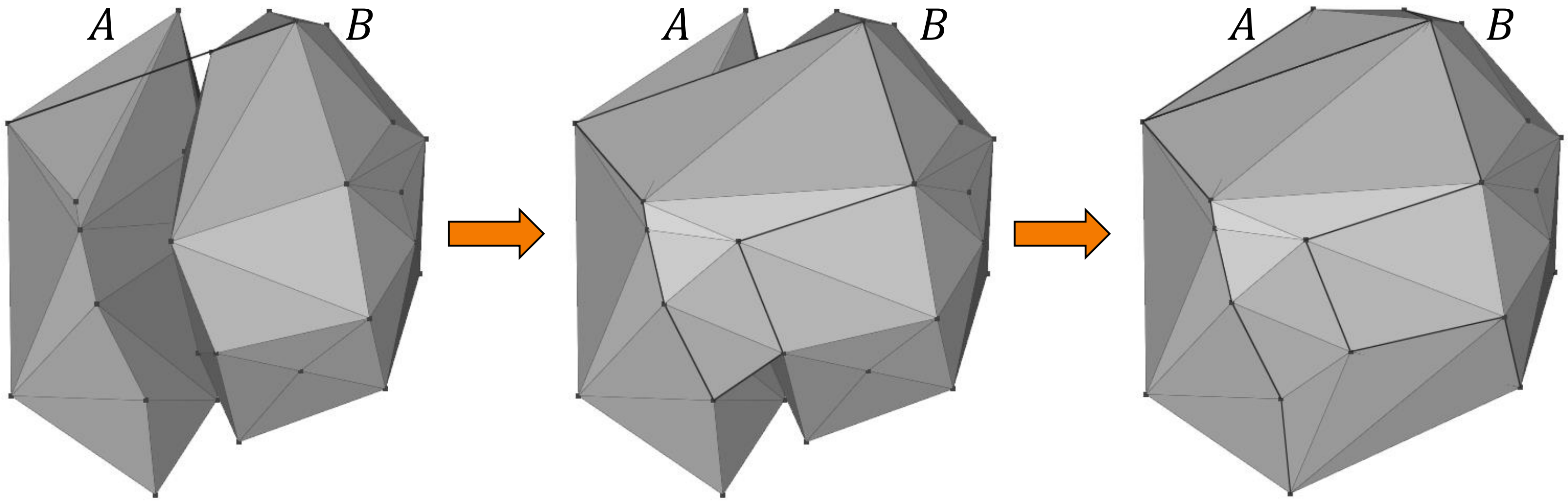
Note:

The fillet has linear complexity since each triangle on the fillet uses an edge from one of the two hulls.

Divide And Conquer

Constructing the Fillet:

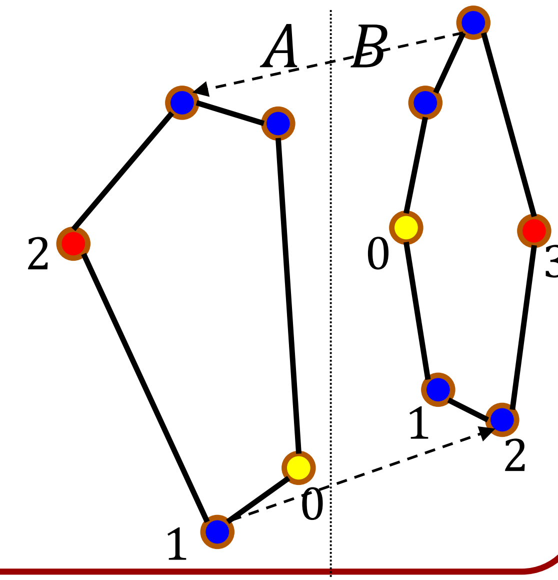
- Find a supporting line
- Pivot around the supporting line



Divide And Conquer

Finding a Supporting Line:

- While computing the 3D hull (recursively), simultaneously compute the 2D hull of the projection of the points onto the xy-plane.
- The supporting lines in 2D correspond to supporting lines in 3D.



