Convex Hulls (2D)

O’Rourke, Chapter 3

Announcements

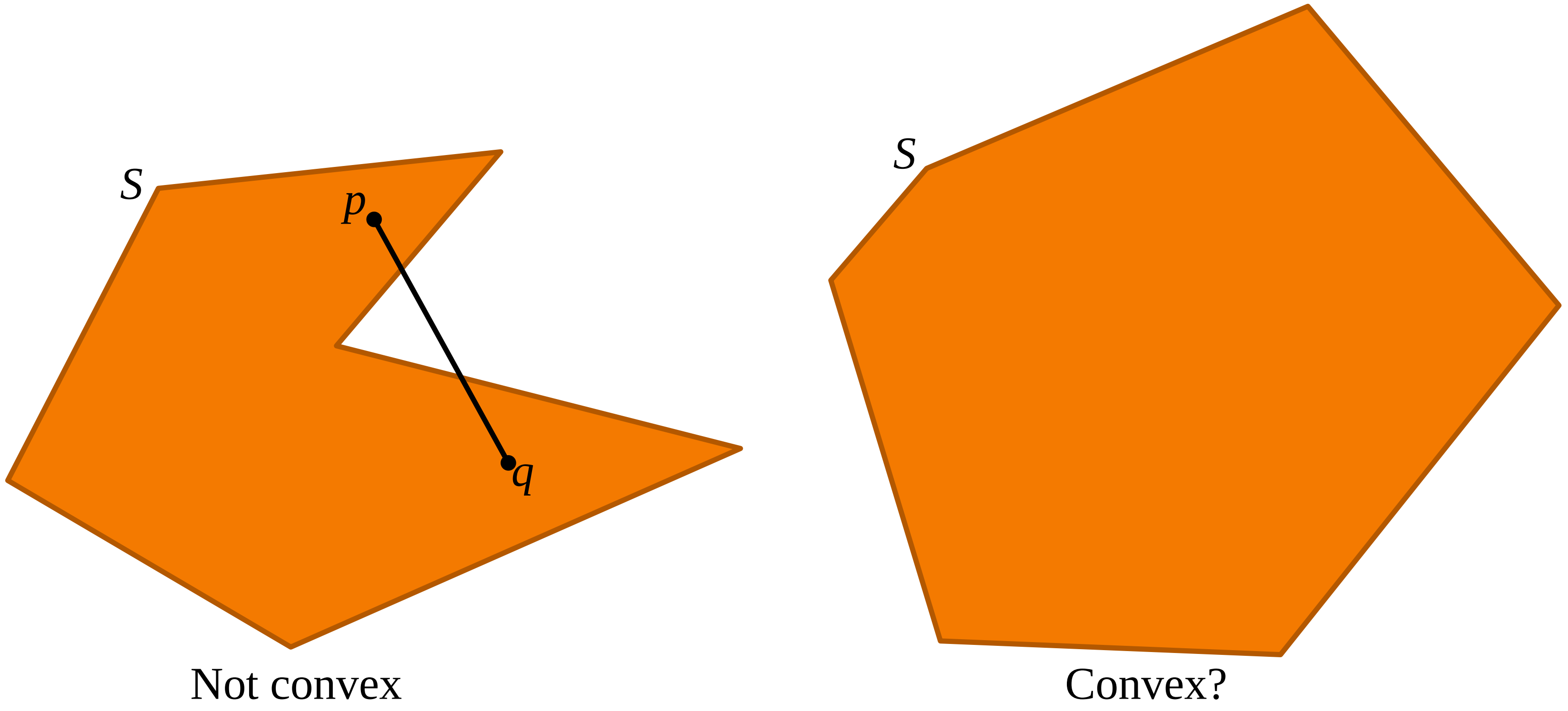
- We are live on Piazza:
  http://piazza.com/jhu/spring2016/600459
- Code-base was updated last Wednesday
  (Fixes a bug where y-values are read in as zero)

Outline

- Convex Hulls
- Algorithms
  - Naïve Implementation(s)
  - Gift Wrapping
  - Quick Hull
  - Graham’s Algorithm
  - Lower bound complexity

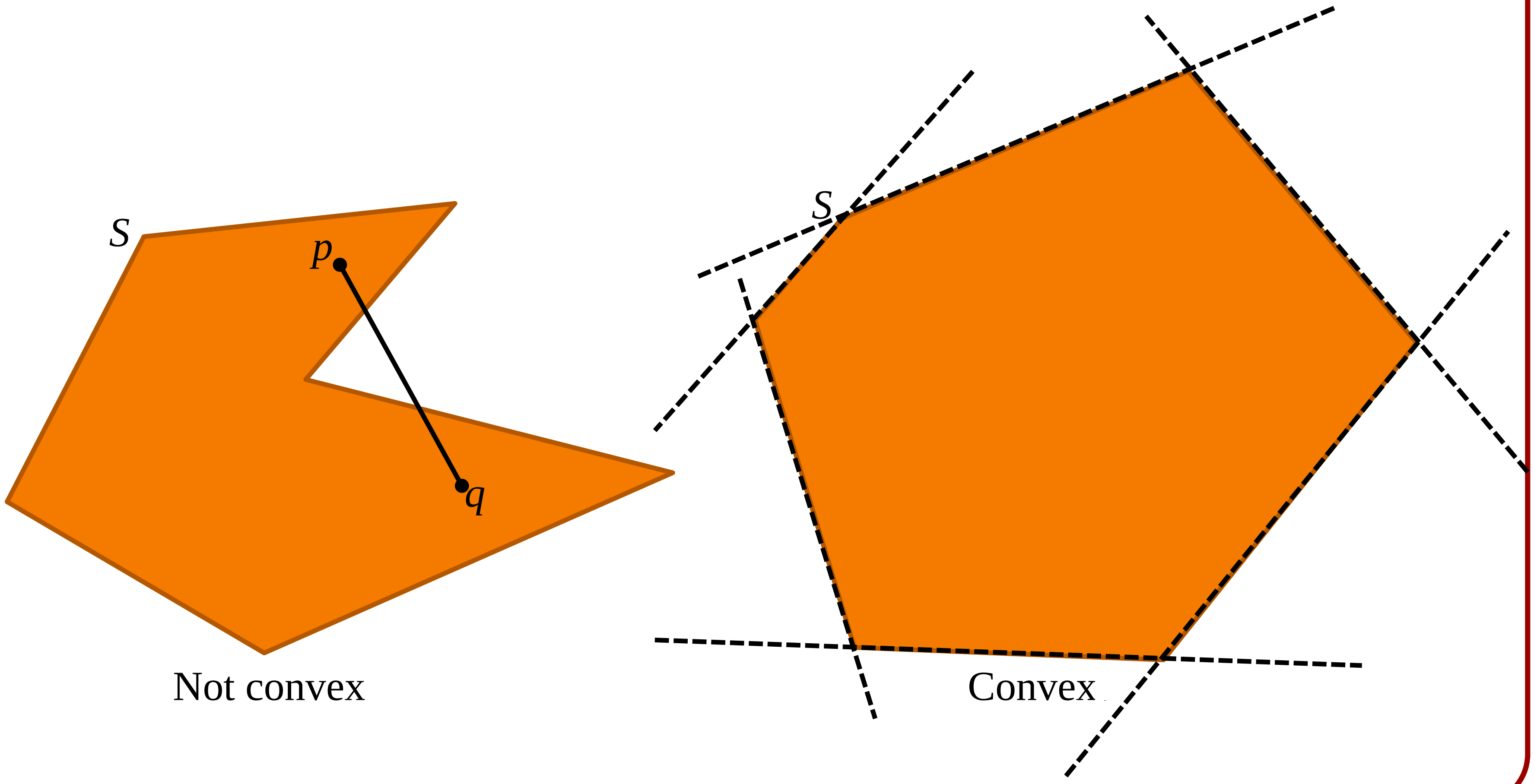
Convexity

A set S is *convex* if for any two points $p, q \in S$ the line segment $\overline{pq} \subset S$.



Convexity

A set S is *convex* if it is the intersection of (possibly infinitely many) half-spaces.



Convexity

Given points $\{p_1, \ldots, p_n\} \subset \mathbb{R}^d$, a point $q \in \mathbb{R}^d$ is a *convex combination* of the points if q can be expressed as the linear sum:

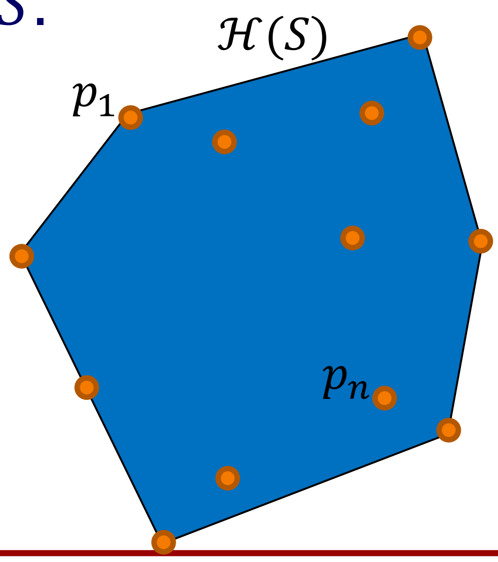
$$q = \sum_{i=1}^{n} \alpha_i \cdot p_i$$

with $\alpha_i \geq 0$ and $\alpha_1 + \cdots + \alpha_n = 1$.

Convex Hull

The *convex hull* of a set of points $S \subset \mathbb{R}^d$, denoted $\mathcal{H}(S)$, is the:

- set of all convex combinations of points in S,
- set of all convex combinations of $d+1$ points in S,
- intersection of all convex sets C w/ $S \subset C$,
- intersection of all half-spaces H w/ $S \subset H$,
- smallest convex polygon containing S.

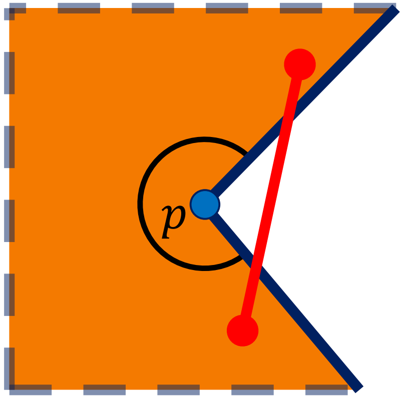


Convex Hull

Note:

If $p \in S \subset \mathbb{R}^2$ and p is a vertex of the convex hull then p must be a convex vertex.

