Polygon Partitioning

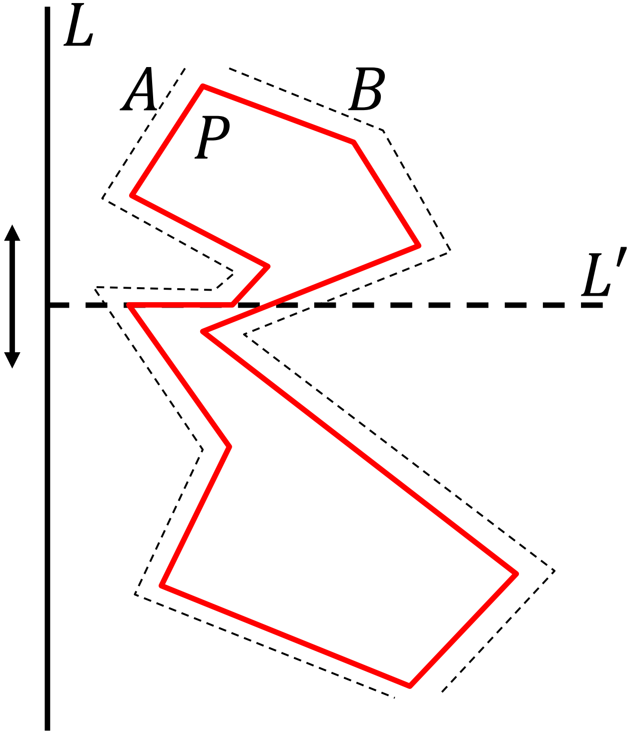
O’Rourke, Chapter 2

Outline

- Triangle Partitions
- Convex Partitions

Monotonicity

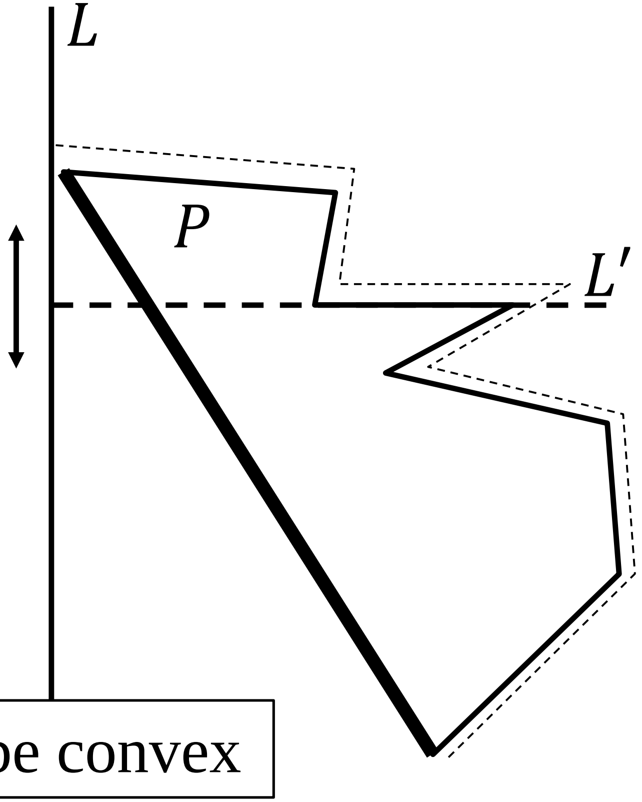
A polygonal P is *monotone w.r.t. a line* L if its boundary can be split into two polygon chains, A and B, such that each chain is monotonic w.r.t. L.



Monotonicity

A polygonal P is *monotone w.r.t. a line* L if its boundary can be split into two polygon chains, A and B, such that each chain is monotonic w.r.t. L.

A polygon P is a *monotone mountain* w.r.t. L if it is monotone w.r.t. L and one of the two chains (the *base*) is a single segment.



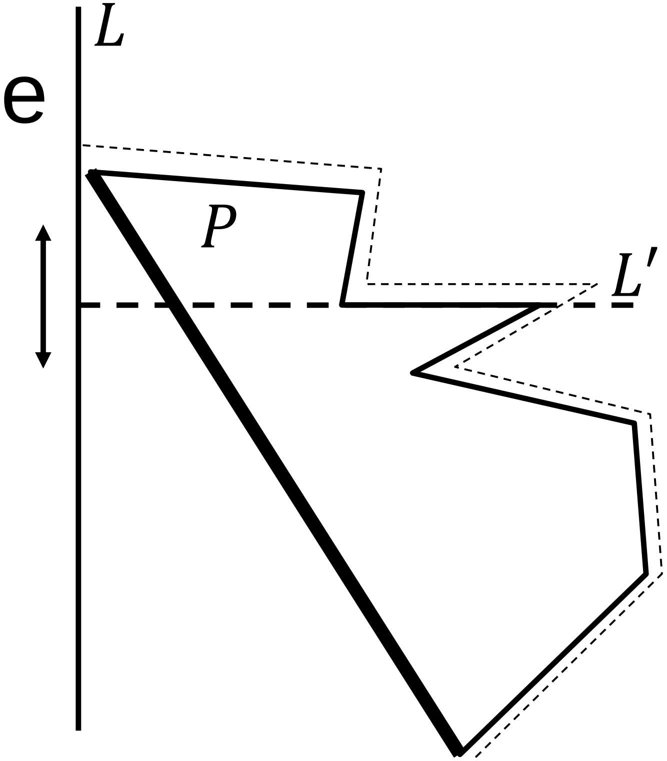
Note: Both endpoints of the base have to be convex

Claim

Every strictly convex vertex of a monotone mountain that is not on the base is an ear tip.

WLOG, we will assume that L is vertical and that the base is to the left.

We will first show that the diagonal has to be interior and then that it cannot intersect the polygon.

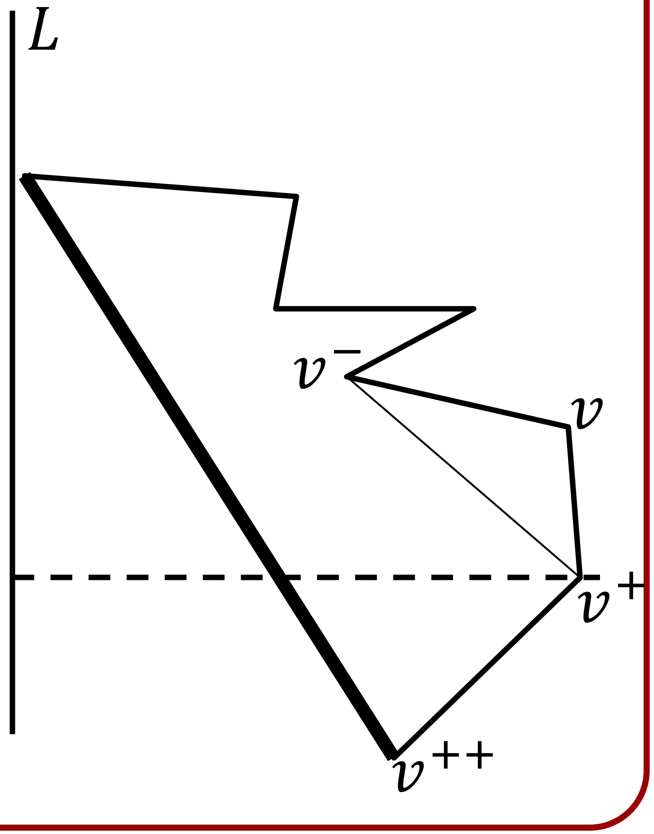


Proofs

Assume $\exists\{v^-, v, v^+\} \subset P$, with v strictly convex, s.t. $\overline{v^- v^+}$ is not an interior diagonal.

$\overline{v^+ v^-}$ cannot be locally exterior because:

- If v^+ is not on the base, then $\overline{v^+ v^{++}}$ must be below v^+. Since v is strictly convex, $\overline{v^+ v^-}$ is to the left of $\overline{v^+ v}$.

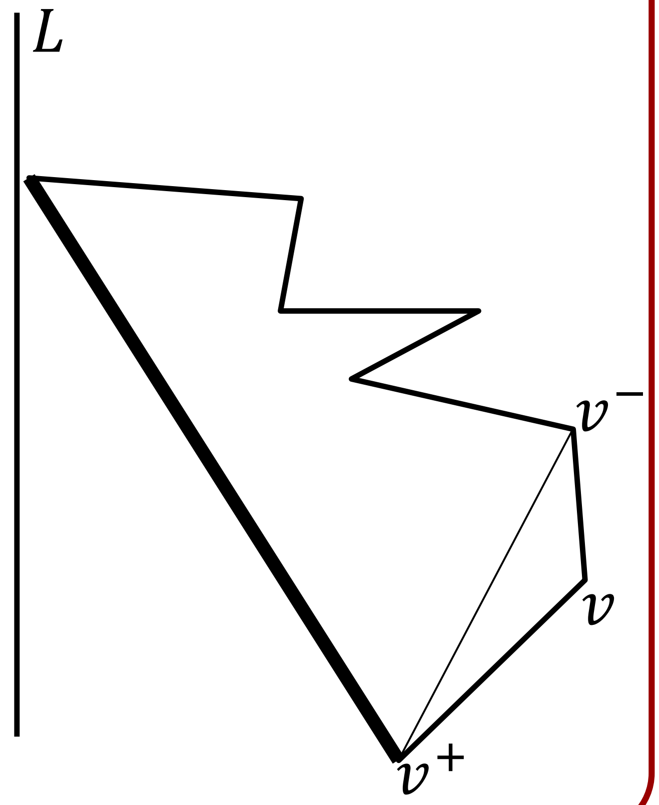


Proofs

Assume $\exists\{v^-, v, v^+\} \subset P$, with v strictly convex, s.t. $\overline{v^- v^+}$ is not an interior diagonal.

$\overline{v^+ v^-}$ cannot be locally exterior because:

- If v^+ is on the base, then $\overline{v^+ v^-}$ is to the left of $\overline{v^+ v}$. But it must also be to the right of the base since v^- is to the right of the base.



