Polygon Triangulation

O'Rourke, Chapter 1

Outline

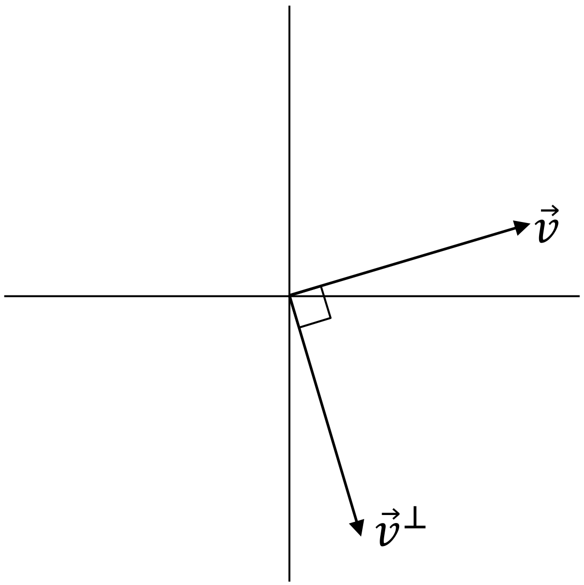
- Polygon Area
- Implementation

Notation

Given a vector $\vec{v} \in \mathbb{R}^2$, we set $\vec{v}^\perp$ to be the clockwise rotation of $\vec{v}$ by $90°$ degrees.

If $\vec{v} = (x, y)$ then we have:

$$\vec{v}^\perp = (y, -x)$$

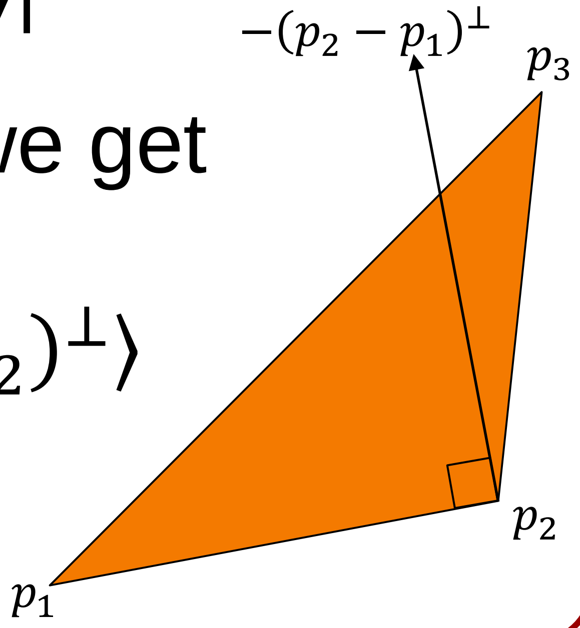


Triangle Area

Given a triangle $T = \{p_1, p_2, p_3\}$, the area of the triangle is half the base times the height:

$$2 \cdot |T| = \|p_2 - p_1\| \cdot \left| \left\langle p_3 - p_2, \frac{(p_1 - p_2)^\perp}{\|(p_1 - p_2)^\perp\|} \right\rangle \right|$$

$$= |\langle p_3 - p_2, (p_1 - p_2)^\perp \rangle|$$



If we drop the absolute value, we get the *signed area*:

$$2 \cdot |T| = \langle p_3 - p_2, (p_1 - p_2)^\perp \rangle$$

This is positive if the vertices are in CCW order.

Triangle Area

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Unless otherwise noted, we will use $|\cdot|$ to denote the <u>signed</u> area.