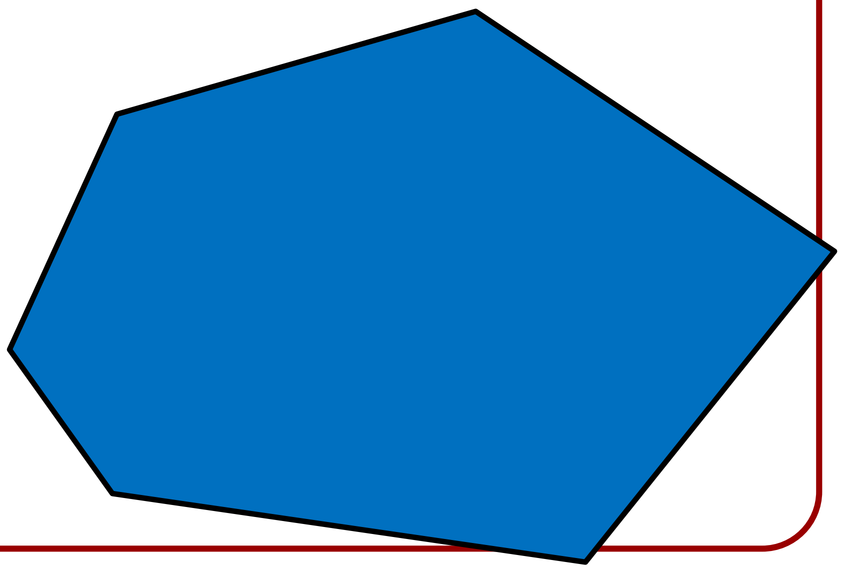
Otherwise, we could create a line segment with vertices inside of the hull but which isn't strictly interior.



Convex Hull

Claim:

If $P \subset \mathbb{R}^2$ is a polygon whose vertices are all convex, then P is convex.

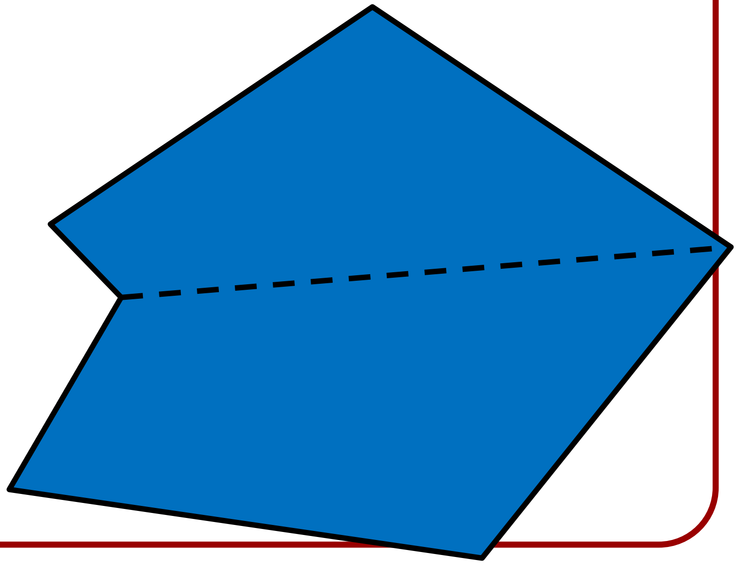


Convex Hull

<u>Proof (by induction)</u>:

Otherwise, we could add a diagonal.

⇒ By induction, each half is convex.

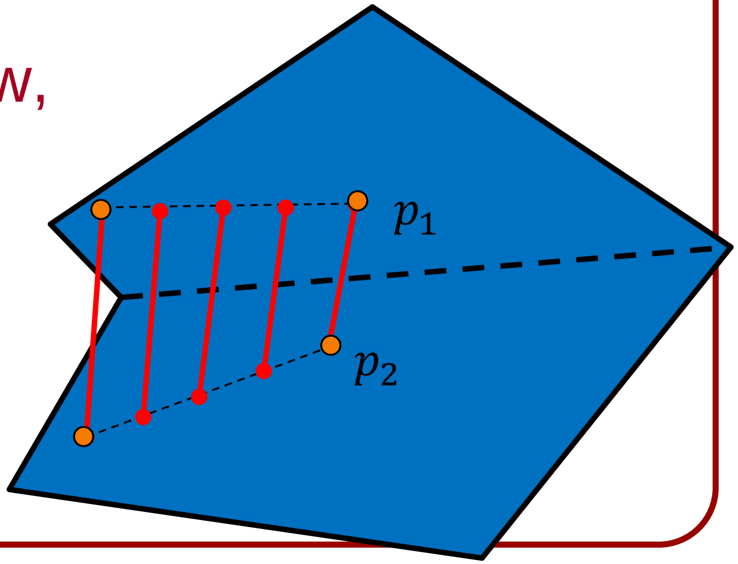


Convex Hull

Proof (by induction):
Otherwise, we could add a diagonal.

⇒ If P is not convex there must be a segment between the two parts that exits P.

Choose p_1 and p_2 above/below the diagonal.
Evolve the segment to $\overline{p_1p_2}$.
Since p_1 and p_2 are above/below, $\overline{p_1p_2}$ crosses the diagonal and is entirely inside P.



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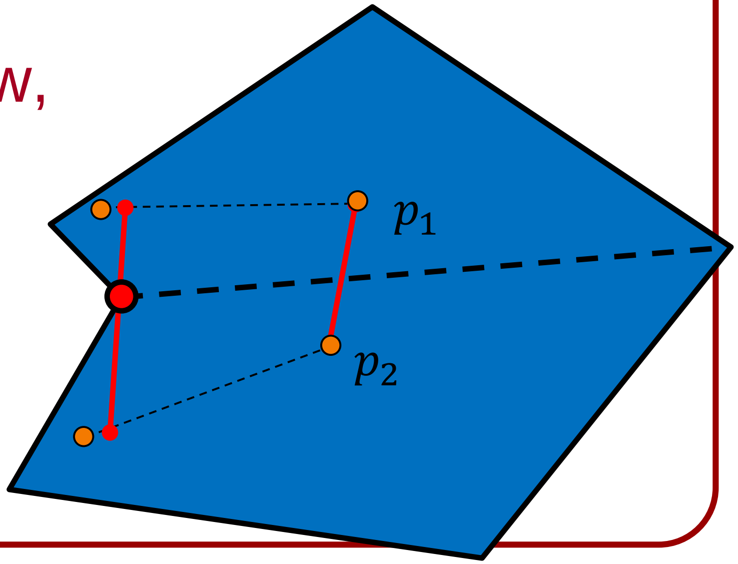
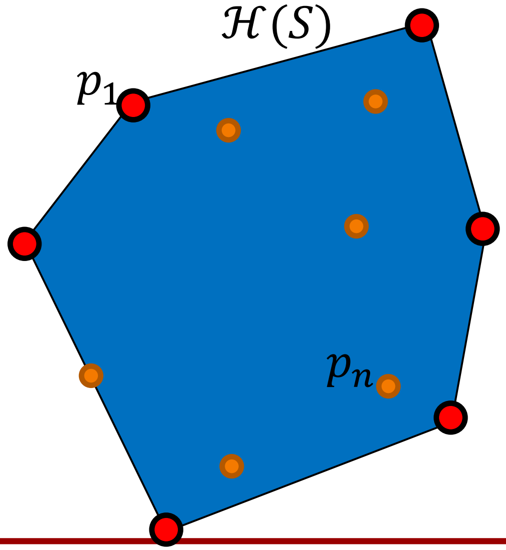
The last point at which the evolving segment is not fully inside must be a reflex vertex.



Convex Hull

The *extreme points* of a set of points $S \subset \mathbb{R}^2$ are the points which are on the convex hull and have interior angle strictly less than π.

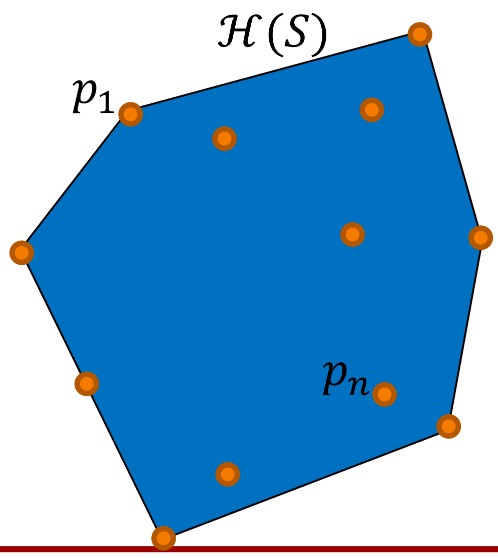


Convex Hull

Goal:

Given a set of points $S = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$, compute the convex hull $\mathcal{H}(S)$ efficiently.

- Do we want all points on the hull or just the extreme ones?
- Do the output vertices need to be sorted or is the set of (extreme) vertices sufficient?



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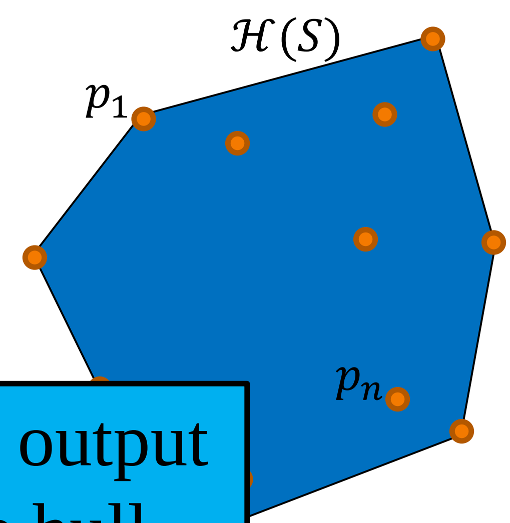
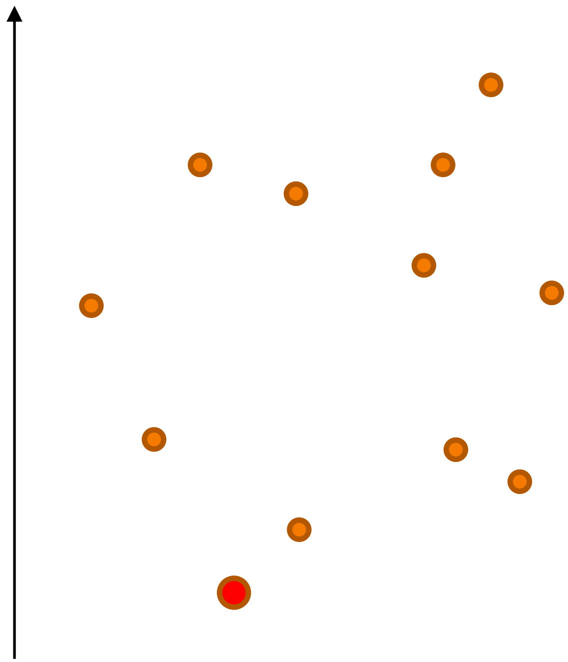


We will focus on the ordered output of the extreme points on the hull.