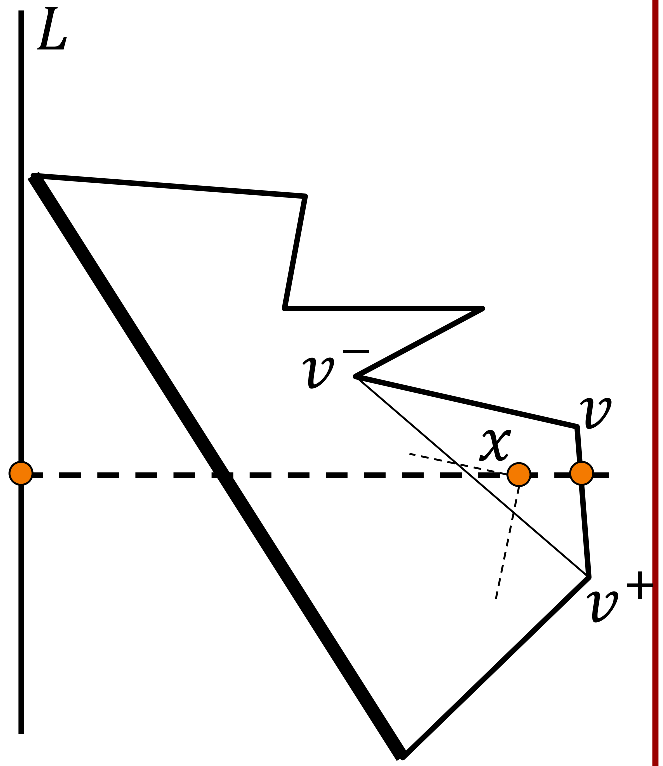
Proofs

Assume $\exists\{v^-, v, v^+\} \subset P$, with v strictly convex, s.t. $\overline{v^- v^+}$ is not an interior diagonal.

$\Rightarrow$ If $\overline{v^- v^+}$ is not an interior diagonal then $\exists x \in P$, reflex, with x interior to $\Delta v^+ v v^-$.

- Interior $\Rightarrow$ it cannot lie on the chain $\overline{v^+ v v^-}$.
- Reflex $\Rightarrow$ it is not on the base.

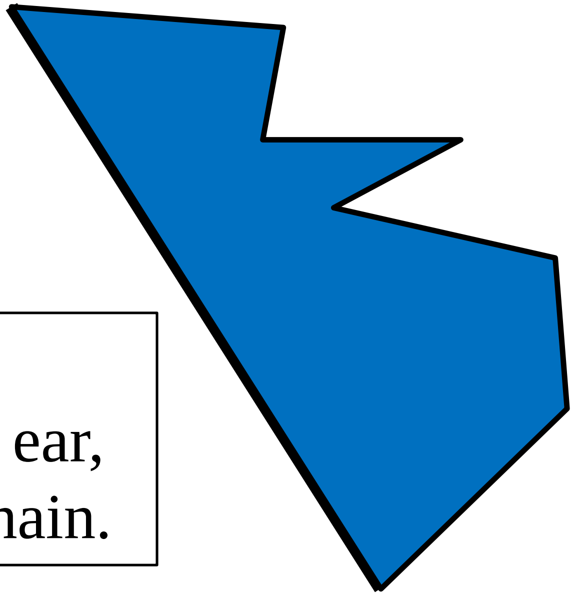
$\Rightarrow$ The horizontal through x intersects P in three points.



Monotone Mountain Triangulation

Approach:

1. Find a convex vertex not on the base.
2. Remove it.
3. Go to step 1.

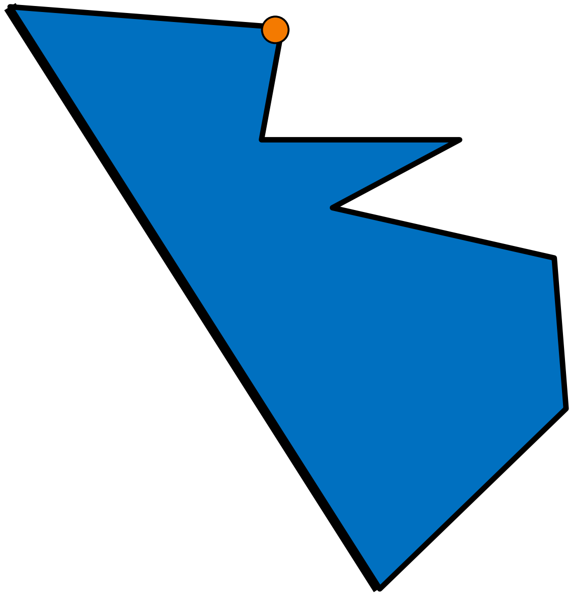


Note:
We can induct because when we remove an ear, we do not violate the monotonicity of the chain.

Monotone Mountain Triangulation

Approach:

1. **Find a convex vertex not on the base.**
2. Remove it.
3. Go to step 1.



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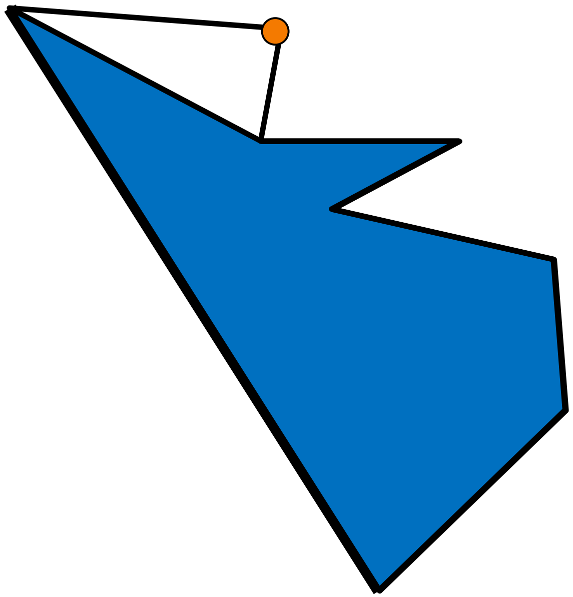

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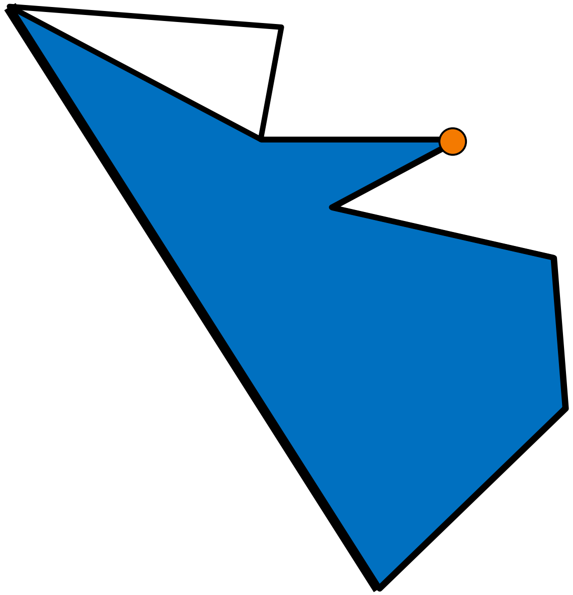

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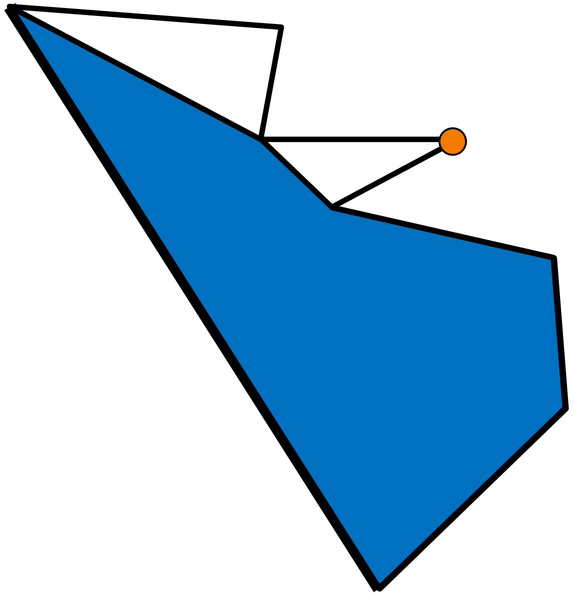

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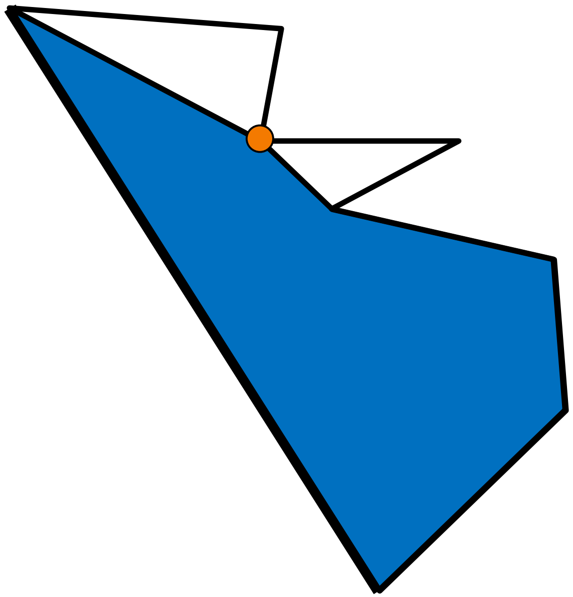

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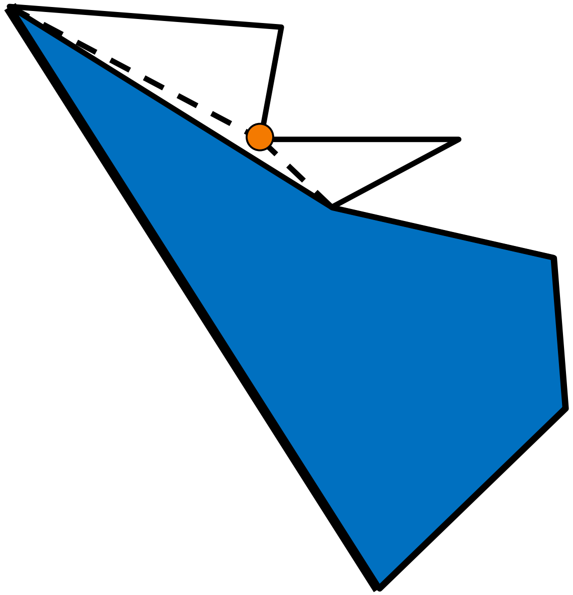