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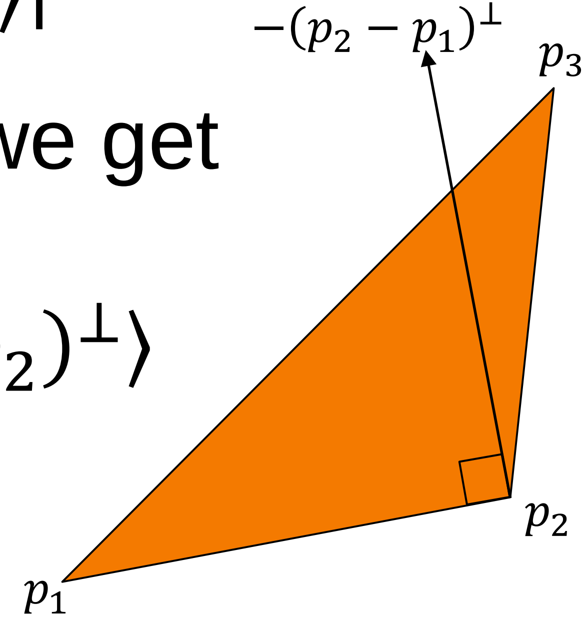
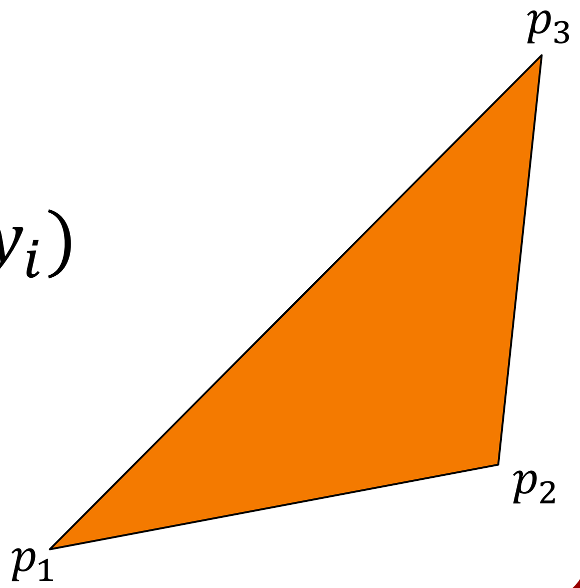


Triangle Area

$$2 \cdot |T| = \langle p_3 - p_2, (p_1 - p_2)^{\perp} \rangle$$

Setting $p_i = (x_i, y_i)$, this gives:

$$\begin{aligned} 2 \cdot |T| &= \langle (x_3 - x_2, y_3 - y_2), (y_1 - y_2, x_2 - x_1) \rangle \\ &= \sum_{i=1}^{3} x_i \cdot y_{i+1} - x_{i+1} \cdot y_i \\ &= \sum_{i=1}^{3} (x_{i+1} + x_i) \cdot (y_{i+1} - y_i) \end{aligned}$$



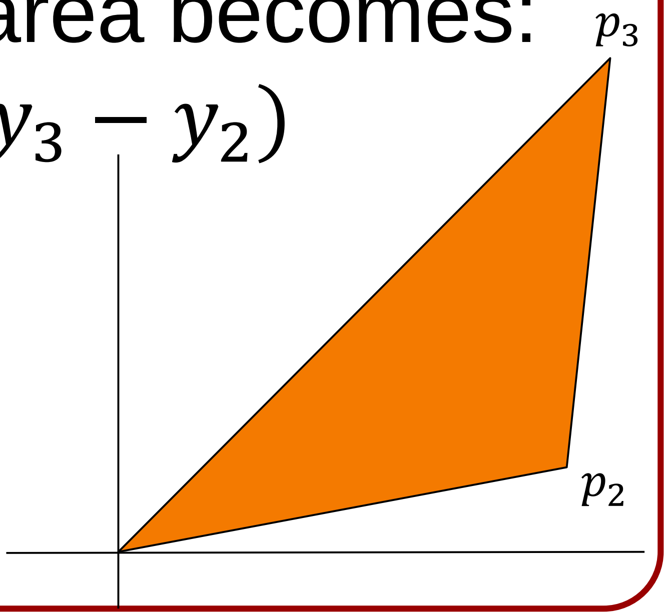
Triangle Area

$$2 \cdot |T| = \sum_{i=1}^{3} (x_{i+1} + x_i) \cdot (y_{i+1} - y_i)$$

Note:

If p_1 is at the origin, then the area becomes:

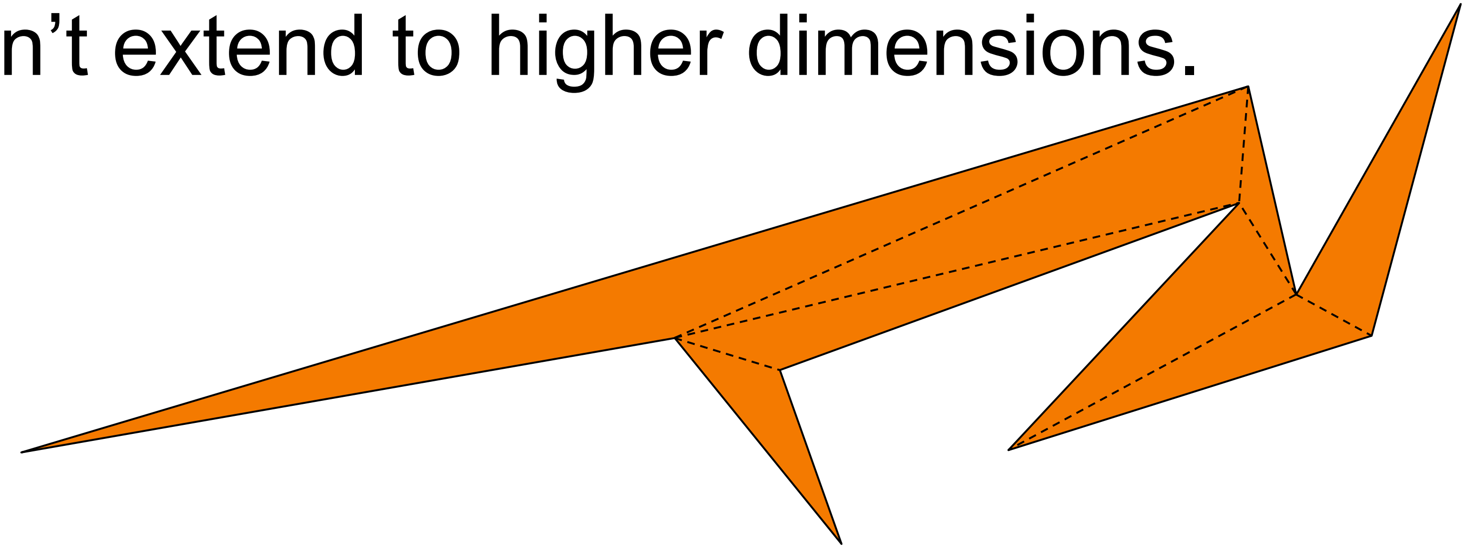
$$2 \cdot |T| = (x_3 + x_2) \cdot (y_3 - y_2)$$



Polygon Area (Take 1)

Triangulate the polygon and compute the sum of the triangle areas.

- ✗ Solving a harder problem than is required.
- ✗ Restricted to “simple” polygons.
- ✗ Doesn’t extend to higher dimensions.



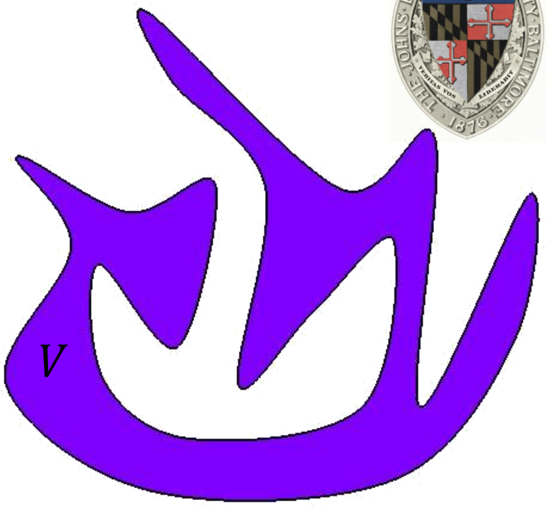
Polygon Area (Take 2)

<u>Divergence Theorem</u>:

Let V be a region in space with boundary ∂V, and let $\vec{F}$ be a vector field on V, then:

$$\int_V \operatorname{div}(\vec{F}) = \int_{\partial V} \langle \vec{F}, \vec{N} \rangle$$

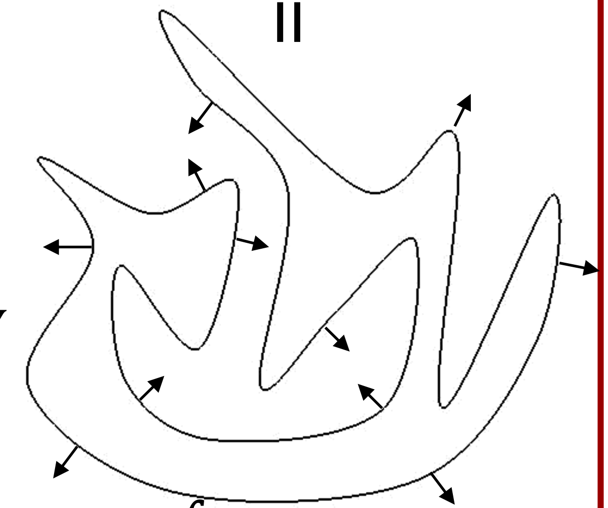
with $\vec{N}$ the normal on the boundary.