Convex Hull (2D)

Note:

We can find a hull vertex in linear time by finding the vertex that is extremal w.r.t. to some direction.



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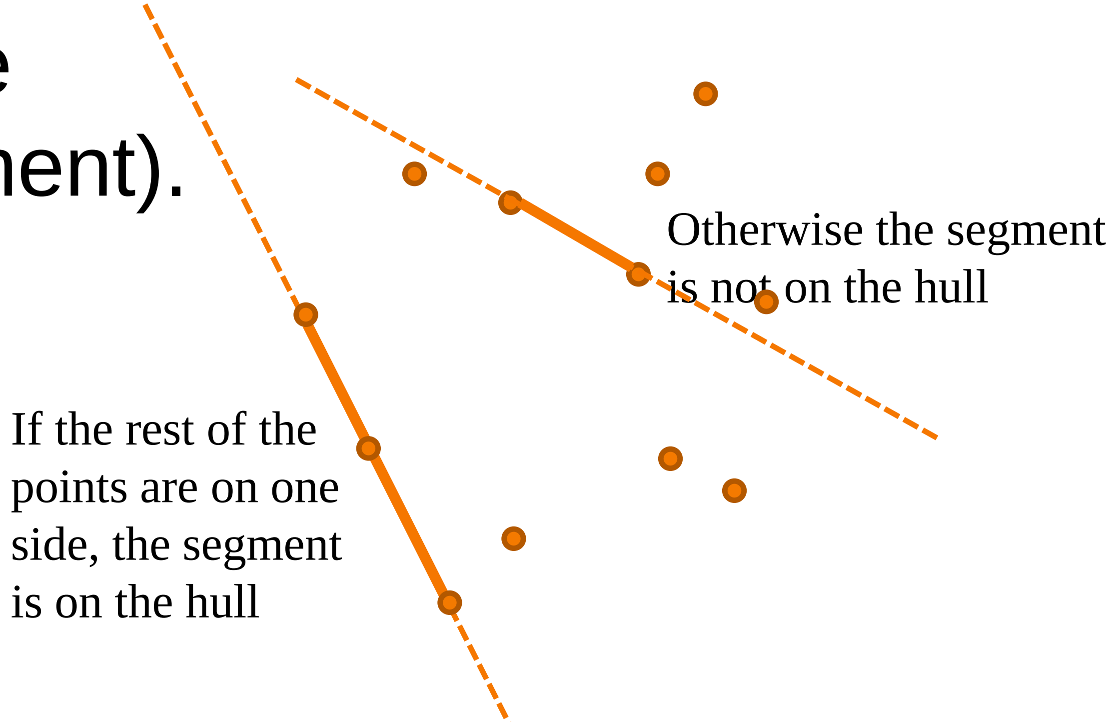
Convex Hull (2D)

Naïve Algorithm:
For each directed edge $e \in S \times S$, check if the half-space to the right of e is empty of points (and there are no points on the line outside the segment).



Convex Hull (2D)

<u>Naïve Algorithm $O(n^3)$</u>:
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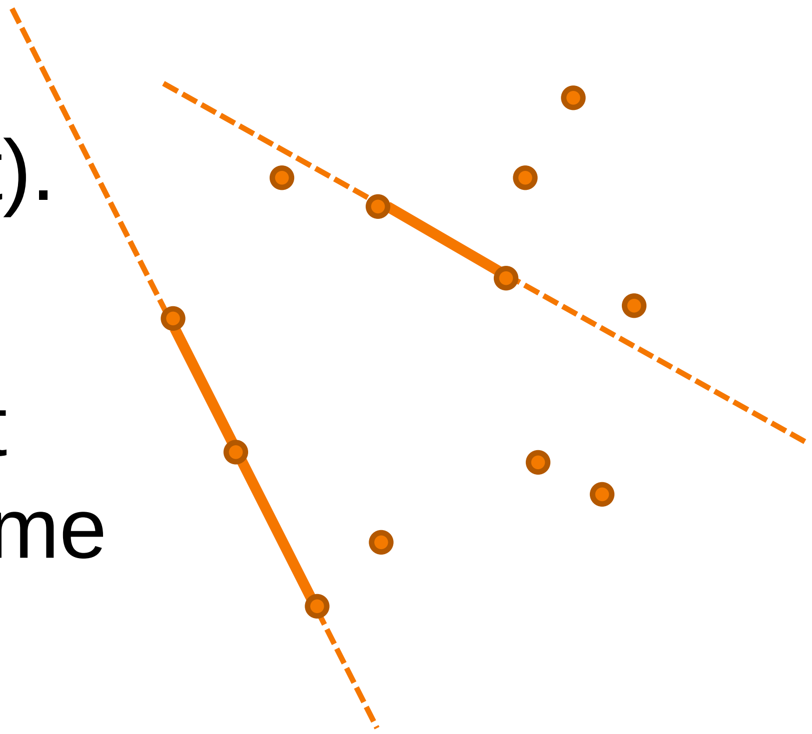

<u>Note</u>:
The output is the set of (unordered) extreme points on the hull.

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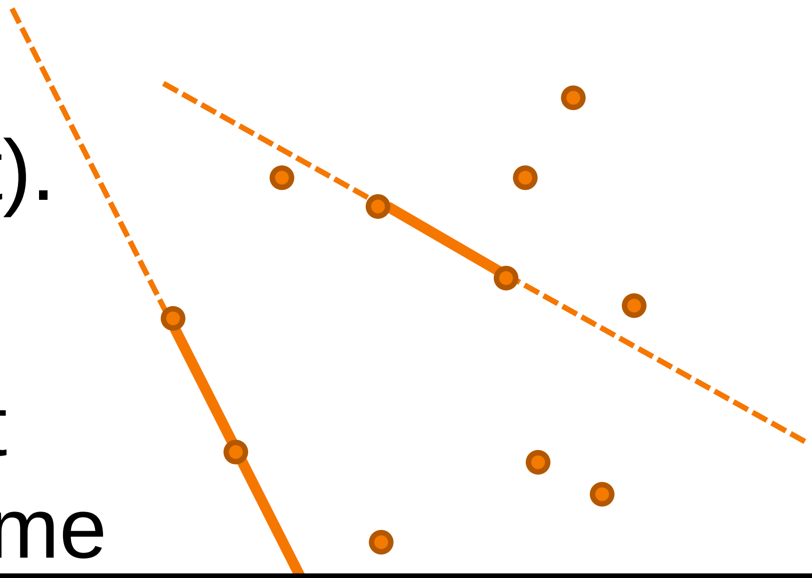

Note:
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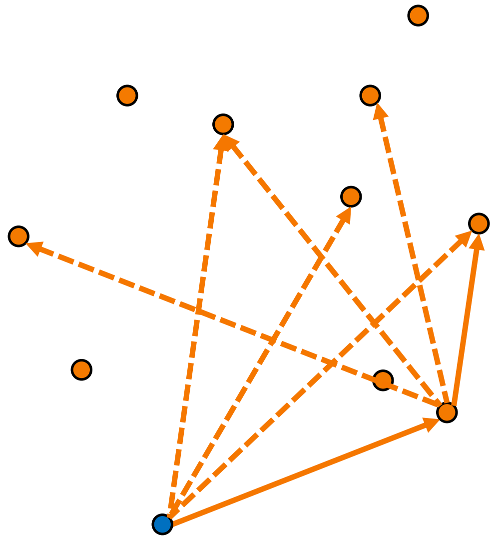
If we want the ordered points, we can stitch the edges together in $\leq O(n^2)$ time in a post-processing step.

Convex Hull (2D)

Naïve Algorithm++:
Grow the hull by starting at a hull vertex and searching for the next edge on the hull by trying all possible edges and testing if they are on the hull.



Convex Hull (2D)

<u>Naïve Algorithm++ $O(n^2h)^*$</u>:
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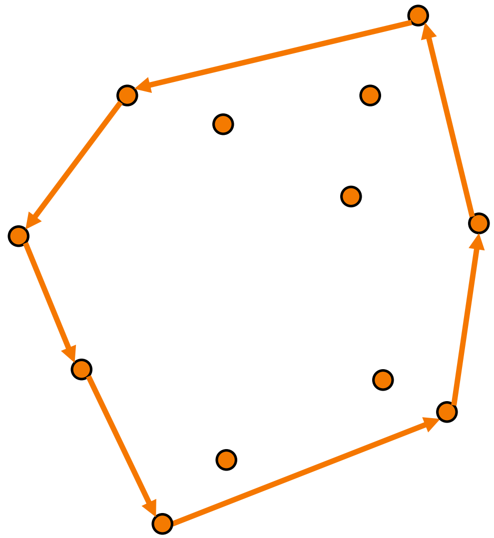
<u>Note</u>:
By explicitly forcing the output to be sorted, we end up with a faster algorithm.



$^*h$ is the number of points on the hull.

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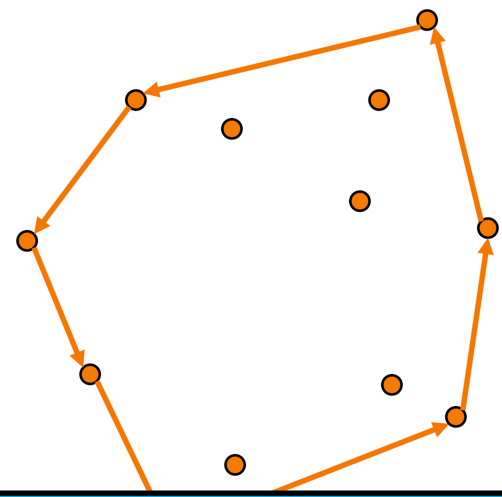

This implementation is *output sensitive*.

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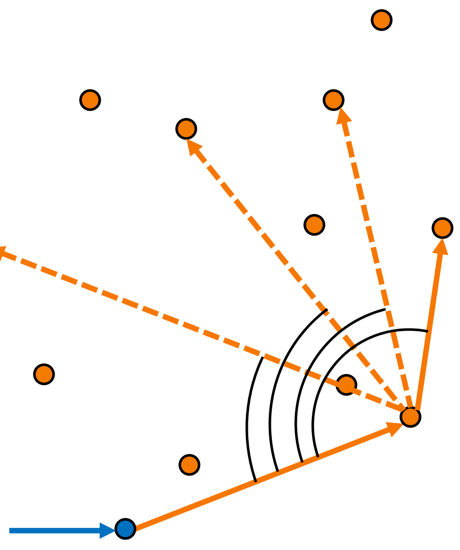
Convex Hull (2D)

Note:
The next edge on the hull is the one making the largest angle. (If two points make the same angle, ignore the closer one.)