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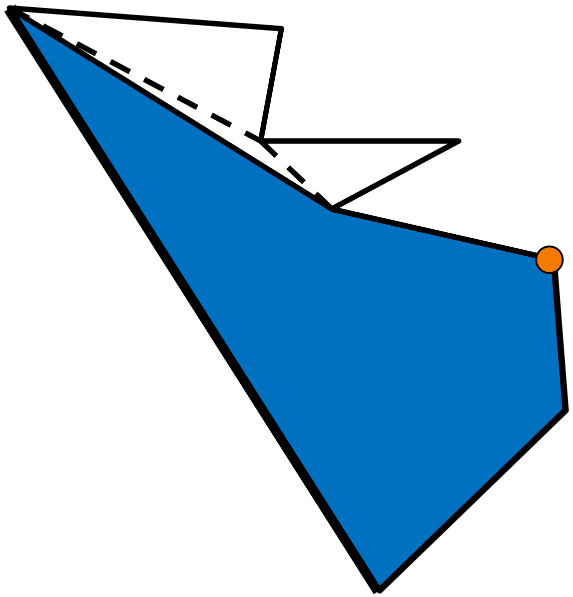

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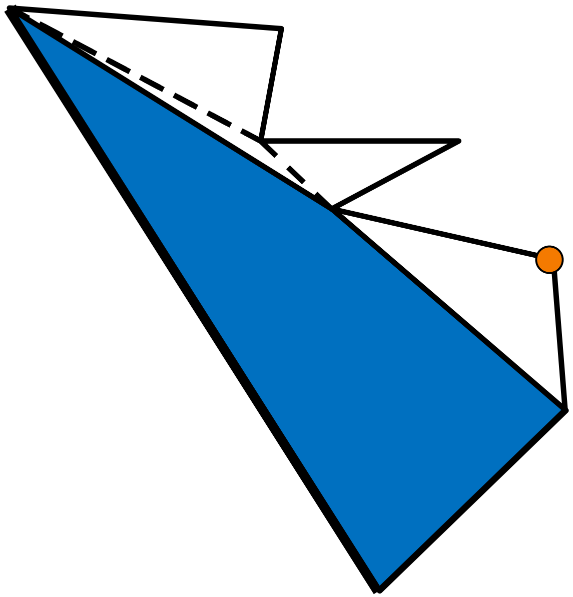

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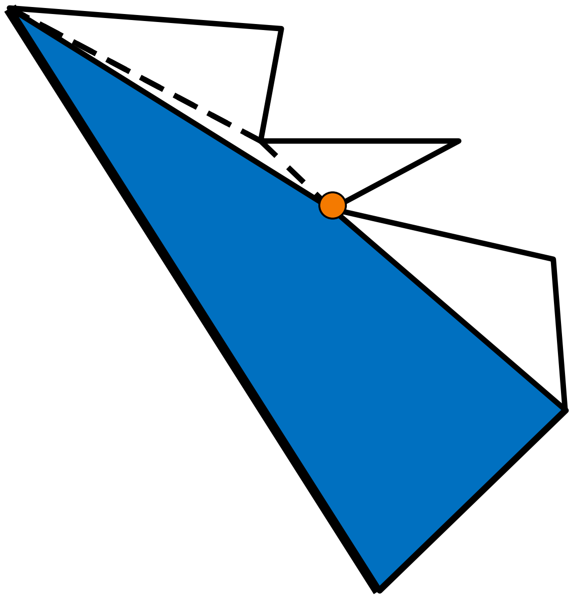

Monotone Mountain Triangulation

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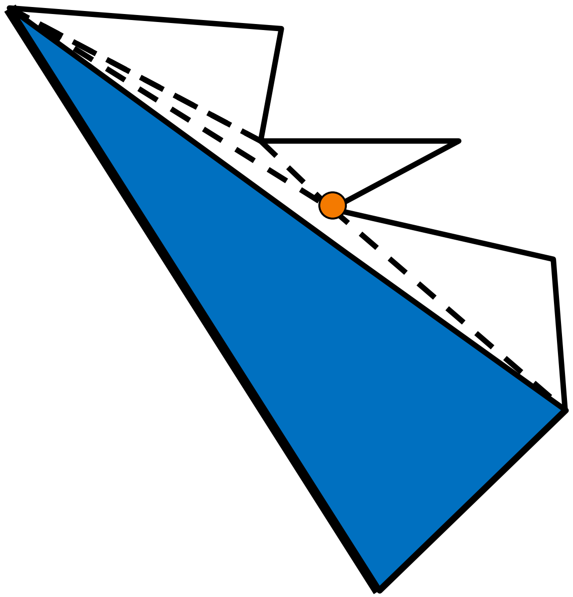
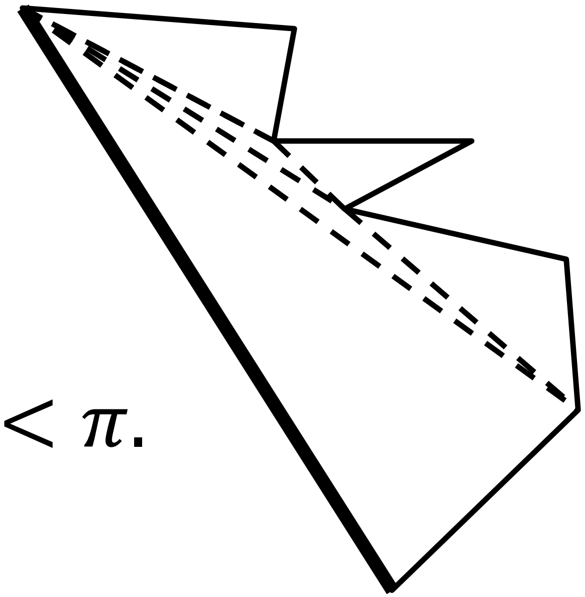


Monotone Mountain Triangulation

Approach:

1. Find a convex vertex not on the base.
2. Remove it.
3. Go to step 1.

To do this, we need to be able to quickly identify the next convex vertex.

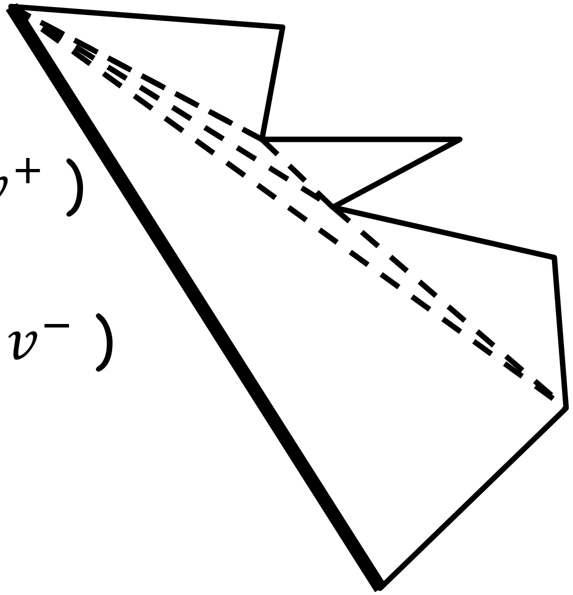


Recall:

Strictly convex ⇔ interior angle < π.

Monotone Mountain Triangulation

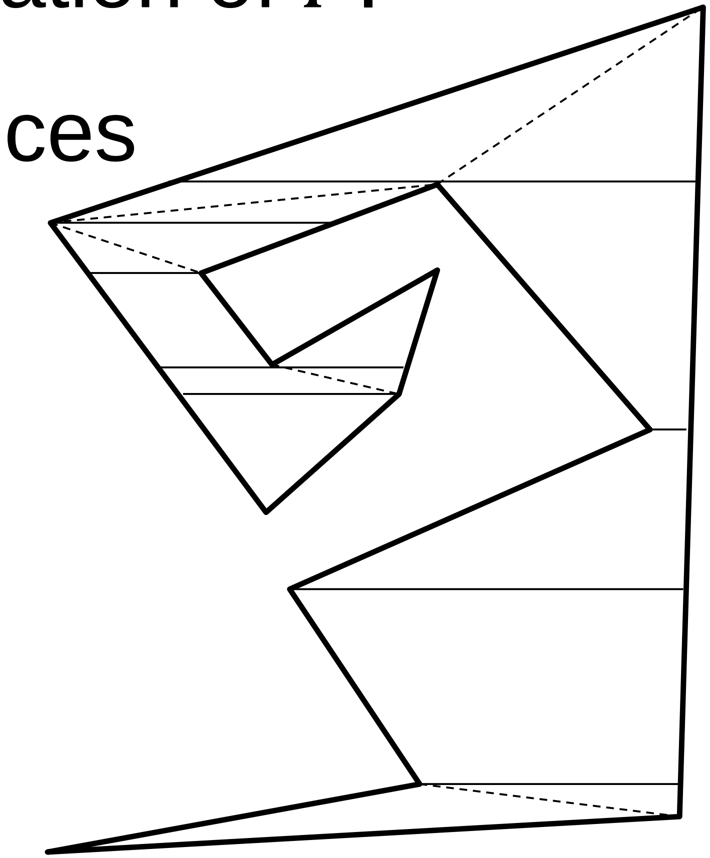
```
○ MonotoneMountainTriangulation( $P$ ):
  – $B$ ←FindBase( $P$ )
  – $C$ ← LinkConvexVertices( $P - B$ )
  – while $C \neq \emptyset$:
      • $v \leftarrow$ First( $C$ )
      • output( $\Delta v^- v v^+$ )
      • $P \leftarrow P - \{v\}$
      • if($v^+ \notin C$ AND $v^+ \notin B$ )
          • if( $\angle v^+ < \pi$ ) $v$.addAfter( $v^+$ )
      • if( $v^- \notin C$ AND $v^- \notin B$ )
          • if( $\angle v^- < \pi$ ) $v$.addBefore( $v^-$ )
      • $C \leftarrow C - \{v\}$
```



Generating Monotone Mountains

Approach:

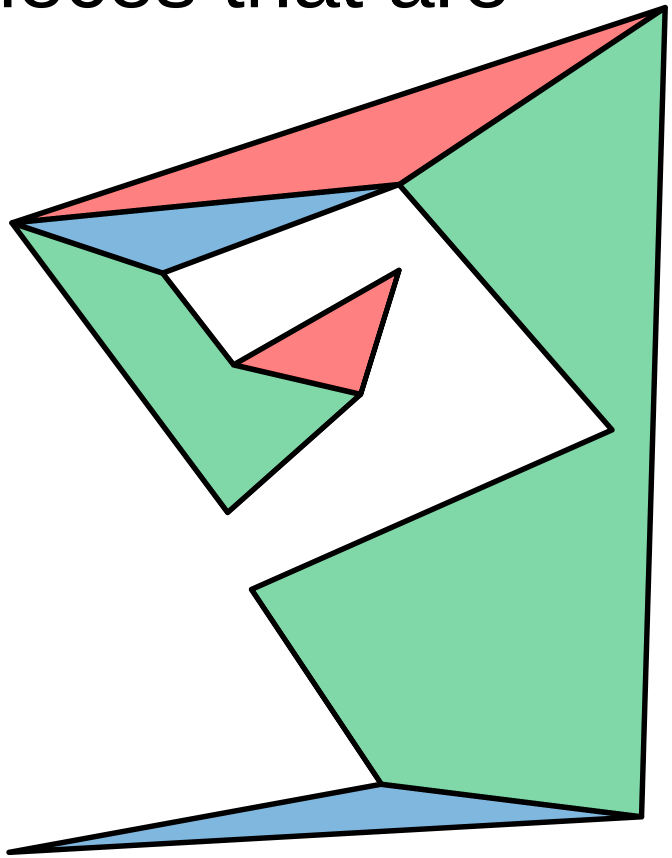
- Compute a trapezoidalization of P.
- Connect supporting vertices that don't come from the same side.