Divide And Conquer

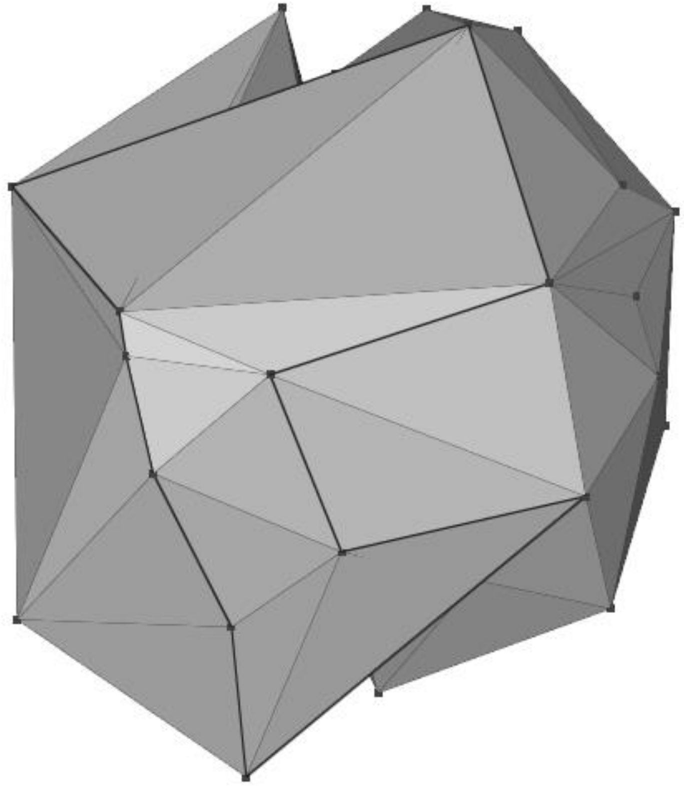
Pivot Around the Supporting Line:

- Proceed as in the gift-wrap algorithm.

Challenge:

- To run in linear time, we can't try all points.

When we pivot, the first point we hit is one of the neighbors of the line's end-points.



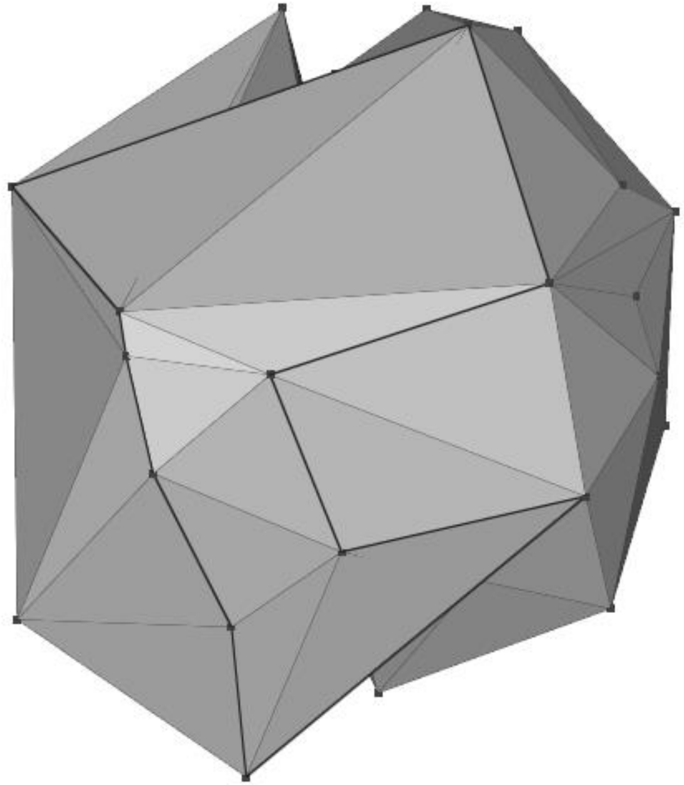
Divide And Conquer

Pivot Around the Supporting Line:

- Proceed as in the gift-wrap algorithm.