Gift Wrapping:
Grow by finding the edge making the largest angle.



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Gift Wrapping $O(nh)$:
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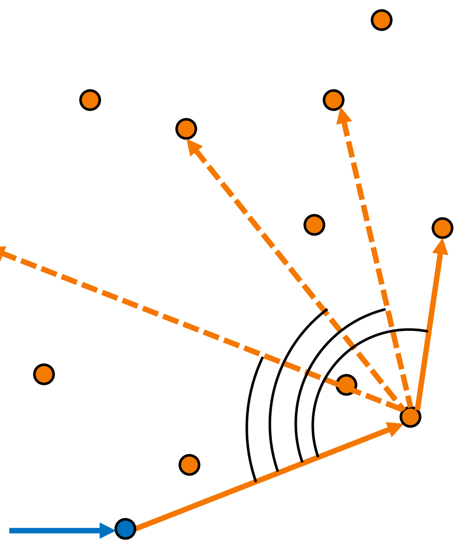

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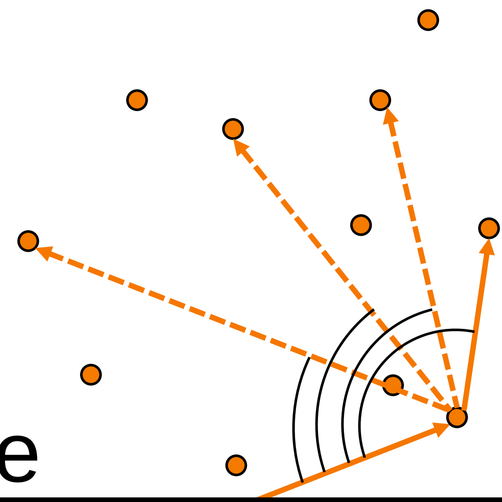

Gift Wrapping $O(nh)$:
Grow by finding the edge
mak

A similar approach makes it possible to find a hull edge in linear time.

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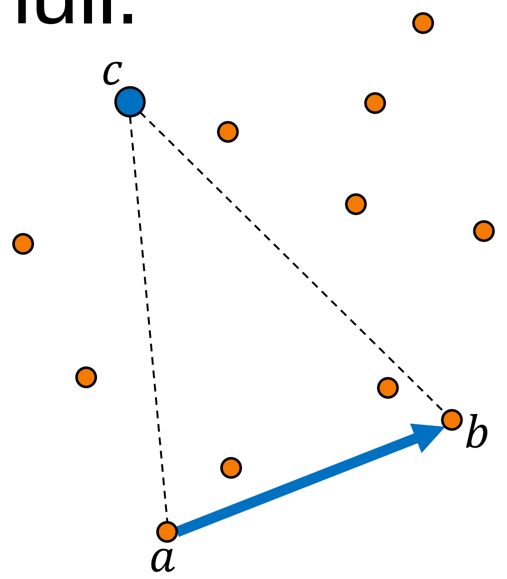
Convex Hull (2D)

Observation:

Given a hull edge (a, b), we can find the point c furthest from the edge in linear time.

1. The point c is on the hull.
2. The triangle Δabc partitions the input into three regions:
   I. Points inside Δabc.
   II. Points to the right of $\overrightarrow{bc}$.
   III. Points to the right of $\overrightarrow{ca}$.



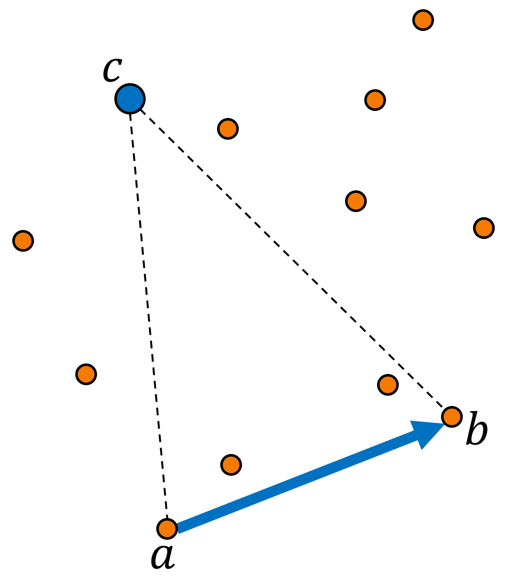
Convex Hull (2D)

Observation:

Given a hull edge (a,b), we can find the point c furthest from the edge in linear time.

$\Rightarrow$ Divide-and-conquer:

- Discard points inside Δabc
- Separately compute the half-hulls to the right of $\overrightarrow{bc}$ and the right of $\overrightarrow{ca}$.
- Merge the two hulls.



Convex Hull (2D)

Observation:

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We don't require $\overrightarrow{ab}$ to be a hull edge.
As long as it's a hull diagonal merging is easy.

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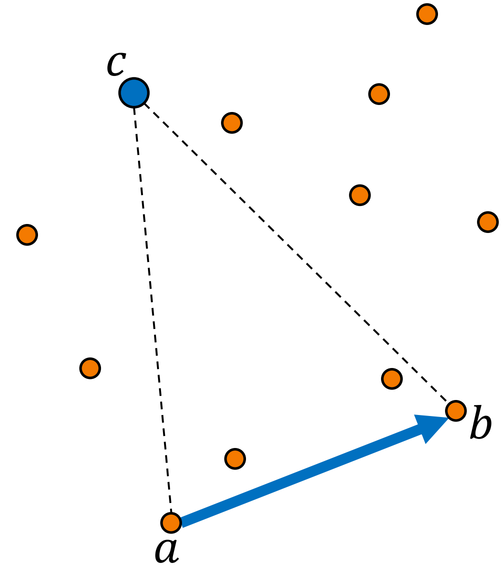

Convex Hull (2D)

QuickHull($S \subset \mathbb{R}^2$)

- $(a, b) \leftarrow$ HorizontalExtrema(S)
- $A \leftarrow$ RightOf(S , $\overrightarrow{ab}$)
- $B \leftarrow$ RightOf(S , $\overrightarrow{ba}$)
- $Q_A \leftarrow$ QuickHalfHull(A , $\overrightarrow{ab}$)
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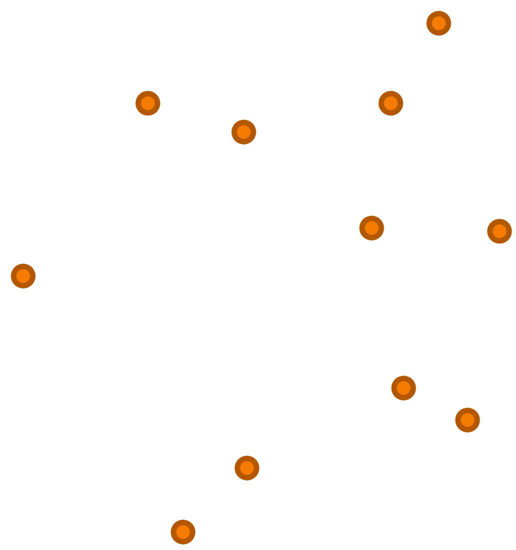
- return $\{a\} \cup Q_A \cup \{b\} \cup Q_B$



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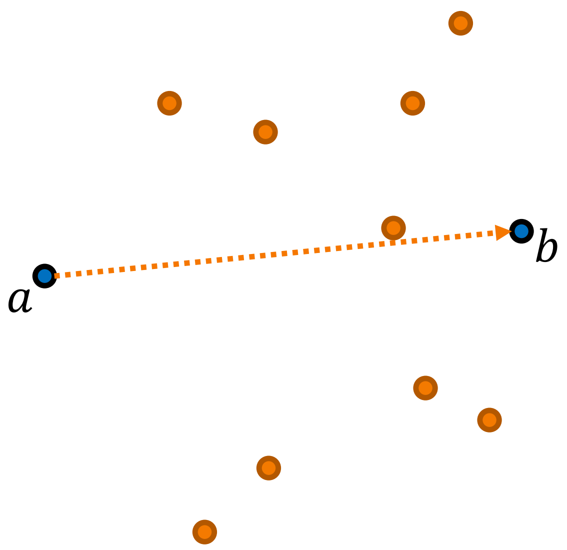

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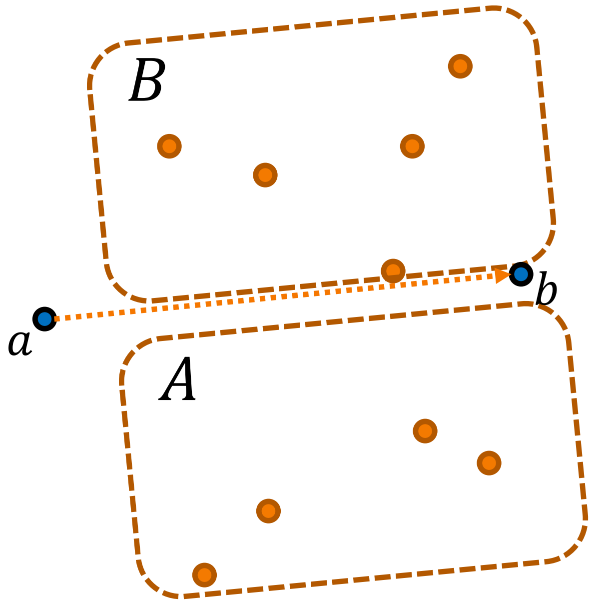

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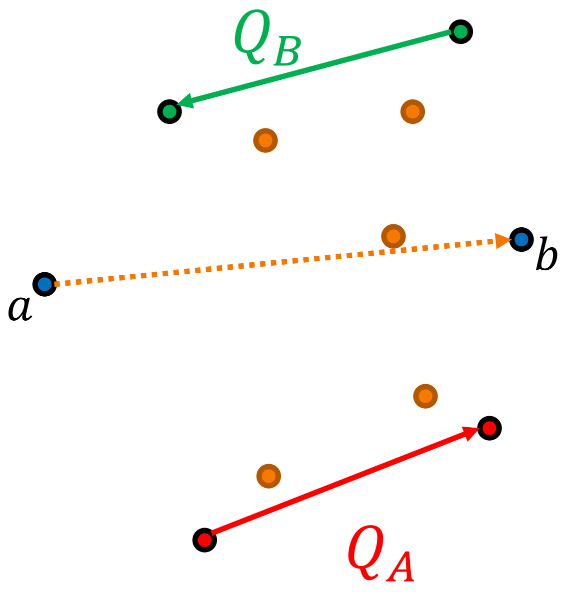