Generating Monotone Mountains

Claim:

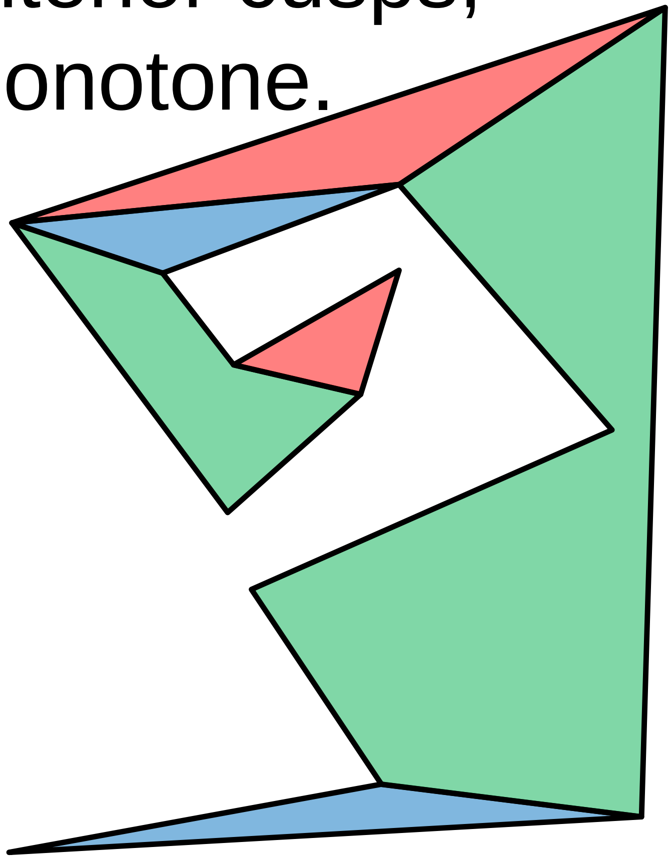
Such a partition generates pieces that are monotone mountains.



Generating Monotone Mountains

Proof:

This algorithm removes all interior cusps, so the partition pieces are monotone.

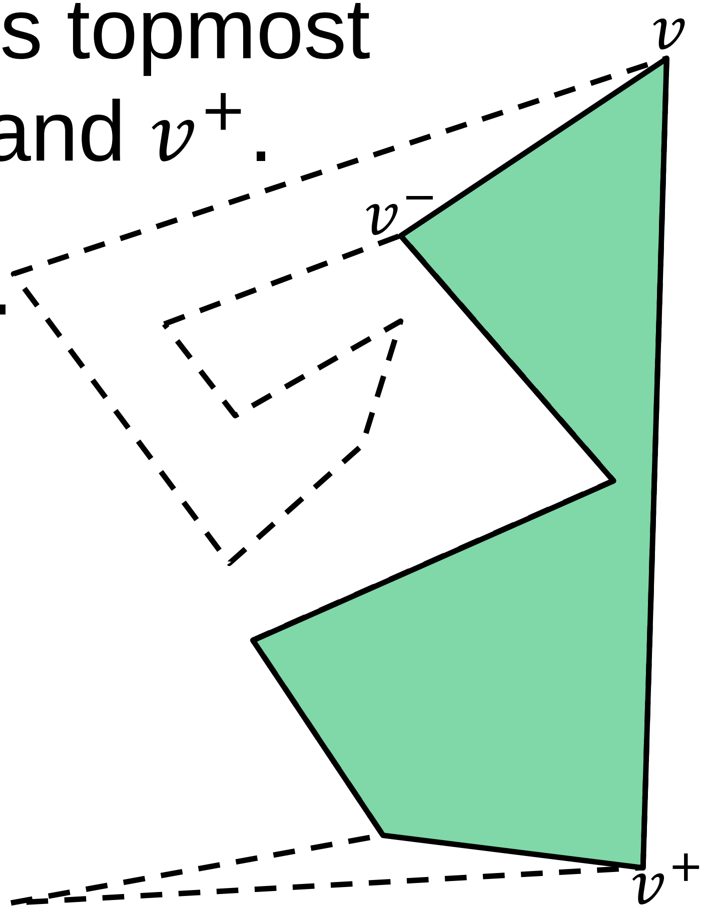


Generating Monotone Mountains

Proof:

Given a piece, we can find its topmost vertex v, with neighbors v^- and v^+.

Assume v^+ is lower than v^-.



Generating Monotone Mountains

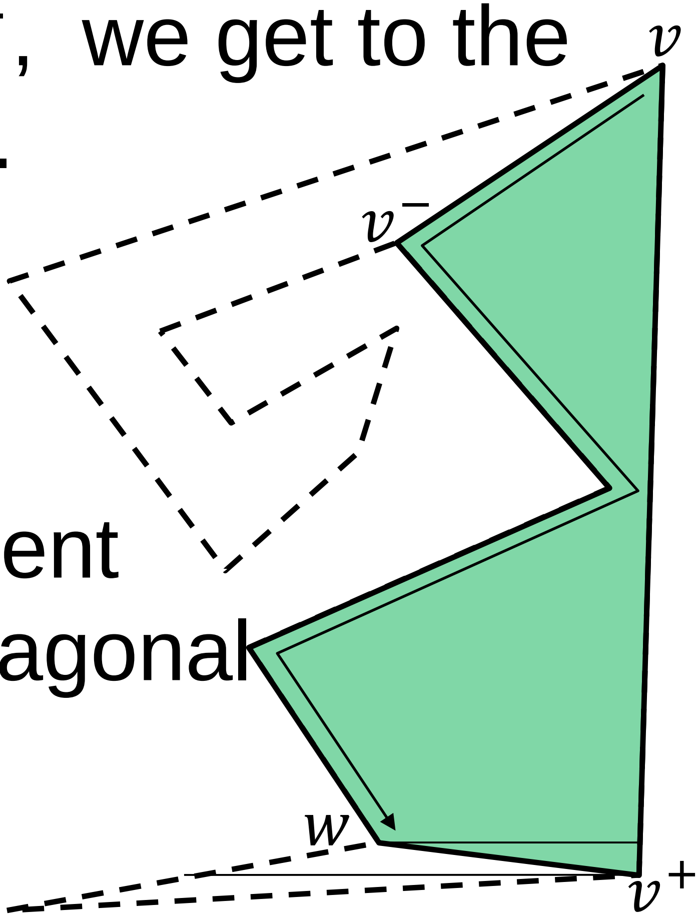
Proof:

Following the chain down v^{-}, we get to the last vertex, $w \in P$, above v^{+}.

There is a trapezoid supported by w and v^{+}.

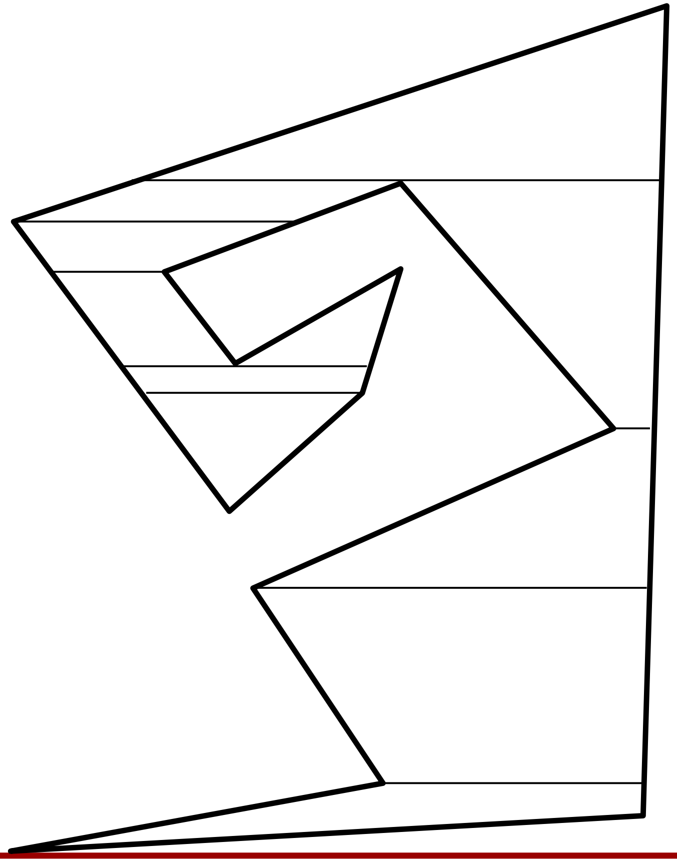
Since w and v^{+} are on different sides of the trapezoid, the diagonal to v^{+} must be added.