Challenge:

- This could still be costly since a vertex can have many neighbors.
  (e.g. If the right endpoint has many neighbors but the pivot keeps hitting a vertex on the left.)



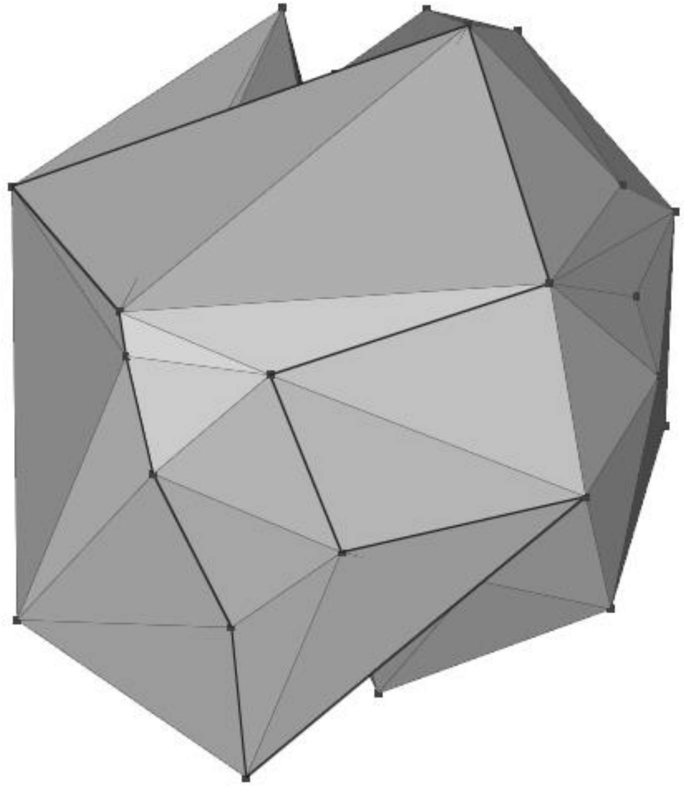
Divide And Conquer

Pivot Around the Supporting Line:

- Proceed as in the gift-wrap algorithm.

Challenge:

- This could still be costly since a vertex can have many neighbors.

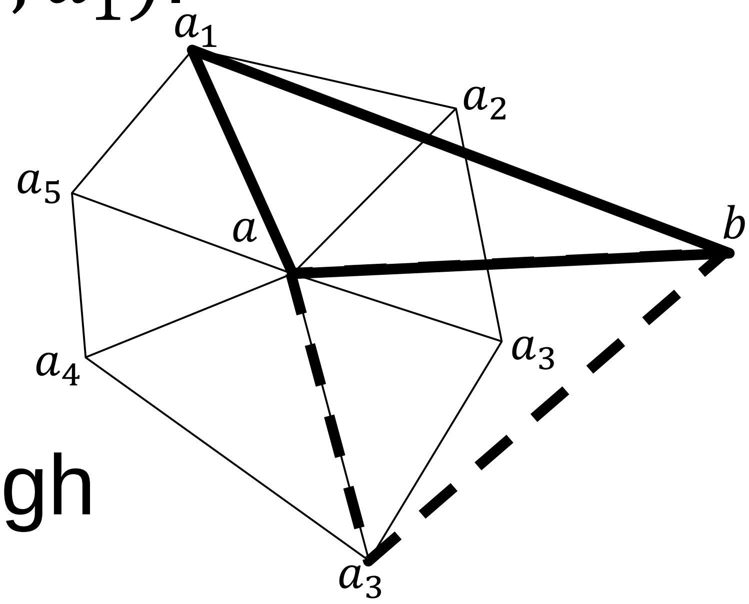


We can use the previous estimated (failed) hit to constrain the next one.