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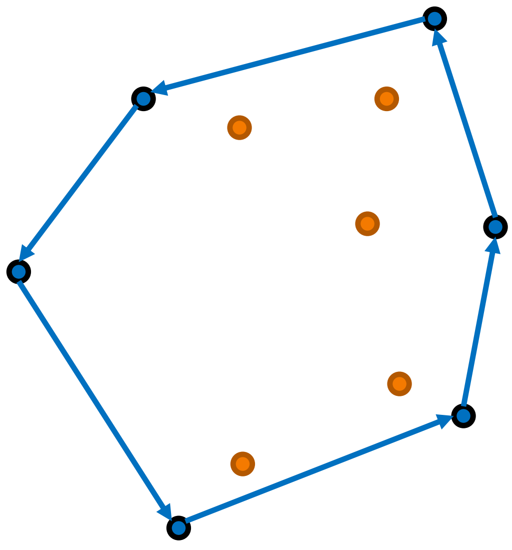

Convex Hull (2D)

(Recursion Level 0)

QuickHalfHull($S \subset \mathbb{R}^2, \overrightarrow{ab} \in S \times S$)

- if($S == \emptyset$) return $\emptyset$
- else
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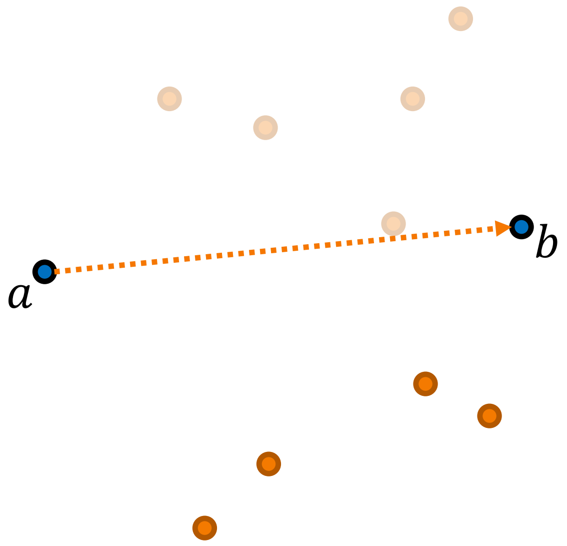

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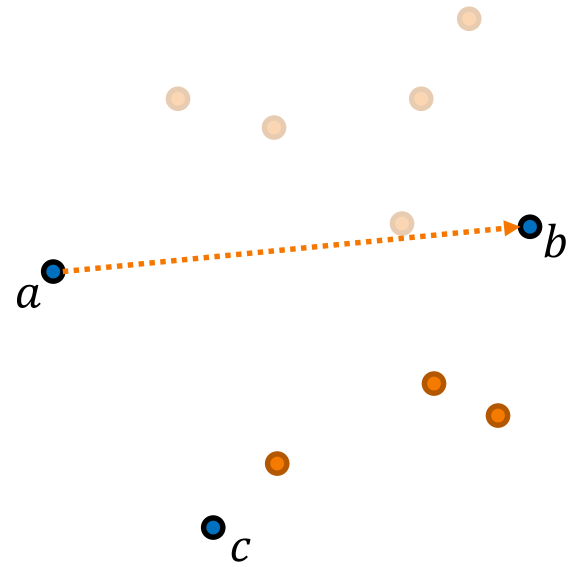

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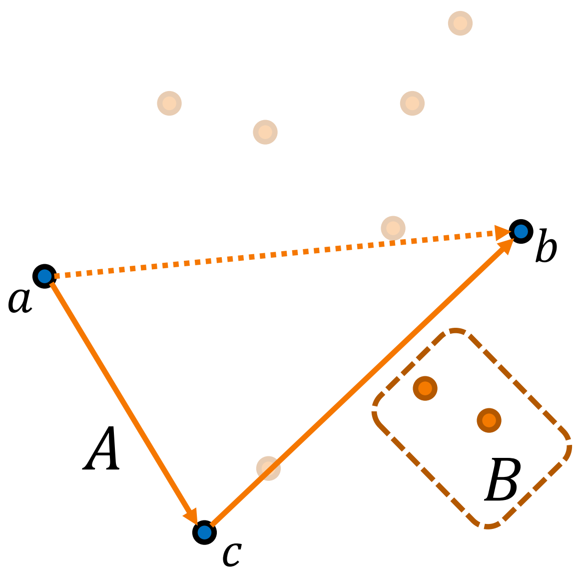

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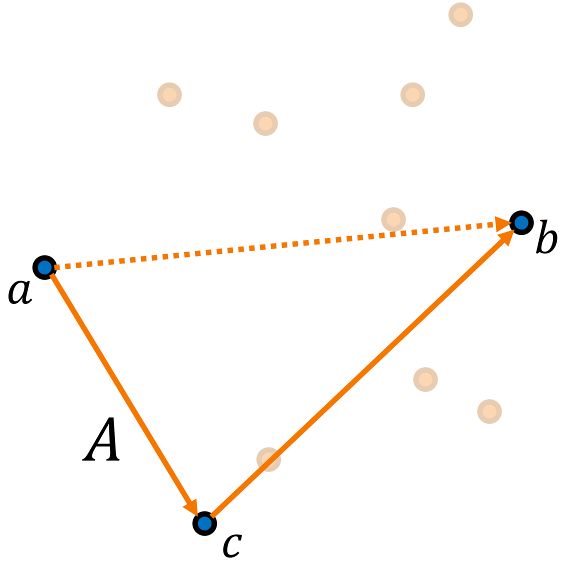

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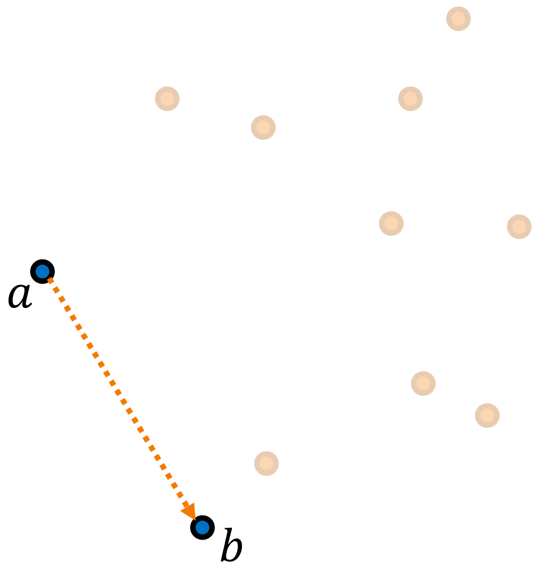

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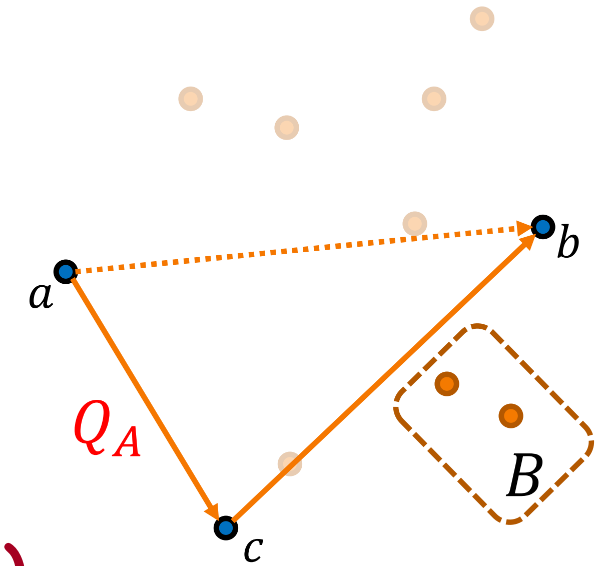

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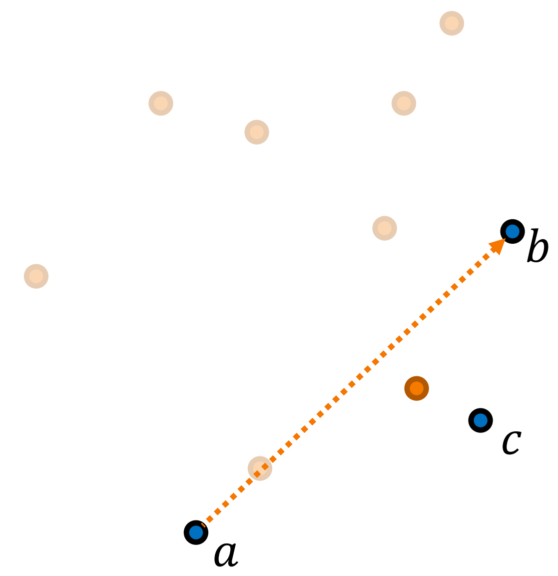

Convex Hull (2D)

(Recursion Level 1)

QuickHalfHull($S \subset \mathbb{R}^2, \overrightarrow{ab} \in S \times S$)

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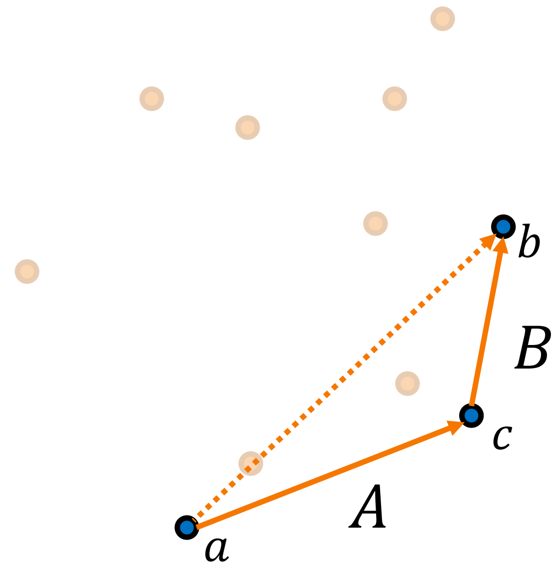

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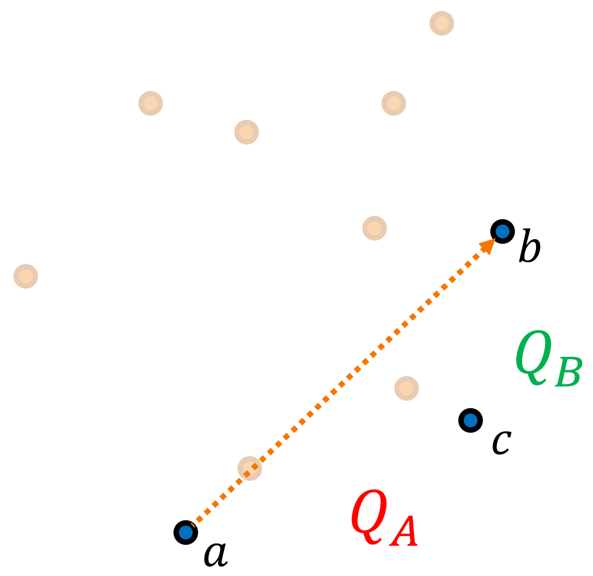