$\Rightarrow v^{+}$ is a partition endpoint



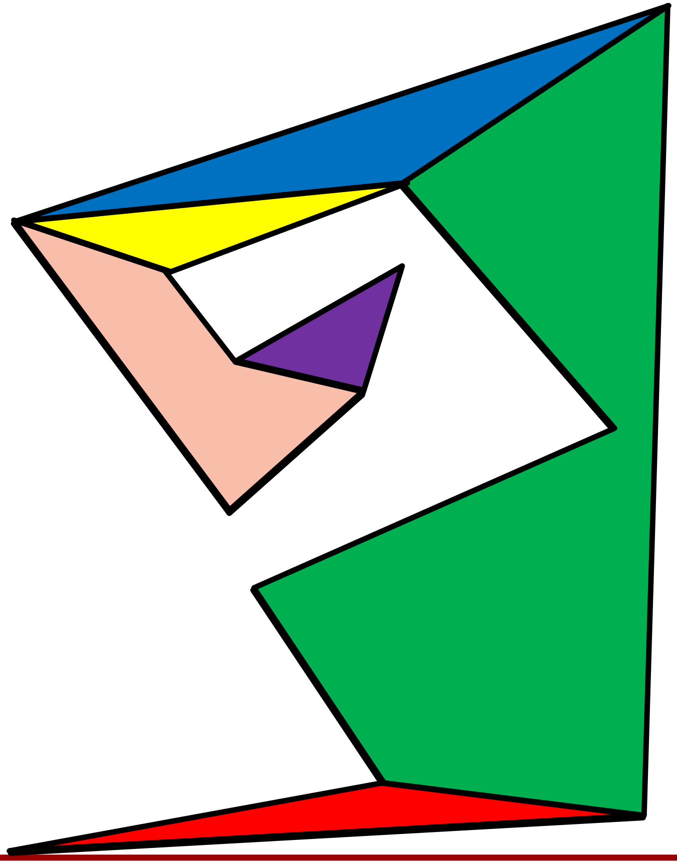
Triangulation

- Triangulate(P):
  - Construct a trapezoidalization



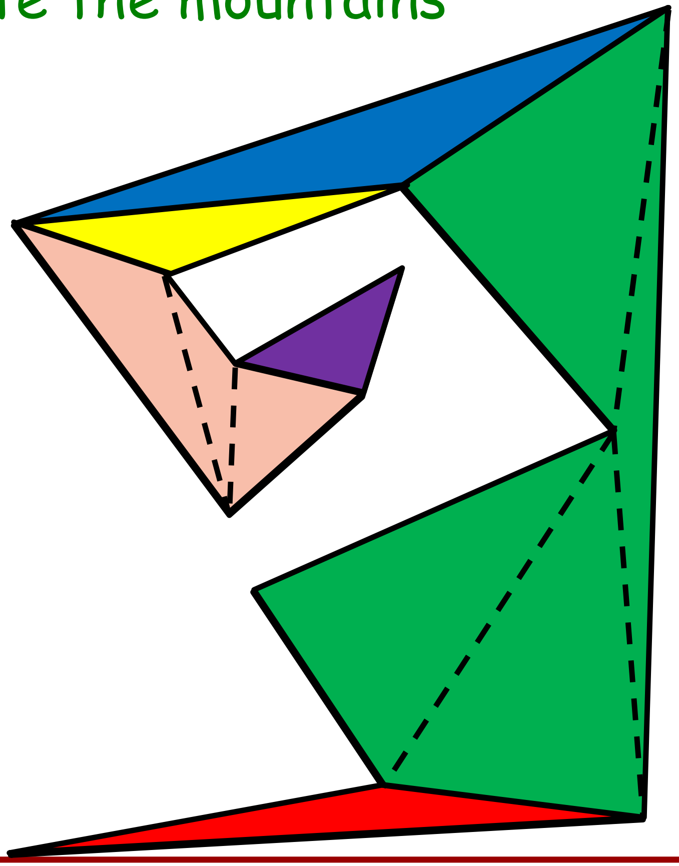
Triangulation

- Triangulate(P):
  - Construct a trapezoidalization
  - Connect supporting vertices from different sides.



Triangulation

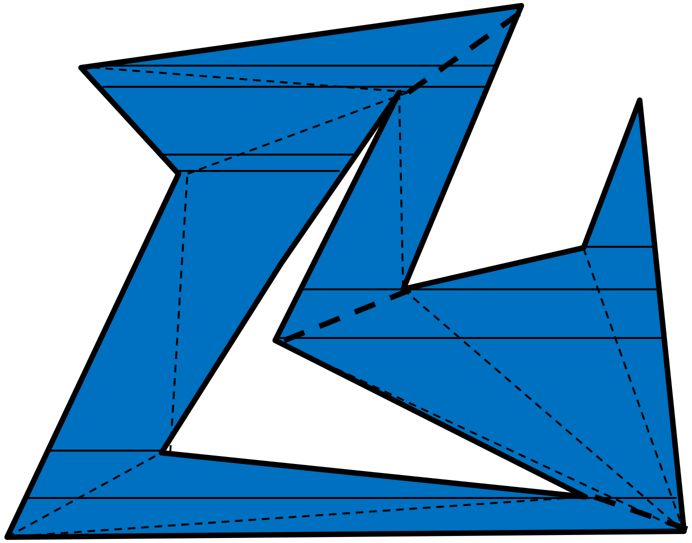
- Triangulate(P):
  - Construct a trapezoidalization
  - Connect supporting vertices from different sides.
  - Triangulate the mountains



Triangulation

Note:

The algorithms for triangulating via monotone polygons and monotone mountains works for polygons with disconnected boundaries.



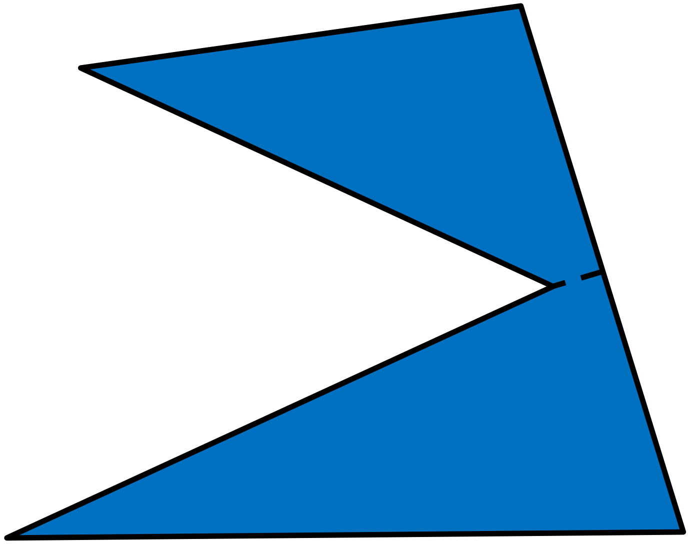
Outline

- Triangle Partitions
- Convex Partitions

Convex Partitions

Definition:

A *convex partition* *by segments* of a polygon P is a decomposition of P into convex polygons obtained by introducing arbitrary segments.

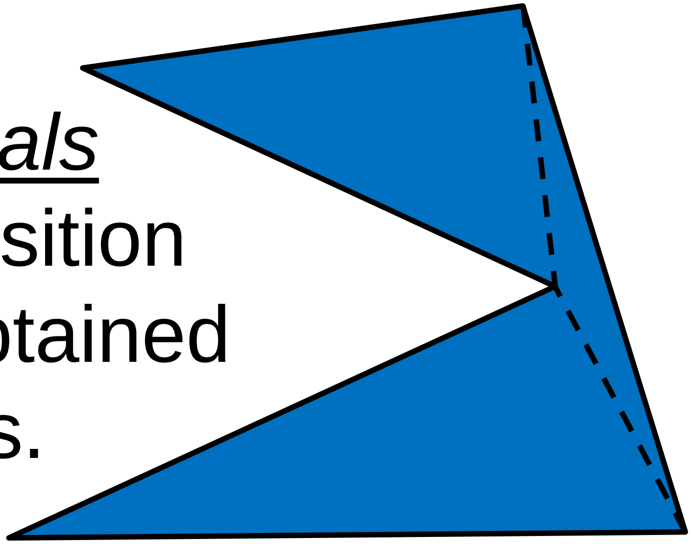


Convex Partitions

Definition:

A *convex partition by segments* of a polygon P is a decomposition of P into convex polygons obtained by introducing arbitrary segments.

A *convex partition by diagonals* of a polygon P is a decomposition of P into convex polygons obtained by only introducing diagonals.



Convex Partitions

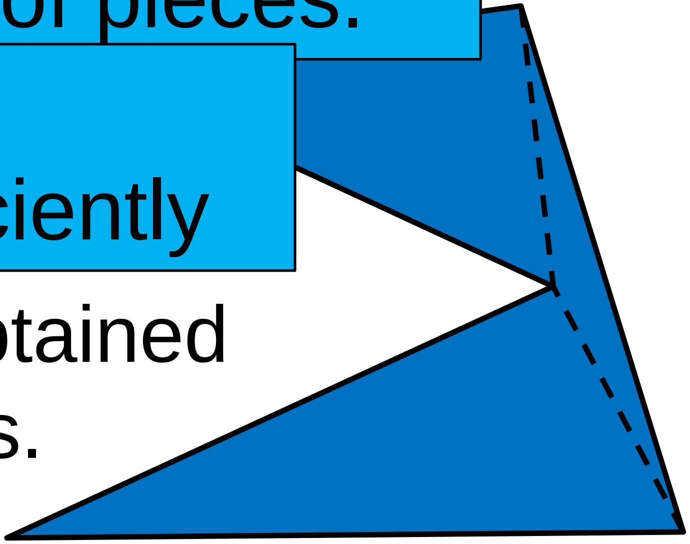
Definition:

A *convex partition by segments* of a polygon P is
a dec s
obtai s.

Challenge:
Compute a convex partition with the smallest number of pieces.

Challenge^2:
Compute it efficiently

A *convex pa*
of a polygo
of P into convex polygons obtained
by only introducing diagonals.