Divide And Conquer

More Specifically:

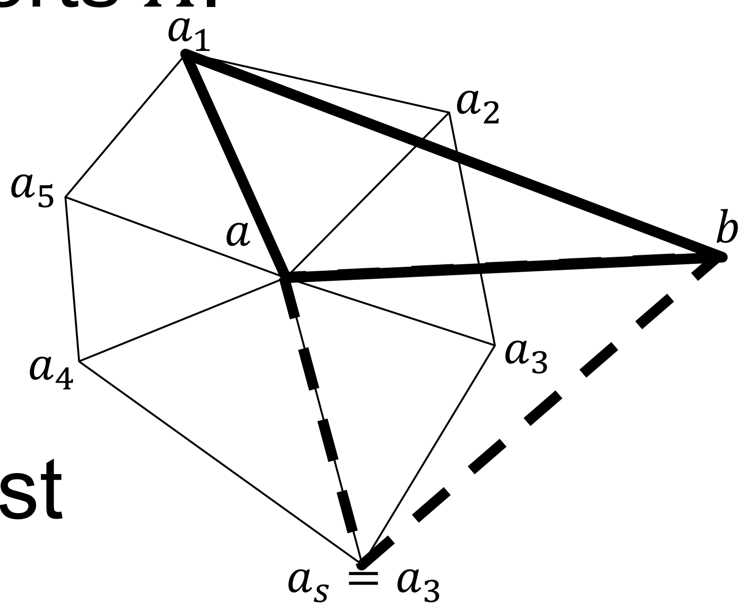
- Assume the fillet is at edge (a, b) having just added triangle (a, b, a_1).
- Sort the neighbors of a CW starting from a_1.
- Let a_s be the neighbor of a s.t. the plane through (b, a, a_s) supports A.



Divide And Conquer

More Specifically:

- Let a_s be the neighbor s.t. the plane through (b, a, a_s) supports A.
- The points $\{a_2, \ldots, a_{s-1}\}$ must be inside the hull.
- Even if we advance on b we won't need to retest these points.



Divide And Conquer

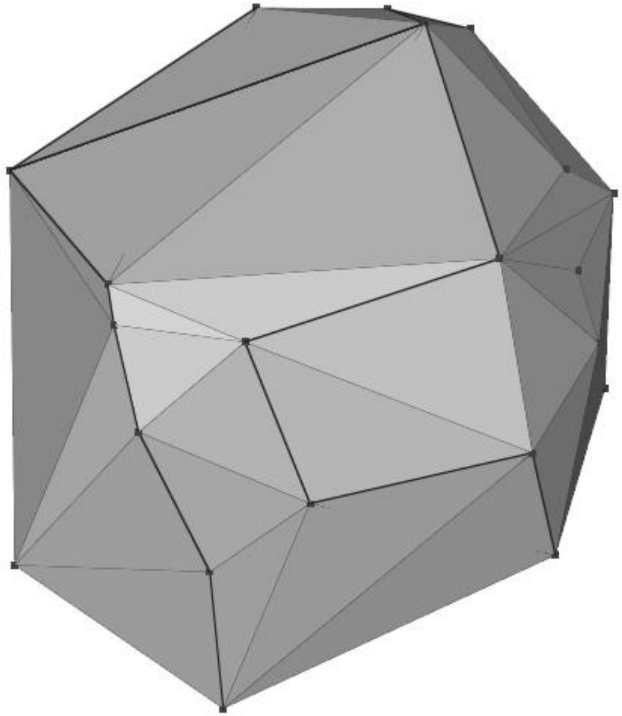
Merge(H_1 , H_2):