$\int_V \operatorname{div}(\vec{F})$

$\|$

∂V



$\int_V \langle \vec{F}, \vec{N} \rangle$

Polygon Area (Take 2)

Divergence Theorem:

$$\int_V \mathrm{div}(\vec{F}) = \int_{\partial V} \langle \vec{F}, \vec{N} \rangle$$

Taking $\vec{F}(x, y) = (x, y)$, gives:

$$2 \int_V 1 = \int_{\partial V} \langle (x, y), \vec{N} \rangle$$

$$\Downarrow$$

$$2 \cdot |V| = \int_{\partial V} \langle (x, y), \vec{N} \rangle$$

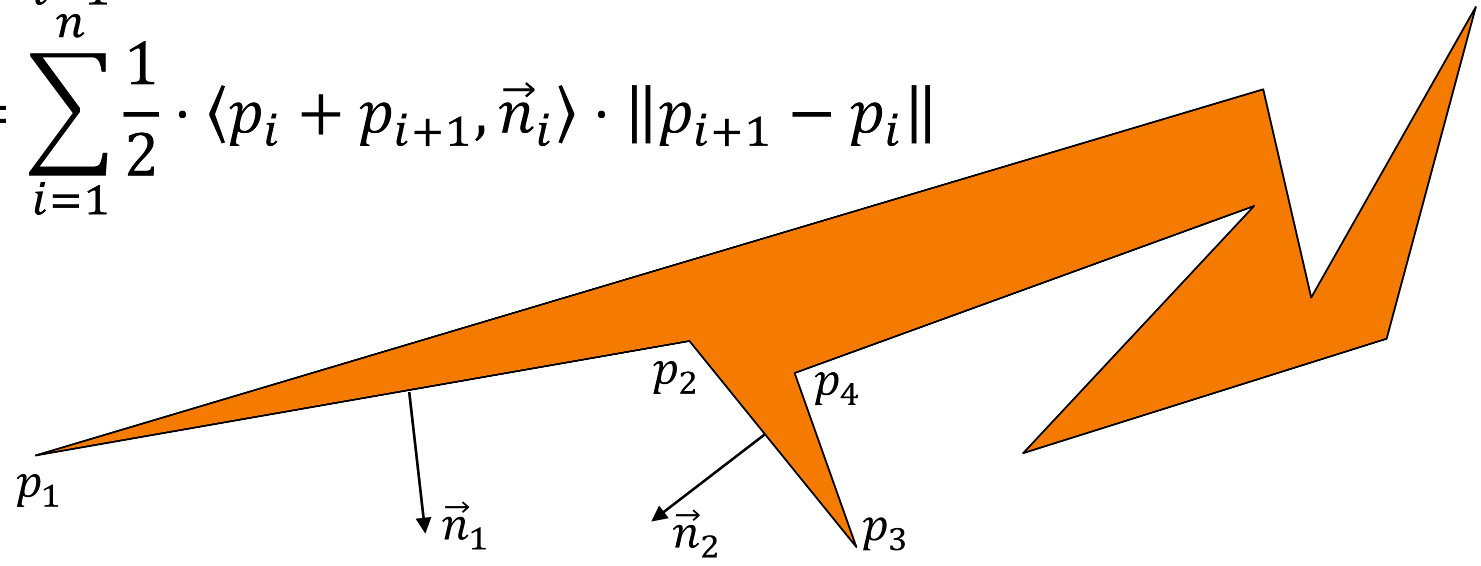
Polygon Area (Take 2)

$$2 \cdot |V| = \int_{\partial V} \langle (x,y), \vec{N} \rangle$$

For a polygon $P = \{p_1, \dots, p_n\}$, we have:

$$2 \cdot |P| = \sum_{i=1}^{n} \int_0^1 \langle (1-t) \cdot p_i + t \cdot p_{i+1}, \vec{n}_i \rangle \cdot \|p_{i+1} - p_i\| \cdot dt$$

$$= \sum_{i=1}^{n} \frac{1}{2} \cdot \langle p_i + p_{i+1}, \vec{n}_i \rangle \cdot \|p_{i+1} - p_i\|$$