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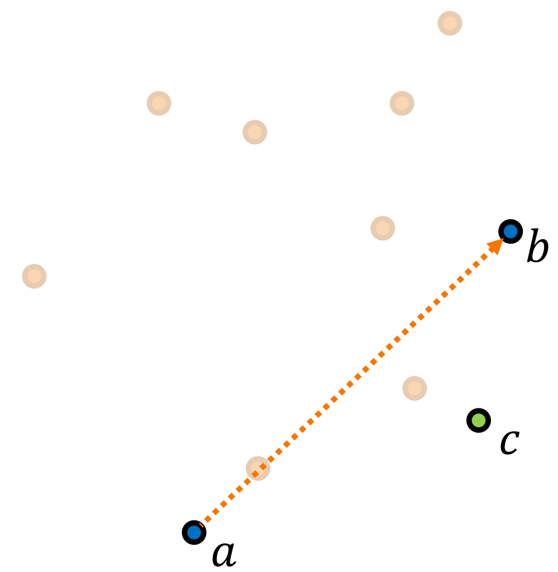

Convex Hull (2D)

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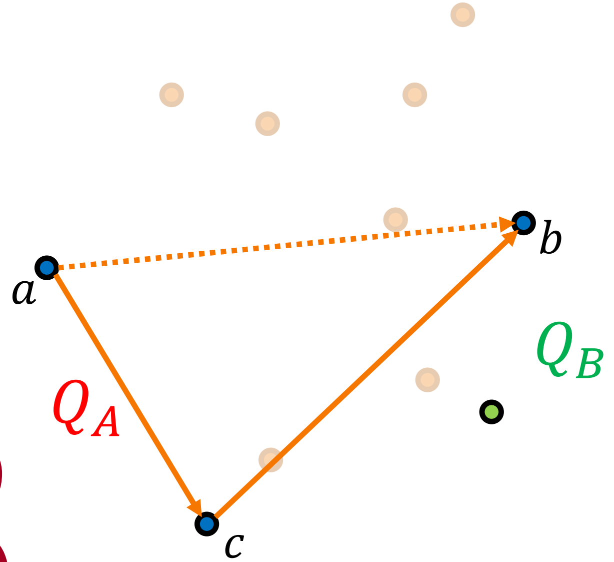

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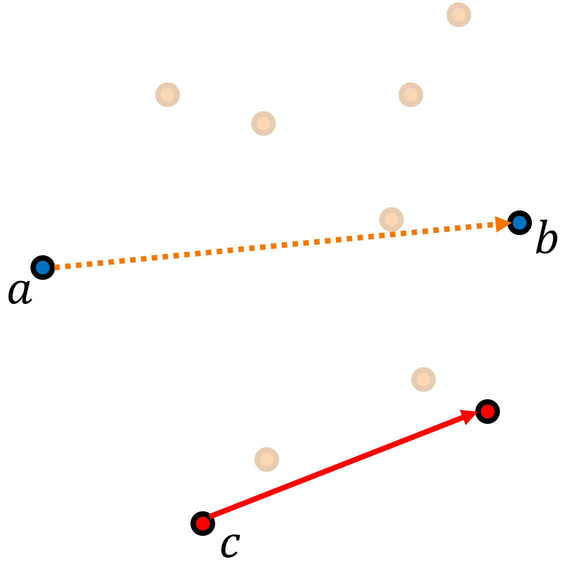

Convex Hull (2D)

QuickHull Complexity:

Like QuickSort:

- In the worst case, the complexity can be $O(n^2)$.

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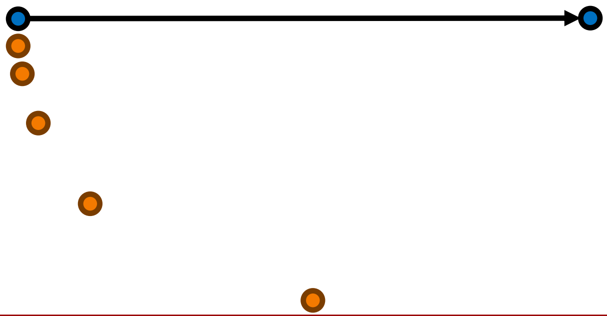

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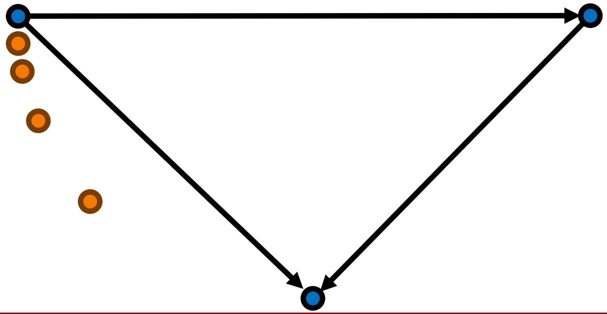

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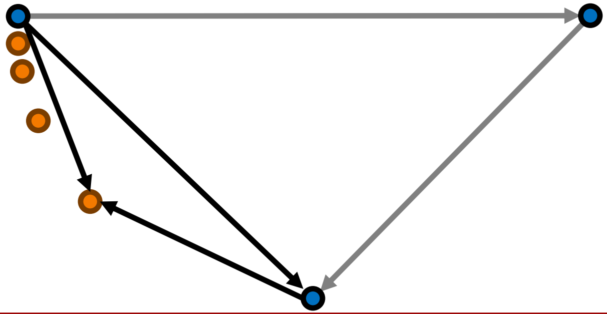

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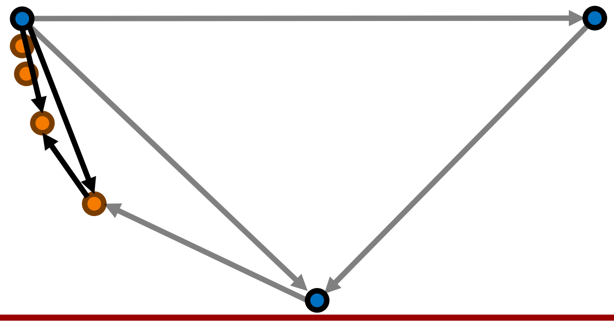


Convex Hull (2D)

QuickHull Complexity:

Like QuickSort:

- In the worst case, the complexity can be $O(n^2)$.
- In practice it is $O(n \log n)$.
- The implementation is output sensitive.

Does it extend to higher dimensions?

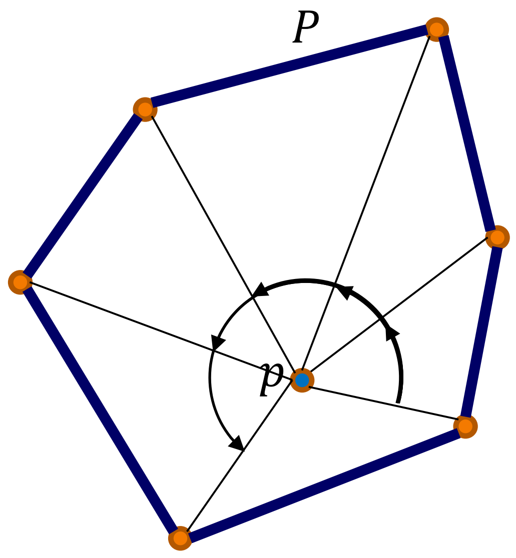
Outline

- Convex Hulls
- Algorithms
  - Naïve Implementation(s)
  - Gift Wrapping
  - Quick Hull
  - Graham’s Algorithm
  - Lower bound complexity

Convex Hull (2D)

Graham's Observation:

If $P \subset \mathbb{R}^2$ is a convex polygon and $p \in P$ is a point in the interior of the polygon, then the angle of the line segments between p and the ordered vertices of P is monotonic.



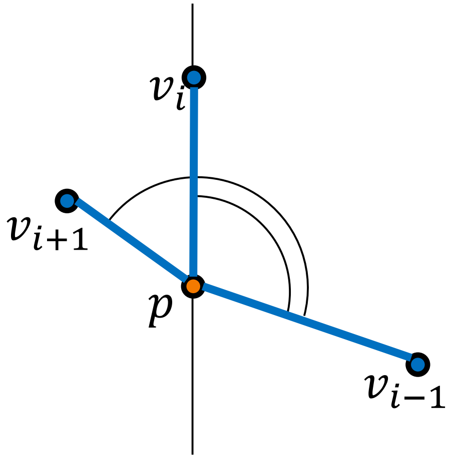
Convex Hull (2D)

Graham's Observation:

WLOG assume p and v_i lie on a vertical line with p below v_i.

Since the polygon is convex, p is to the left of $\overrightarrow{v_i v_{i+1}}$.
$\Rightarrow v_{i+1}$ is to the left of the vertical.
Since the polygon is convex, p is to the left of $\overrightarrow{v_{i-1} v_i}$.
$\Rightarrow v_{i-1}$ is to the right of the vertical.

$\Rightarrow$ The angles $\angle p v_{i-1}$, $\angle p v_i$, $\angle p v_{i+1}$ increase monotonically.



Convex Hull (2D)

GrahamScan($S \subset \mathbb{R}^2$)

- $p \leftarrow$ PointInHull(S)
- $H \leftarrow$ SortByAngle(S , p)
- while(RemoveReflexVertex(H)){ }
- return H

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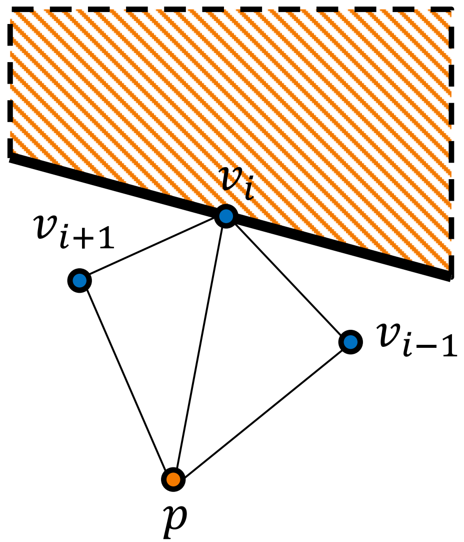

<u>Note</u>:

At every iteration, the vertices of H are sorted by angle relative to p.

$\Rightarrow$ Hull vertices can never be removed because the angle between the previous and next vertex is always convex.