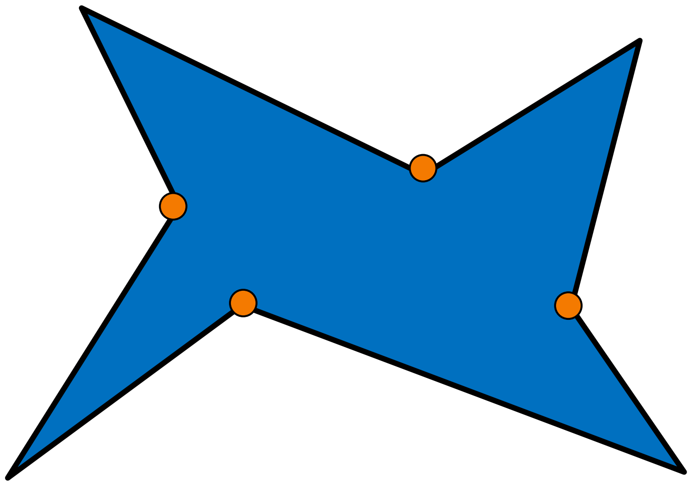


Convex Partitions (by Segments)

Claim (Chazelle):

Assume the polygon P has r reflex vertices.
If Φ is the fewest number of polygons required for a convex partition by segments of P then:

$$\lceil r/2 \rceil + 1 \leq \Phi \leq r + 1$$



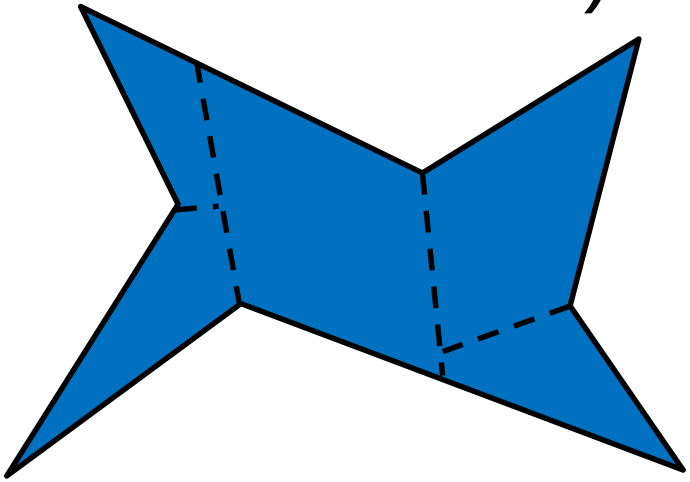
Convex Partitions (by Segments)

Proof ($\Phi \leq r+1$):

For each reflex vertex, add the bisector.

Because the segment bisects, the reflex angle splits into two convex angles. (Angles at the new vertices have to be $< \pi$.)

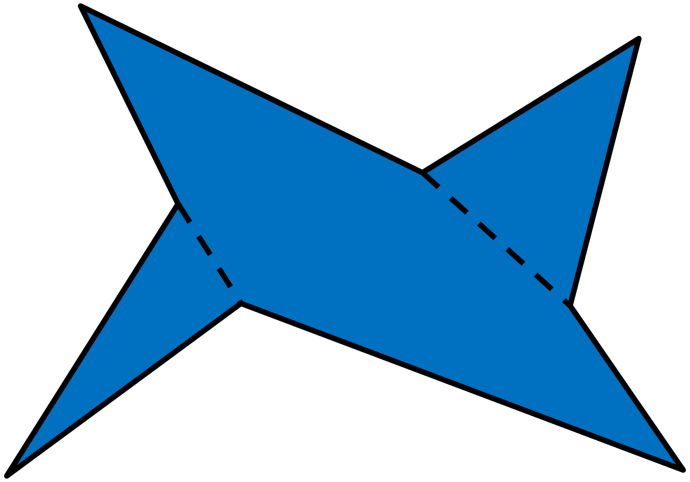
Doing this for each reflex vertices, gives a convex partition with $r+1$ pieces.



Convex Partitions (by Segments)

Proof ($\lceil r/2 \rceil \leq \Phi$):

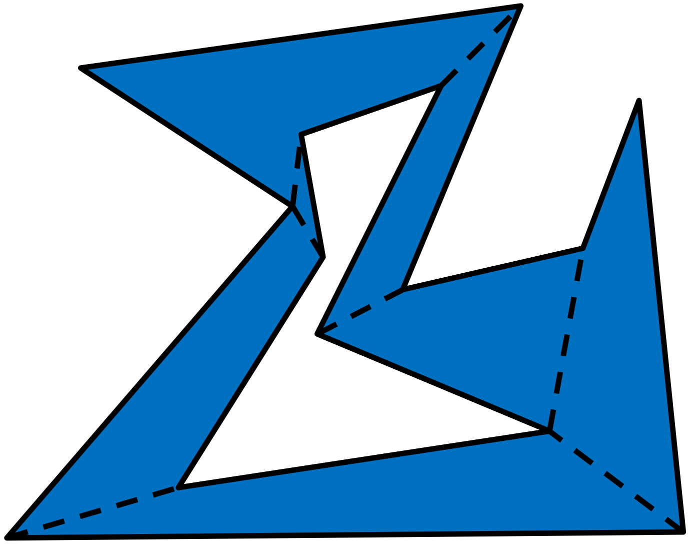
Each reflex vertex needs to be split and each introduced segment can split at most two reflex vertices.



Convex Partitions (by Diagonals)

Definition:

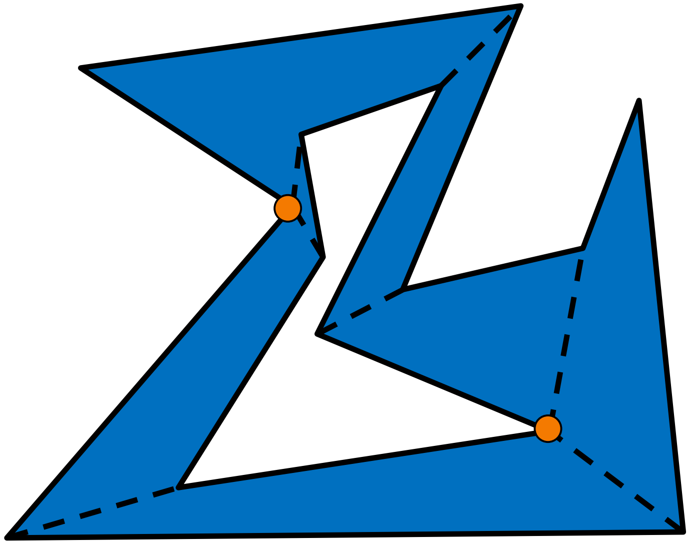
A diagonal in a convex partition is *essential for vertex* $v \in P$ if removing the diagonal creates a piece that is not convex at v.



Convex Partitions (by Diagonals)

<u>Claim</u>:

If v is a reflex vertex, it can have at most two essential diagonals.



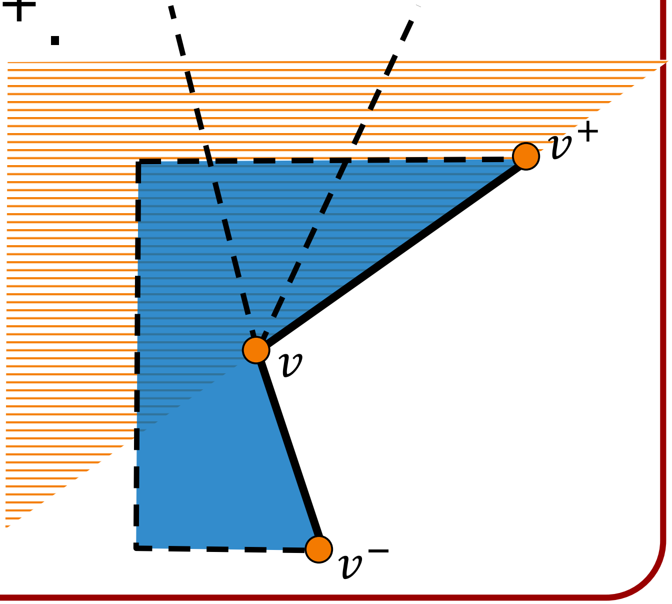
Convex Partitions (by Diagonals)

Proof:

Given a reflex vertex v, let v^- and v^+ be the vertices immediately before and after v in P.

There can be at most one essential segment in the half-space to the right of $\overrightarrow{vv^+}$.

(If there were two, we could remove the one closer to $\overrightarrow{vv^+}$.)

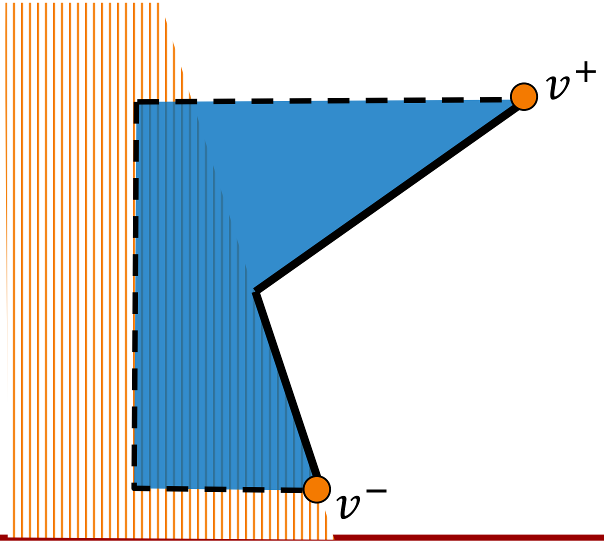


Convex Partitions (by Diagonals)

Proof:

Given a reflex vertex v, let v^- and v^+ be the vertices immediately before and after v in P.

Similarly, there can be at most one essential segment in the half-space to the right of $\overrightarrow{vv^-}$.



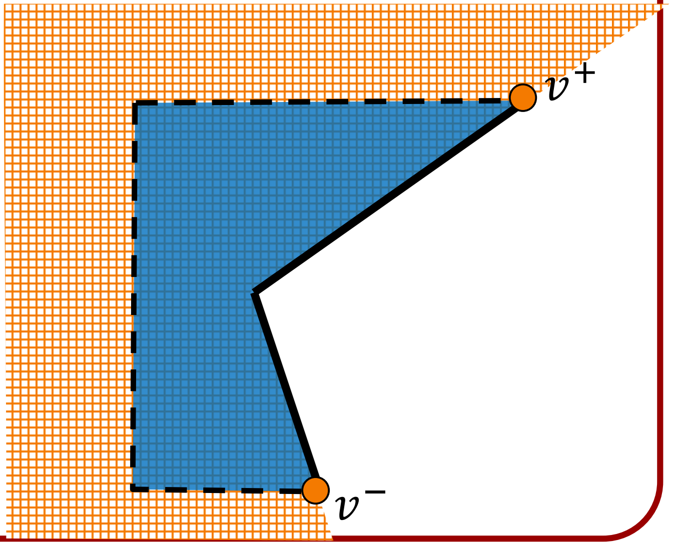
Convex Partitions (by Diagonals)

Proof:

Given a reflex vertex v, let v^- and v^+ be the vertices immediately before and after v in P.

Similarly, there can be at most one essential segment in the half-space to the right of $\overrightarrow{vv^-}$.

Since the two half-spaces cover the interior of the vertex there are at most two essential vertices at v.