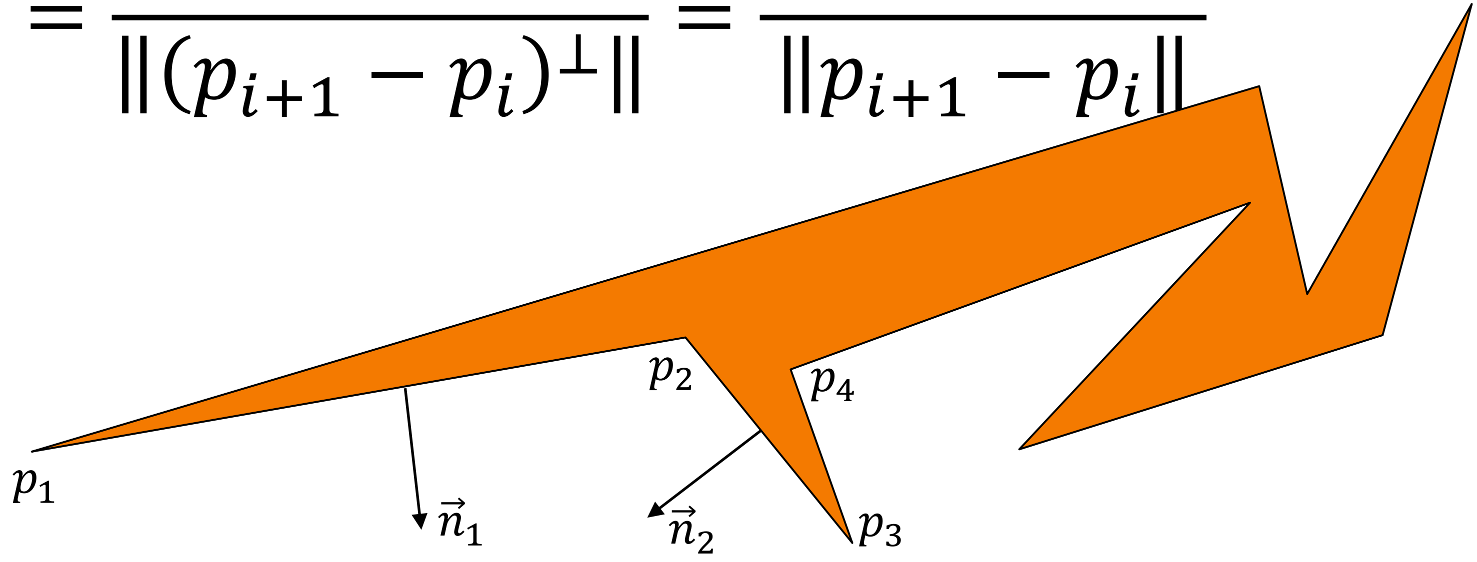


Polygon Area (Take 2)

$$2 \cdot |P| = \sum_{i=1}^{n} \frac{1}{2} \cdot \langle p_i + p_{i+1}, \vec{n}_i \rangle \cdot \|p_{i+1} - p_i\|$$

Writing the normal as the 90° rotation of the difference (normalized):

$$\vec{n}_i = \frac{(p_{i+1} - p_i)^\perp}{\|(p_{i+1} - p_i)^\perp\|} = \frac{(p_{i+1} - p_i)^\perp}{\|p_{i+1} - p_i\|}$$



Polygon Area (Take 2)

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$$\Downarrow$$

$$2 \cdot |P| = \sum_{i=1}^{n} \frac{1}{2} \cdot \langle (p_{i+1} + p_i), (p_{i+1} - p_i)^{\perp} \rangle$$

Polygon Area (Take 2)

$$2 \cdot |P| = \sum_{i=1}^{n} \frac{1}{2} \cdot \langle (p_{i+1} + p_i), (p_{i+1} - p_i)^{\perp} \rangle$$

Noting that $(x, y)^{\perp} = (y, -x)$ and writing $p_i = (x_i, y_i)$, we get:

$$2 \cdot |P| = \sum_{i=1}^{n} x_i \cdot y_{i+1} - x_{i+1} \cdot y_i$$

$$= \sum_{i=1}^{n} (x_{i+1} + x_i) \cdot (y_{i+1} - y_i)$$

Polygon Area (Take 2)

$$2 \cdot |P| = \sum_{i=1}^{n} \frac{1}{2} \cdot \langle (p_{i+1} + p_i), (p_{i+1} - p_i)^{\perp} \rangle$$

Computing the area of a polygon requires two adds and one multiply per vertex.