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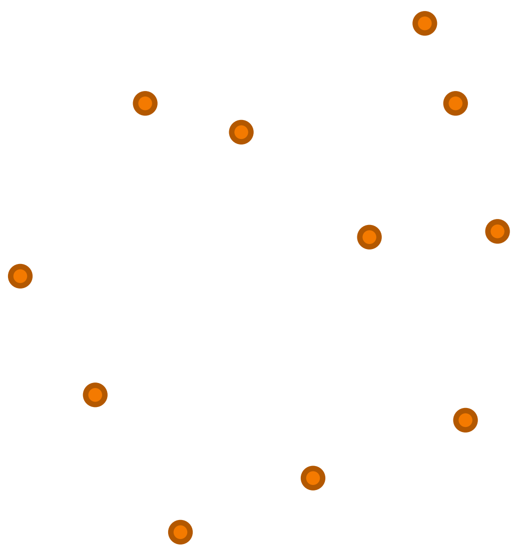
Correctness:

- The output polygon has only convex vertices.
  $\Rightarrow$ It's convex.
  $\Rightarrow H \subset \mathcal{H}(S)$.
- All hull vertices are in H.
  $\Rightarrow \mathcal{H}(S) \subset H$.

Convex Hull (2D)

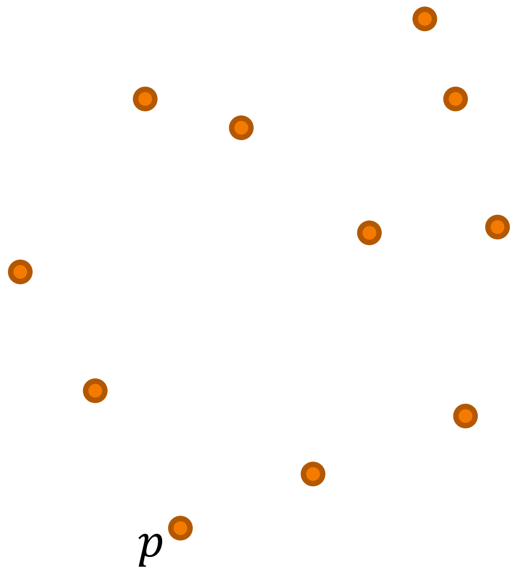
GrahamScan($S \subset \mathbb{R}^2$)



Convex Hull (2D)

GrahamScan($S \subset \mathbb{R}^2$)

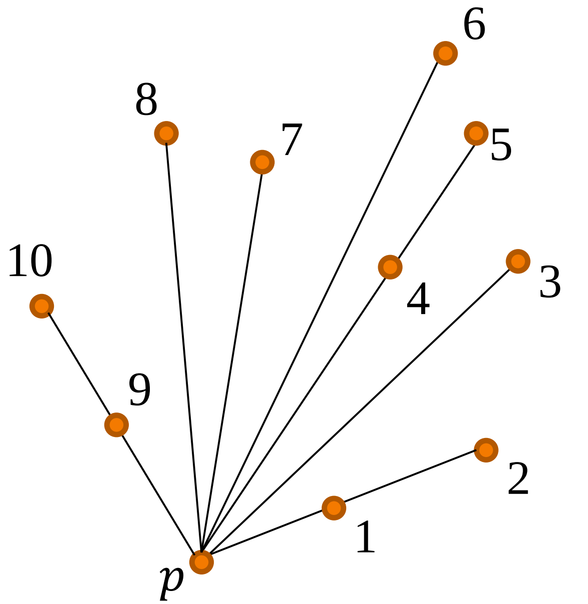
- $p \leftarrow$ BottommostRightmost(S)



Convex Hull (2D)

GrahamScan($S \subset \mathbb{R}^2$)

- $p \leftarrow$ BottommostRightmost(S)
- $\tilde{S} \leftarrow$ SortByAngleAndLength(p , $S - \{p\}$)



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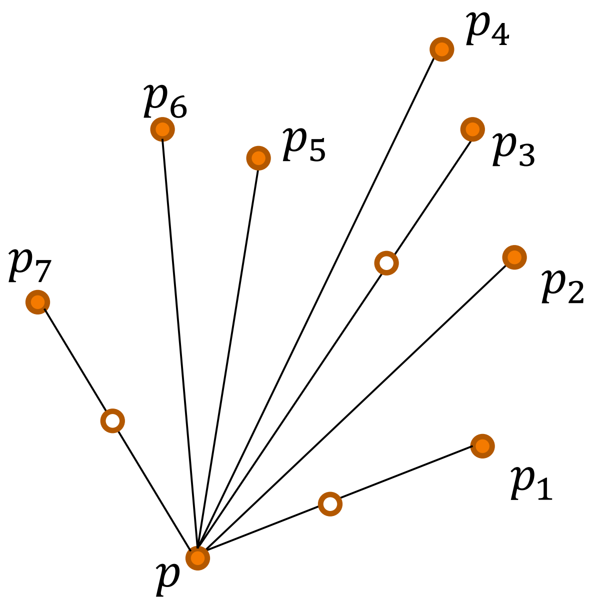

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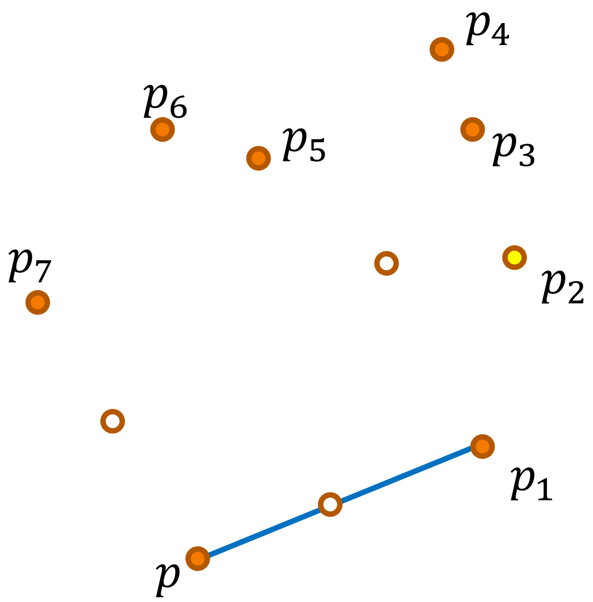
<u>Note</u>:
Since p is bottom-(right)-most, vertices are sorted by angle in $(0, \pi]$, and non-extreme points are removed, $\overline{pp_1}$ is on the hull.



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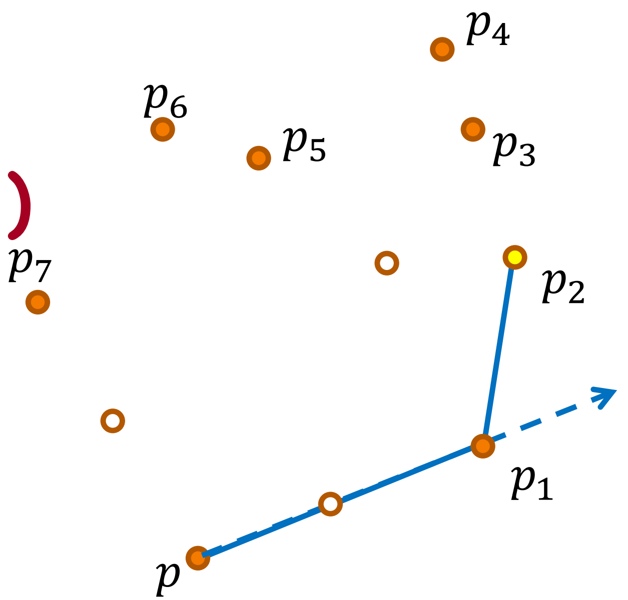

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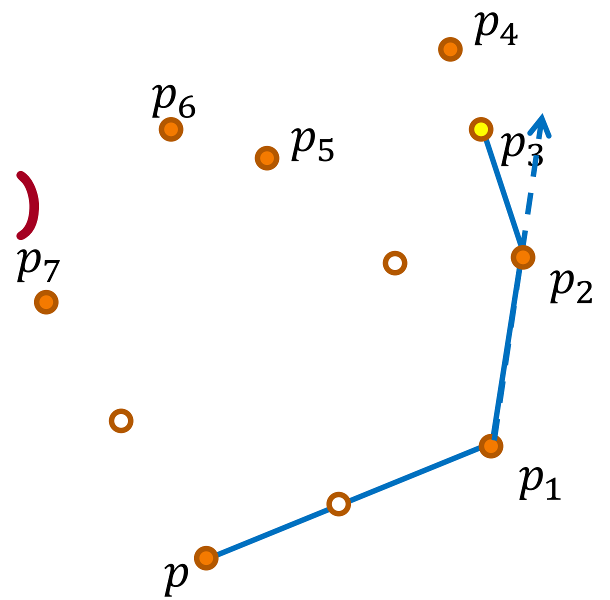

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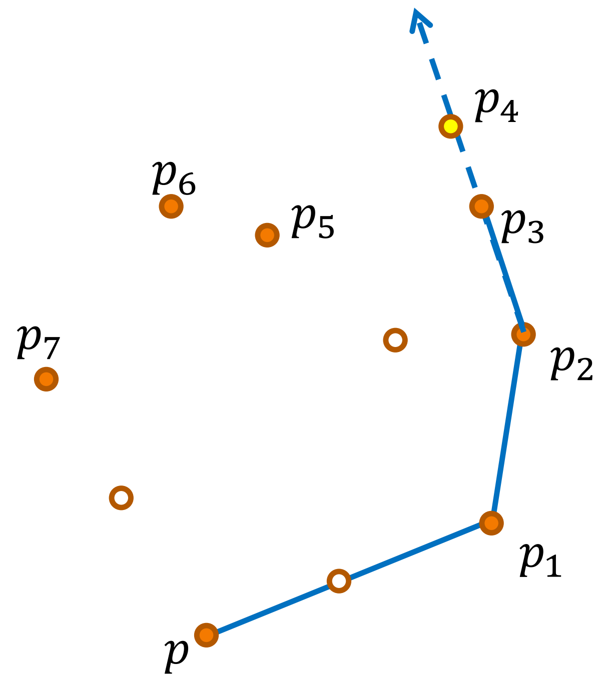

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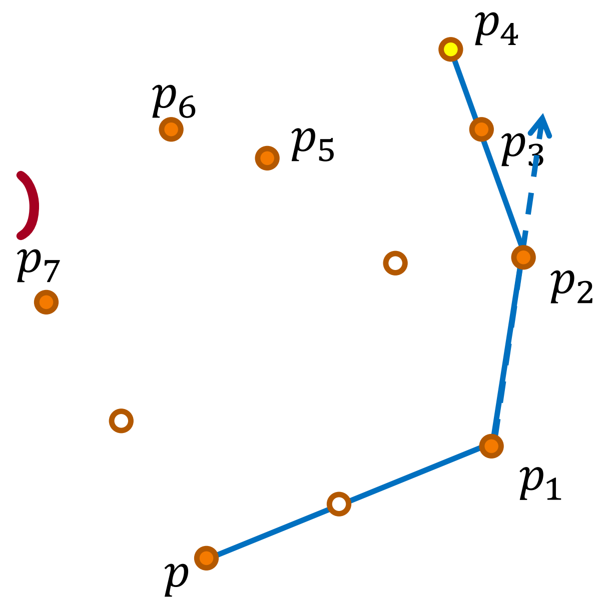