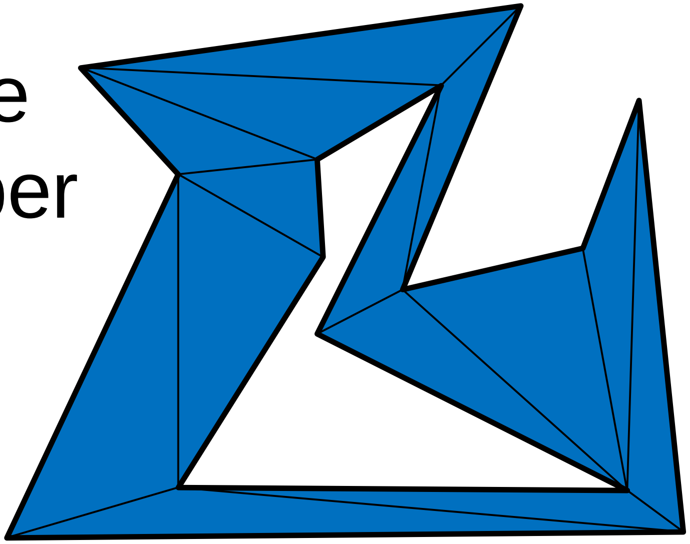
Convex Partitions (by Diagonals)

Algorithm (Hertl & Mehlhorn):

Start with a triangulation and remove inessential diagonals.

Claim:

This algorithm is never worse than $4 \times$ optimal in the number of convex pieces.



Convex Partitions (by Diagonals)

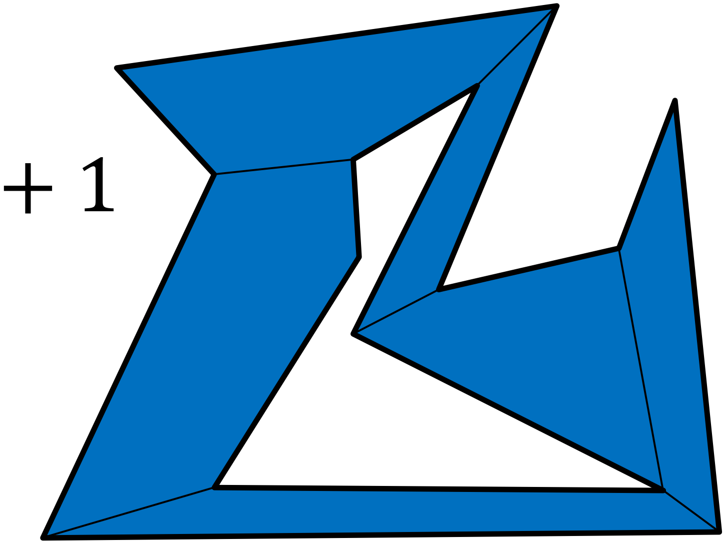
Proof:

When the algorithm terminates, every remaining diagonal is essential for some (reflex) vertex.

Each reflex vertex can have at most two essential diagonals.

$\Rightarrow$ There can be at most $2r+1$ pieces in the partition.

Since at least $\lceil r/2 \rceil + 1$ are required, the result is within $4 \times$ optimal.



Convex Partitions

Why do we care?

Convex polygons are easier to intersect against.

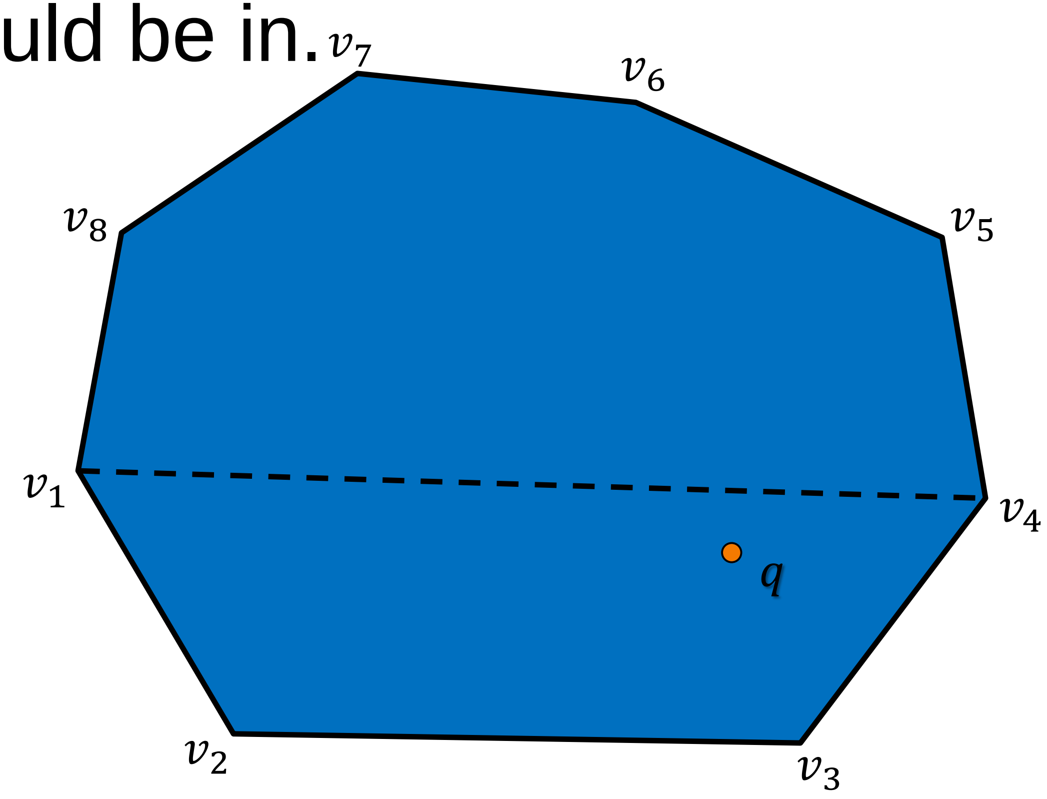
For a polygon with n vertices:

- Testing if a point is inside is $O(\log n)$.
- Testing if a line intersects is $O(\log n)$.
- Testing if two polygons intersect is $O(\log n)$.

Point/Convex-Polygon Intersection

Algorithm:

Recursively split the polygon in half and test the half the point could be in.



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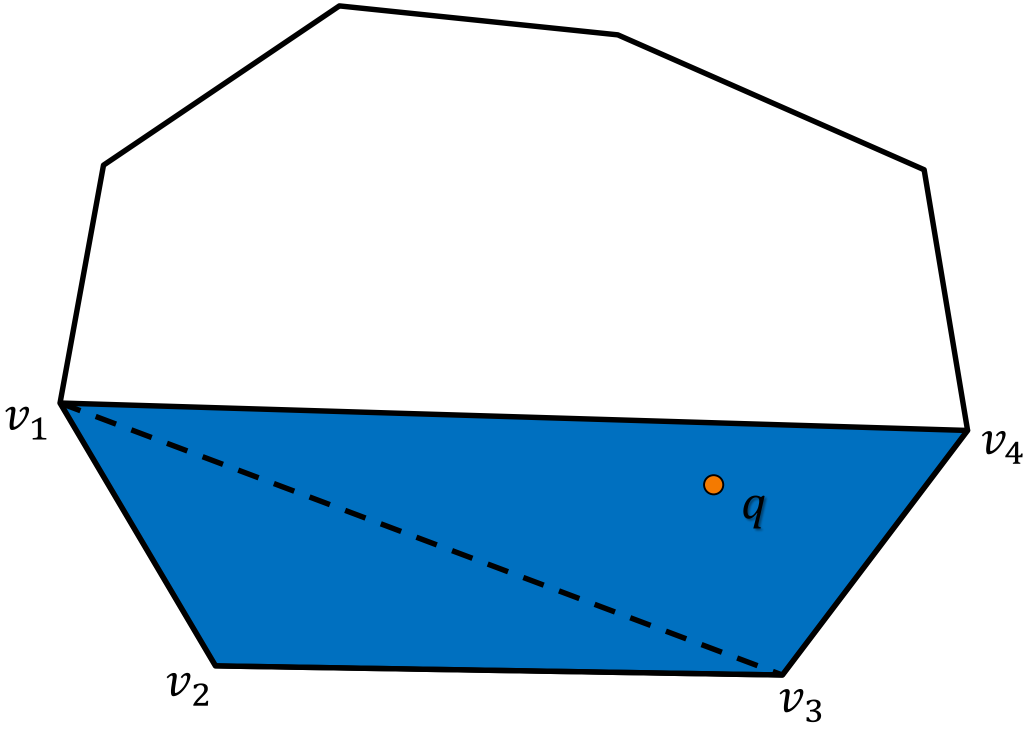

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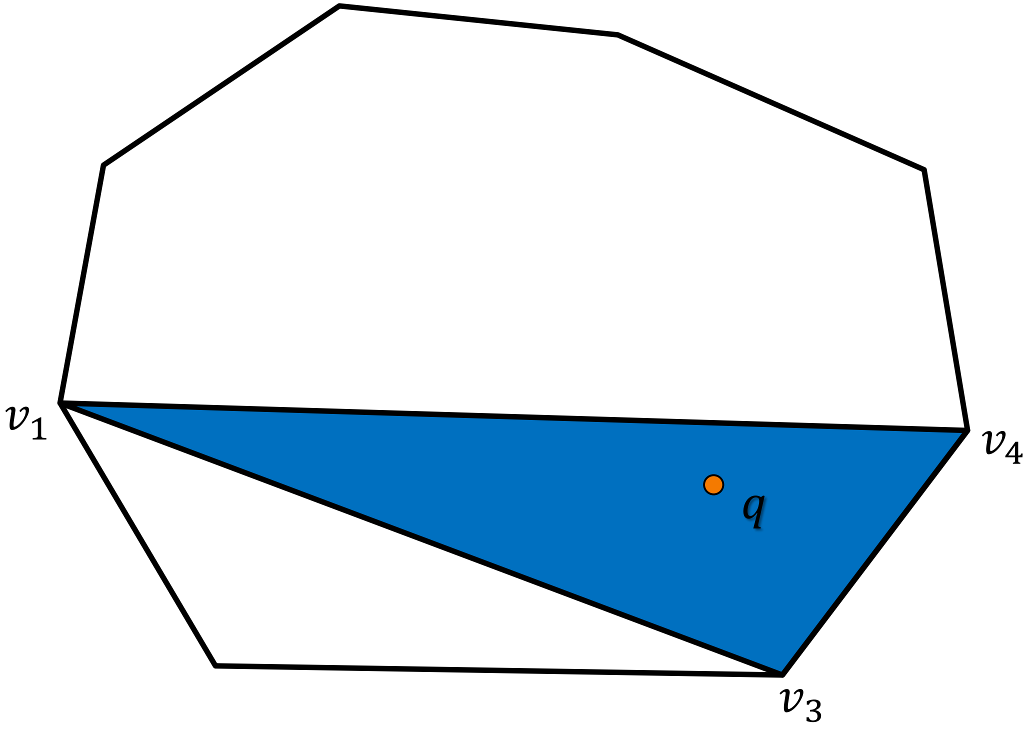