$$2 \cdot |P| = \sum_{i=1}^{n} x_i \cdot y_{i+1} - x_{i+1} \cdot y_i$$

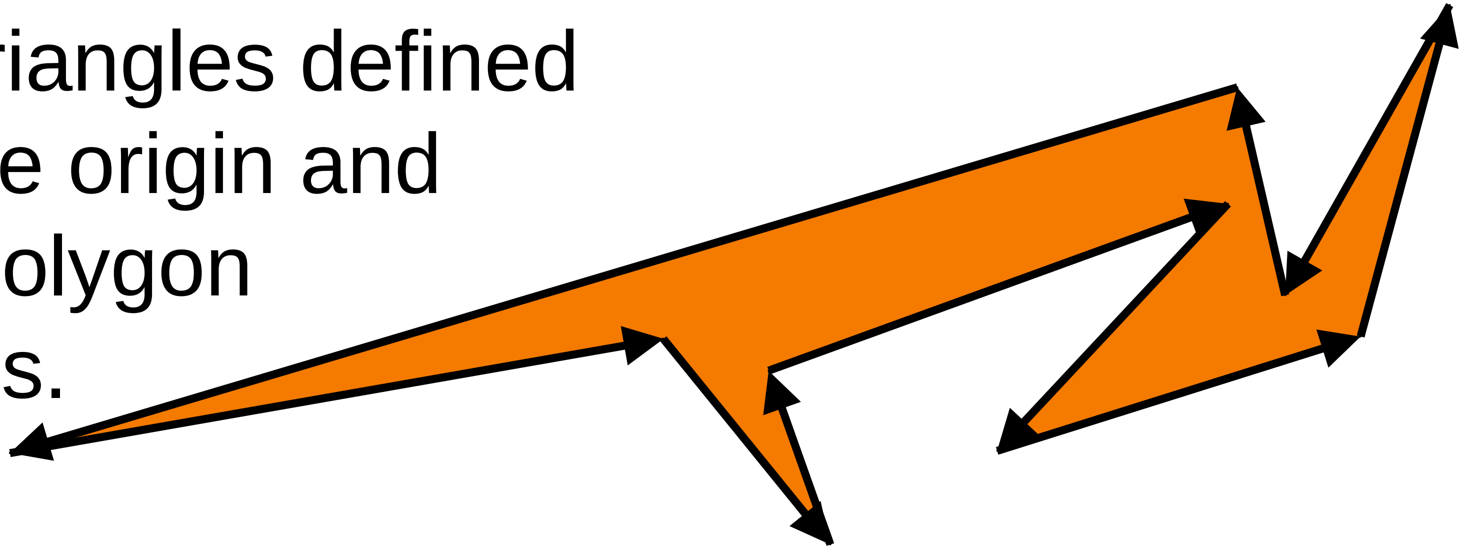
$$= \sum_{i=1}^{n} (x_{i+1} + x_i) \cdot (y_{i+1} - y_i)$$

Polygon Area (Take 2)

$$2 \cdot |P| = \sum_{i=1}^{n} (x_{i+1} + x_i) \cdot (y_{i+1} - y_i)$$

Q: What's really going on?

A: Sum the areas of the triangles defined by the origin and the polygon edges.



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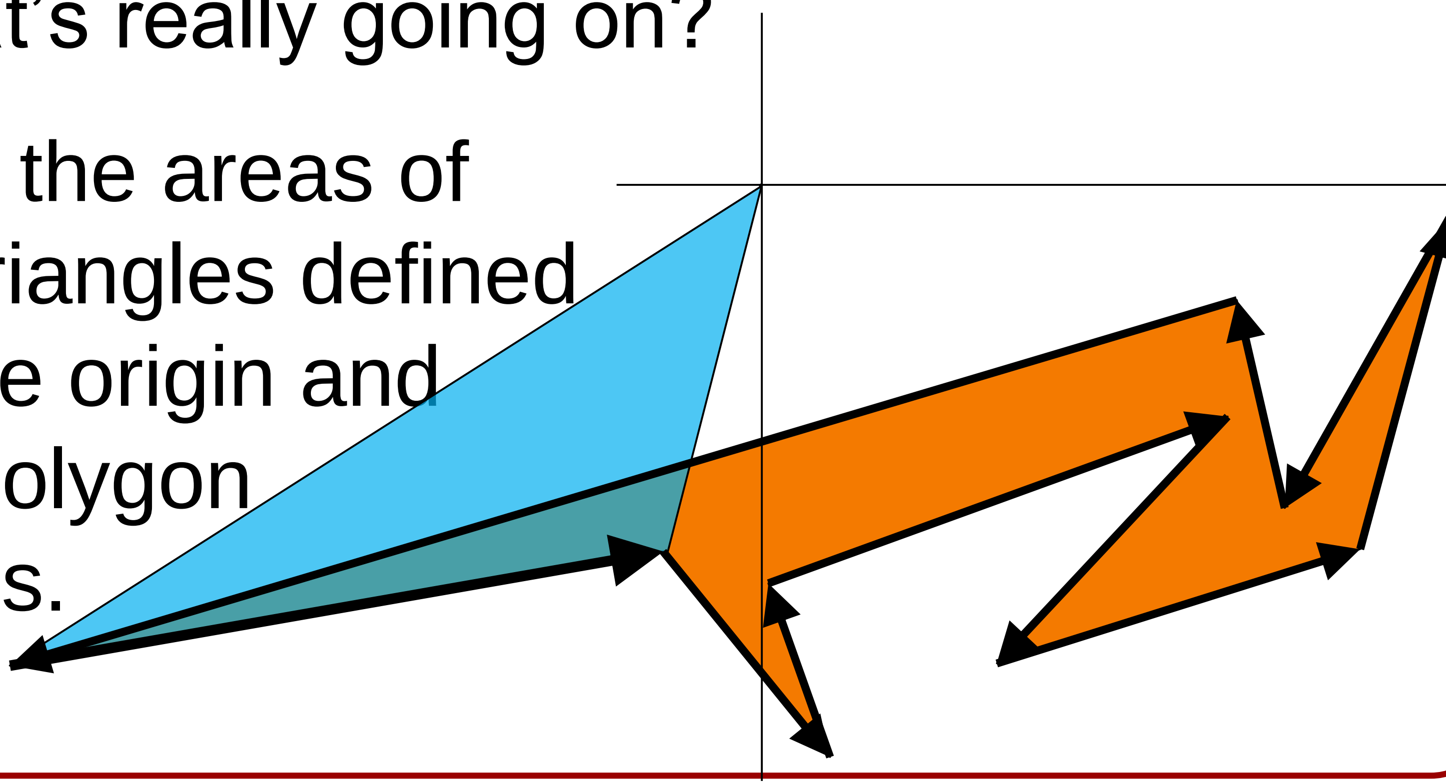