- $(v_1, v_2) \leftarrow$ FindSupportingLine(H_1 , H_2)
- $Q \leftarrow \{(v_1, v_2)\}$
- $F \leftarrow \emptyset$
- While ($Q \neq \emptyset$)
  - » $e \leftarrow Q$.pop_back()
  - » if($e \neq \{v_2, v_1\}$)
    - – $t \leftarrow$ SupportingTriangle(H_1 , H_2 , e)
    - – $F \leftarrow F \cup \{t\}$
    - – $Q \leftarrow Q \cup$ CrossingEdges(t) / $\{e\}$
- CleanUp

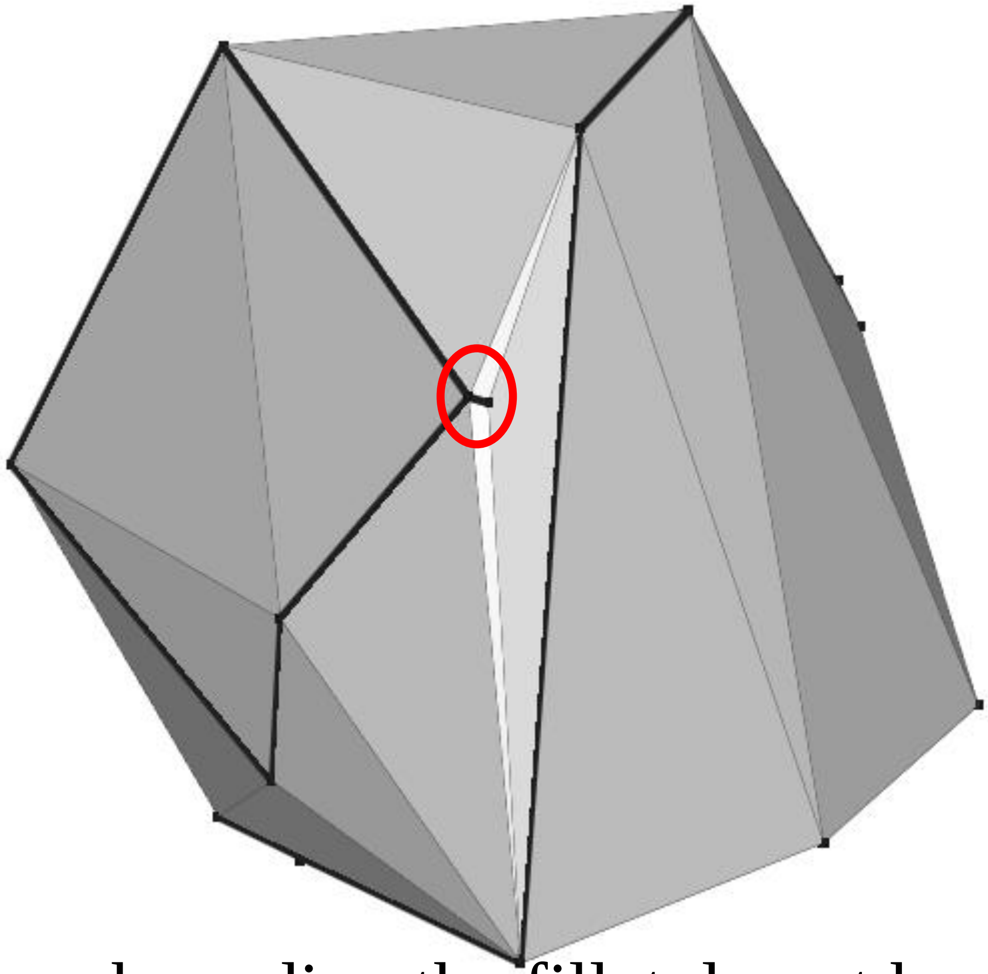
Divide And Conquer

Clean-Up:

- Represent the two hulls with a winged-edge data structure.
- Replace the opposite edges of the silhouette with the edges of the new triangles.
- Flood-fill to find interior triangles.



Divide And Conquer



Note: The curves bounding the fillet do not have to be simple