Point/Convex-Polygon Intersection

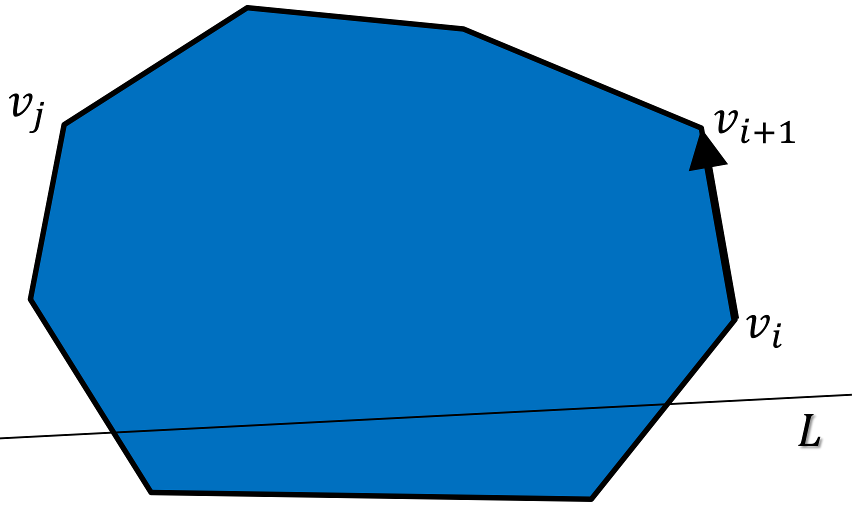
```
InConvexPolygon( $q$ , $\{v_1, \dots, v_n\}$ )
    ◦ if( $n == 3$ )
        » return InTriangle( $q$ , $\{v_1, v_2, v_3\}$ );
    ◦ if( Left( $v_1, v_{n/2}$ , $p$ ) )
        » return InConvexPolygon( $q$ , $\{v_{n/2}, \dots, v_1\}$ );
    ◦ else
        » return InConvexPolygon( $q$ , $\{v_1, \dots, v_{n/2}\}$ );
```

Line/Convex-Polygon Intersection

Note:

Given a convex polygon P, a line segment L, and vertices $v_i, v_j \in P$ on the same side of L.[*]

If the vector $\overrightarrow{v_i v_{i+1}}$ points away from L, then L can only intersect P along the chain $\{v_j, v_{j+1}, \dots, v_i\}$.



Otherwise, L can only intersect P along the chain $\{v_i, v_{i+1}, \dots, v_j\}$.

[*] Assume, WLOG that v_i is closer to L than v_j.

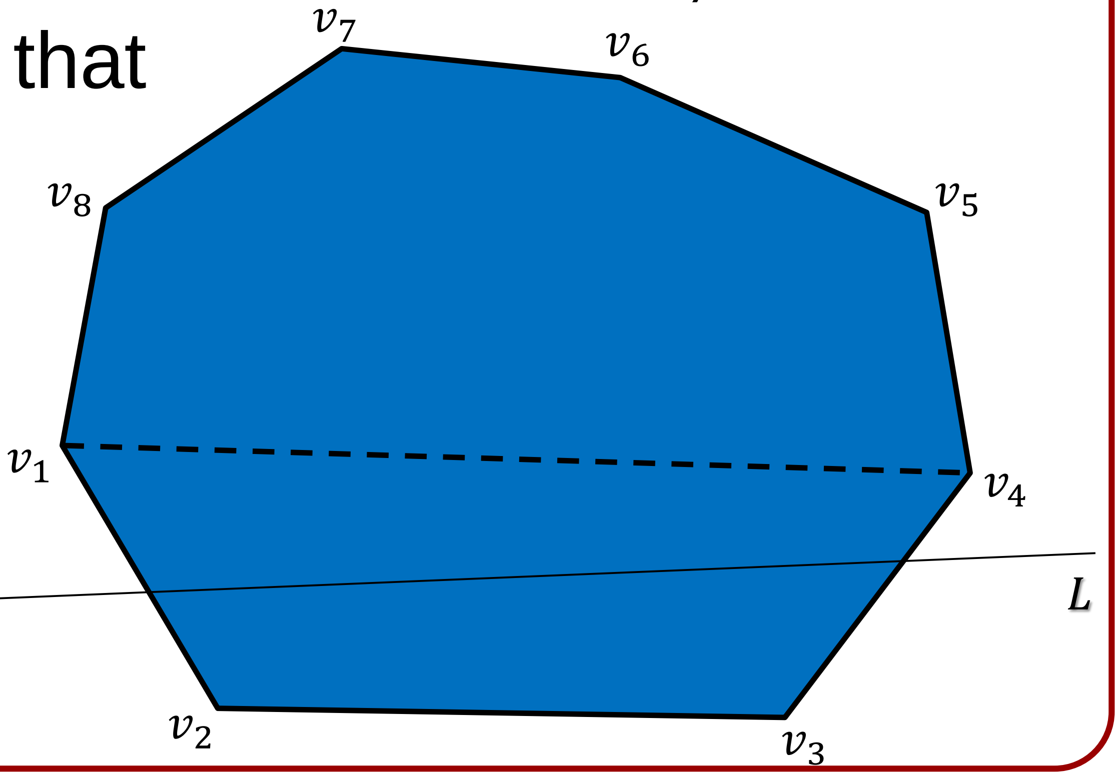
Line/Convex-Polygon Intersection

Algorithm:

Recursively split the polygon in half.

If the split points are on the same side, test the half of the polygon that could intersect.

Otherwise there is an intersection.



Line/Convex-Polygon Intersection

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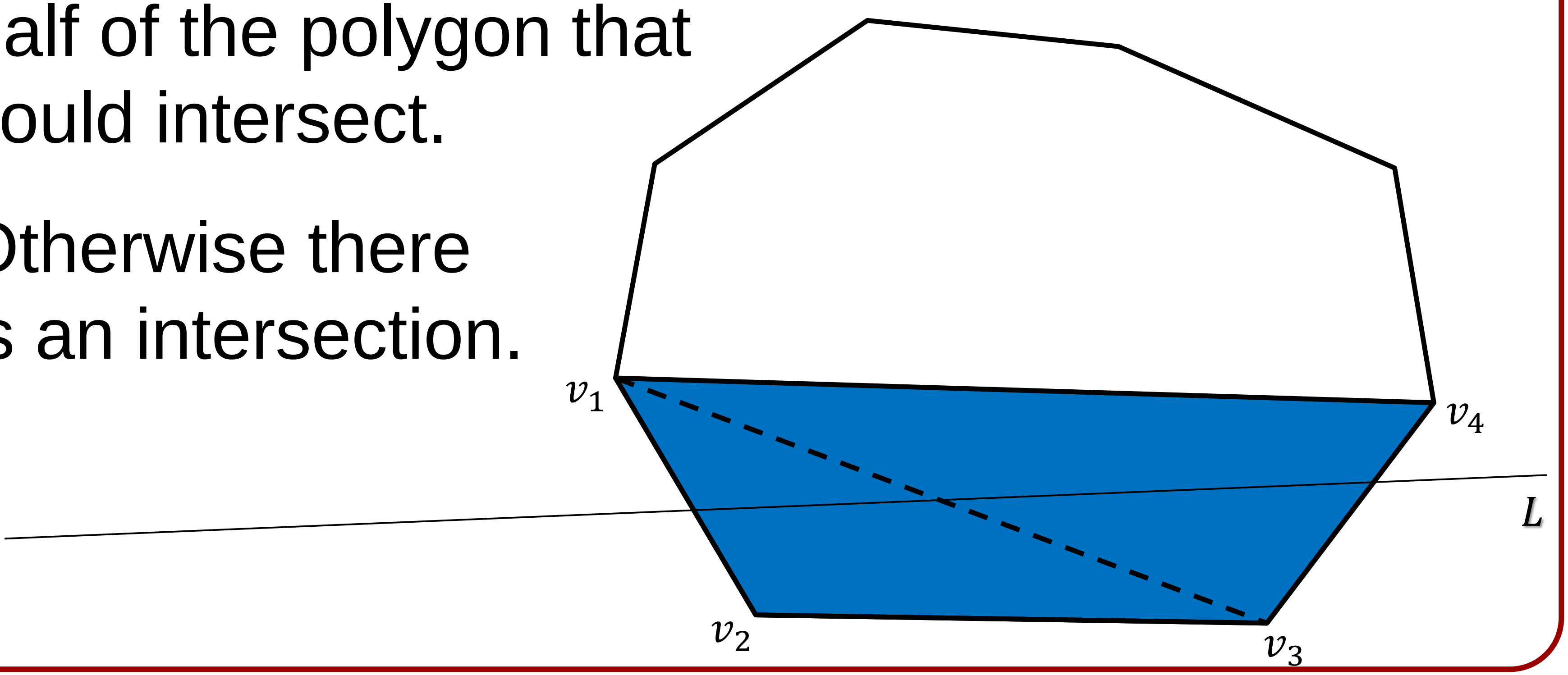
If the split points are on the same side, test the half of the polygon that could intersect.

Otherwise there is an intersection.



Line/Convex-Polygon Intersection

```
IsectConvexPolygon( $\{l_1, l_2\}$ , $\{v_1, \ldots, v_n\}$ )
    ◦ if( $n == 3$ )
        » return IsectTriangle( $\{l_1, l_2\}$ , $\{v_1, v_2, v_3$ );
    ◦ if( Left( $l_1$ , $l_2$ , $v_1$ )!=Left( $l_1$ , $l_2$ , $v_{n/2}$ ) )
        » return true;
    ◦ else
        » if( Dist( $\{l_1, l_2\}$ , $v_1$ )<Dist( $\{l_1, l_2\}$ , $v_{n/2}$ )
            – if( Dist( $\{l_1, l_2\}$ , $v_1$ )<Dist( $\{l_1, l_2\}$ , $v_2$ )
                • return IsectConvexPolygon( $\{l_1, l_2\}$ , $\{v_{n/2}, \ldots, v_1\}$ );
            – else
                • return IsectConvexPolygon( $\{l_1, l_2\}$ , $\{v_1, \ldots, v_{n/2}\}$ );
        » else…
```