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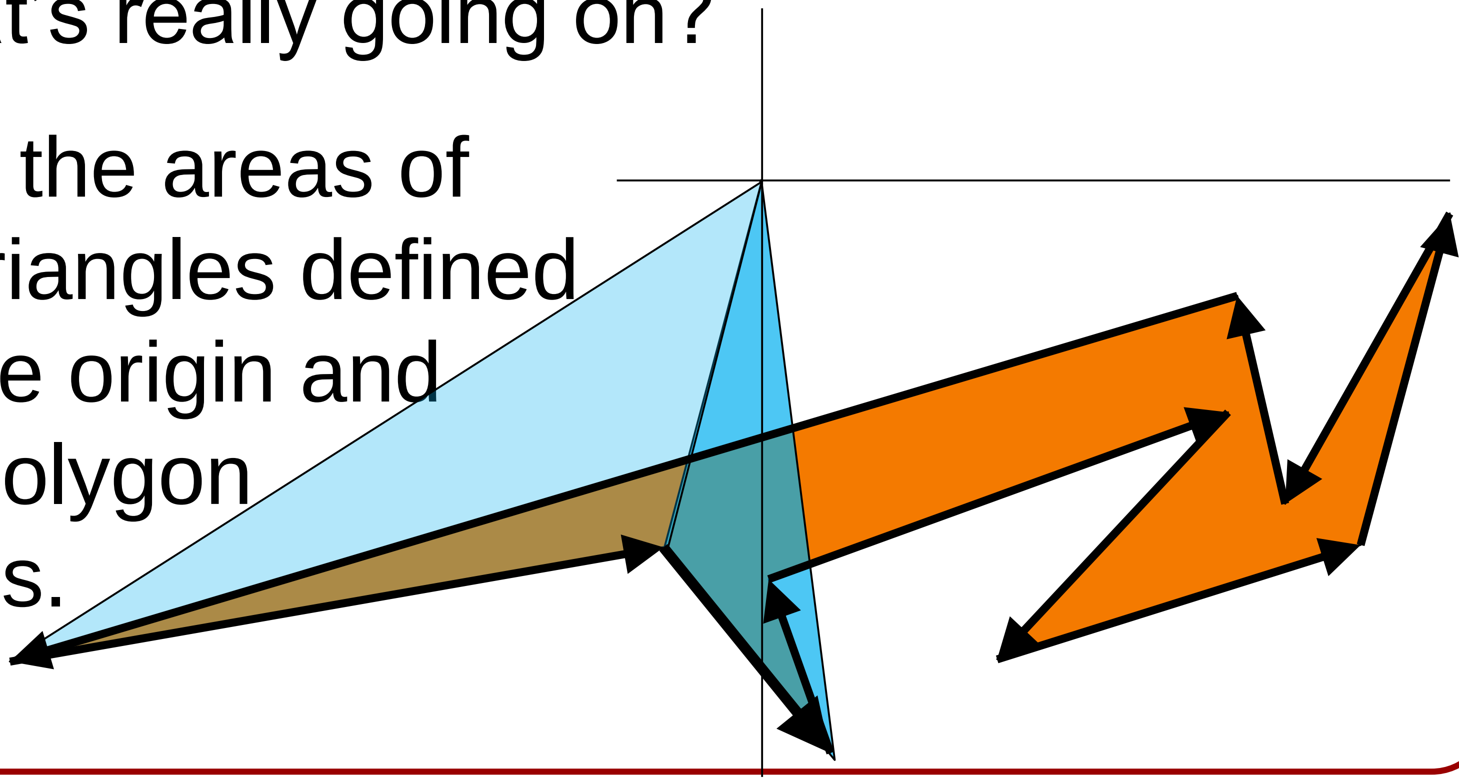