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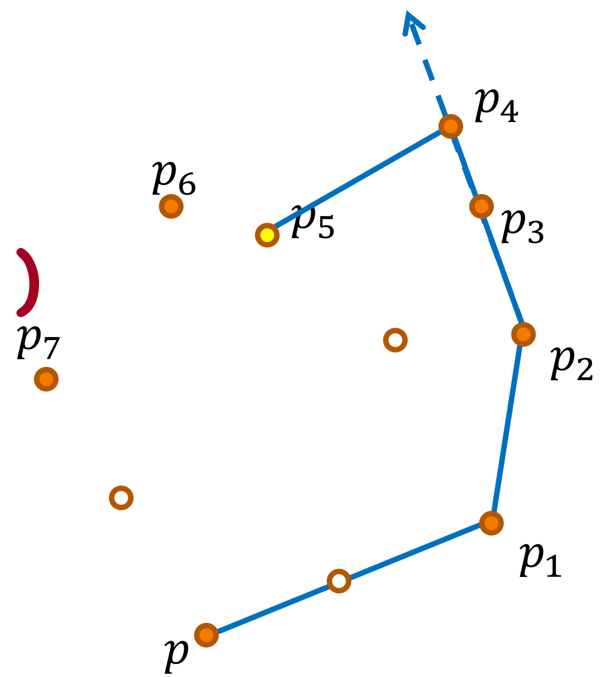

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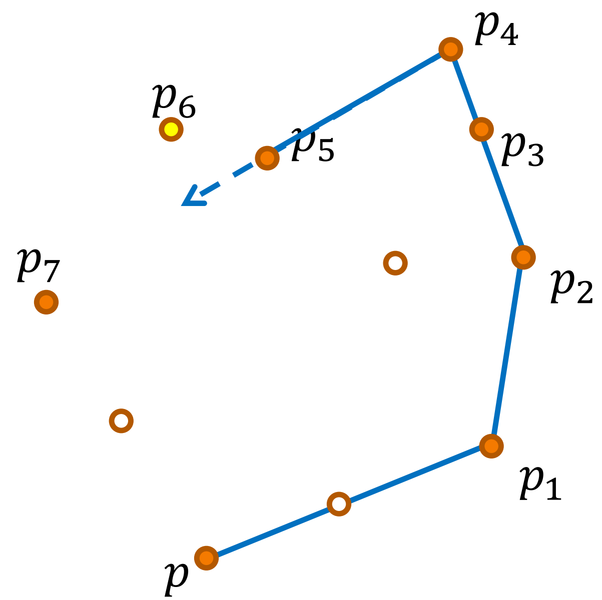

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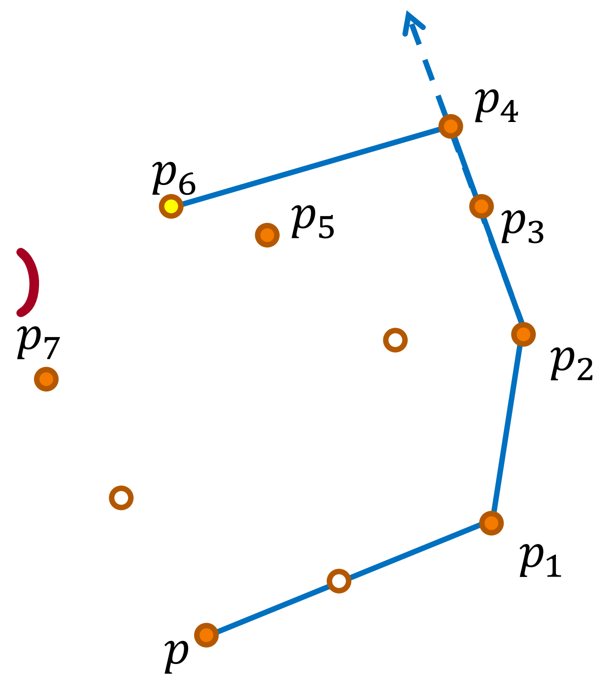

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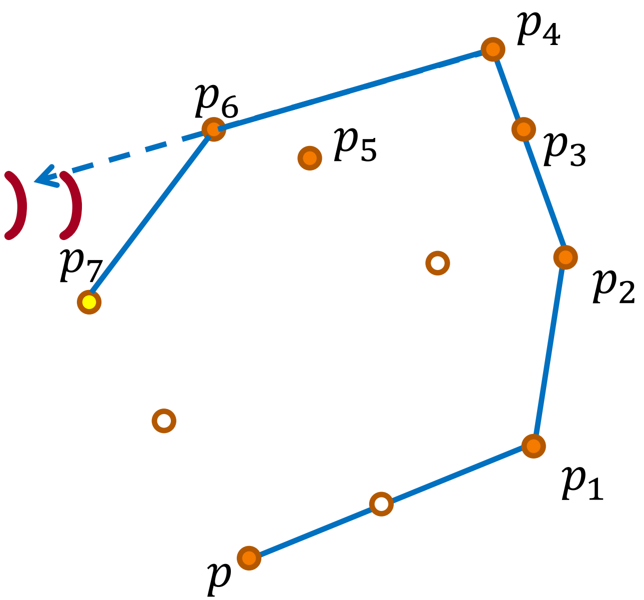

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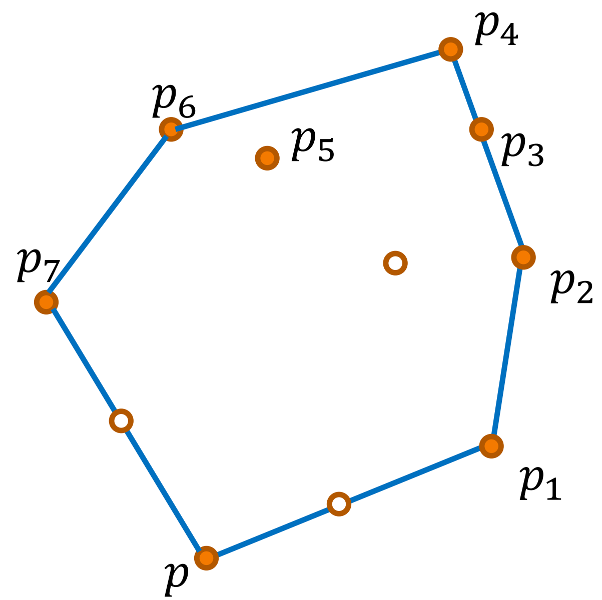

Convex Hull (2D)

GrahamScan($S \subset \mathbb{R}^2$)

- $p \leftarrow$ BottommostRightmost(S) $\qquad O(n)$
- $\tilde{S} \leftarrow$ SortByAngleAndLength(p , $S - \{p\}$) $\qquad O(n \log n)$
- if(angle(p_i) == angle(p_{i+1})): $\tilde{S} \leftarrow \tilde{S} - \{p_i\}$ $\qquad O(n)$
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$O(n)$

Outline

- Convex Hulls
- Algorithms
  - Naïve Implementation(s)
  - Gift Wrapping
  - Quick Hull
  - Graham's Algorithm
  - Lower bound complexity

Lower Bound Complexity

Recall:

Sorting n numbers has lower bound complexity $O(n \log n)$.

Approach:

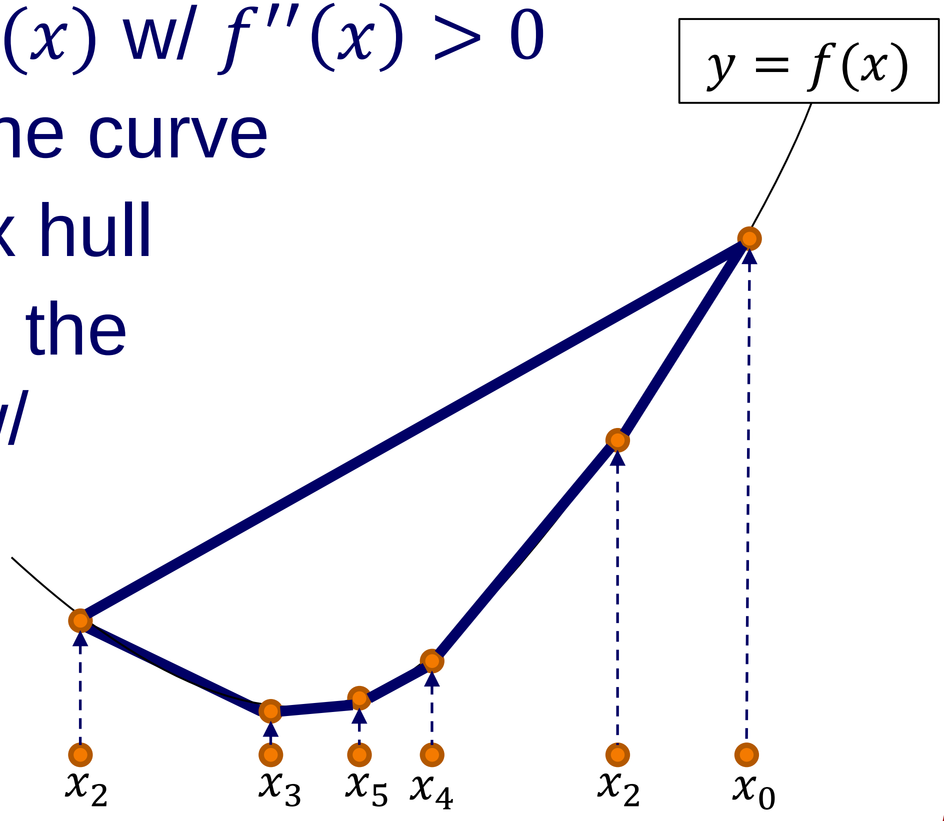
We will show that computing the 2D hull has the same complexity by reducing sorting to the problem of computing the convex hull.

Lower Bound Complexity

Sorting → Convex Hull Reduction (Shamos):

Given a set of points $\{x_i\} \subset \mathbb{R}$:

- Choose a function $f(x)$ w/ $f''(x) > 0$
- Lift the points onto the curve
- Compute the convex hull
- Return the points on the lower hull, starting w/ the left-most.



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- L
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The reduction assumes that the hull vertices are output in order.