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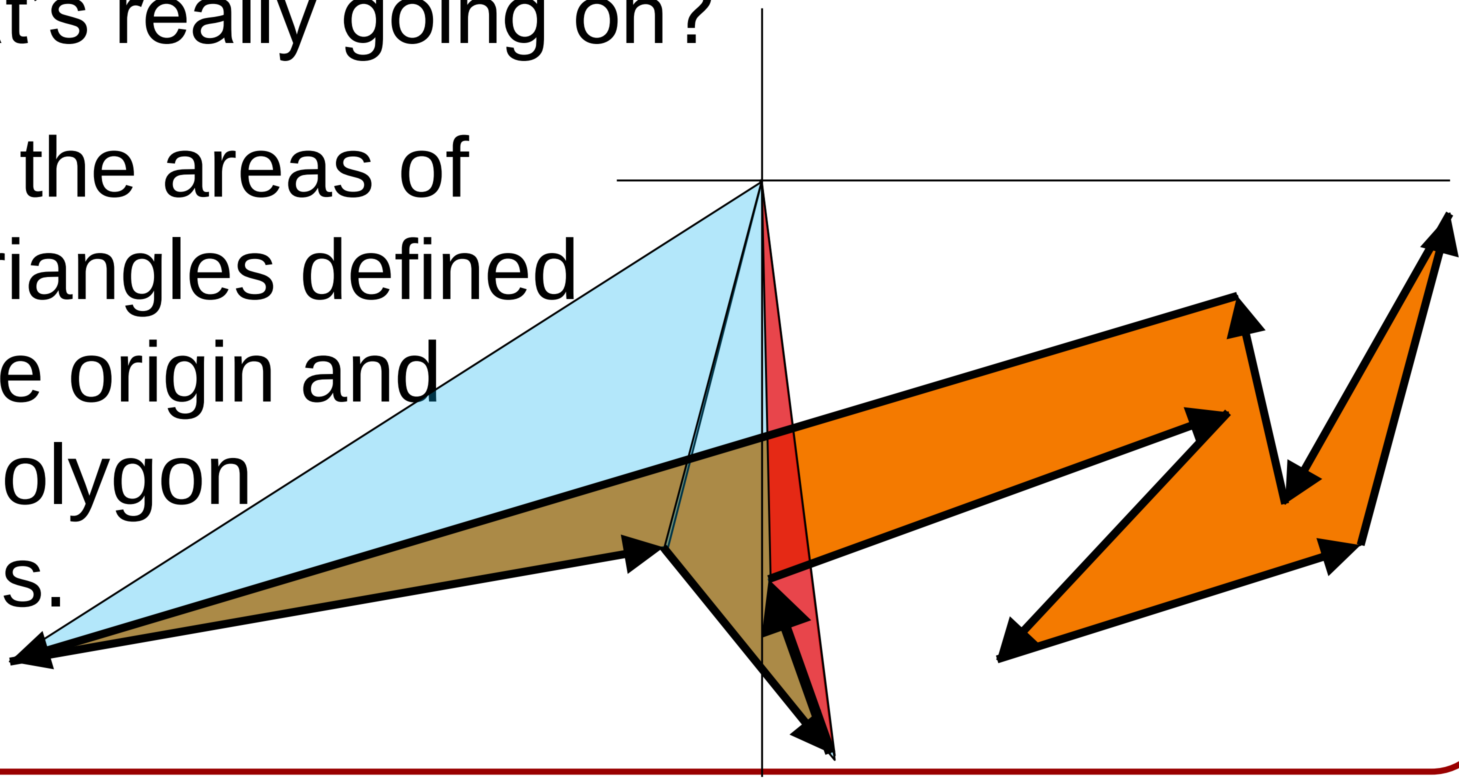

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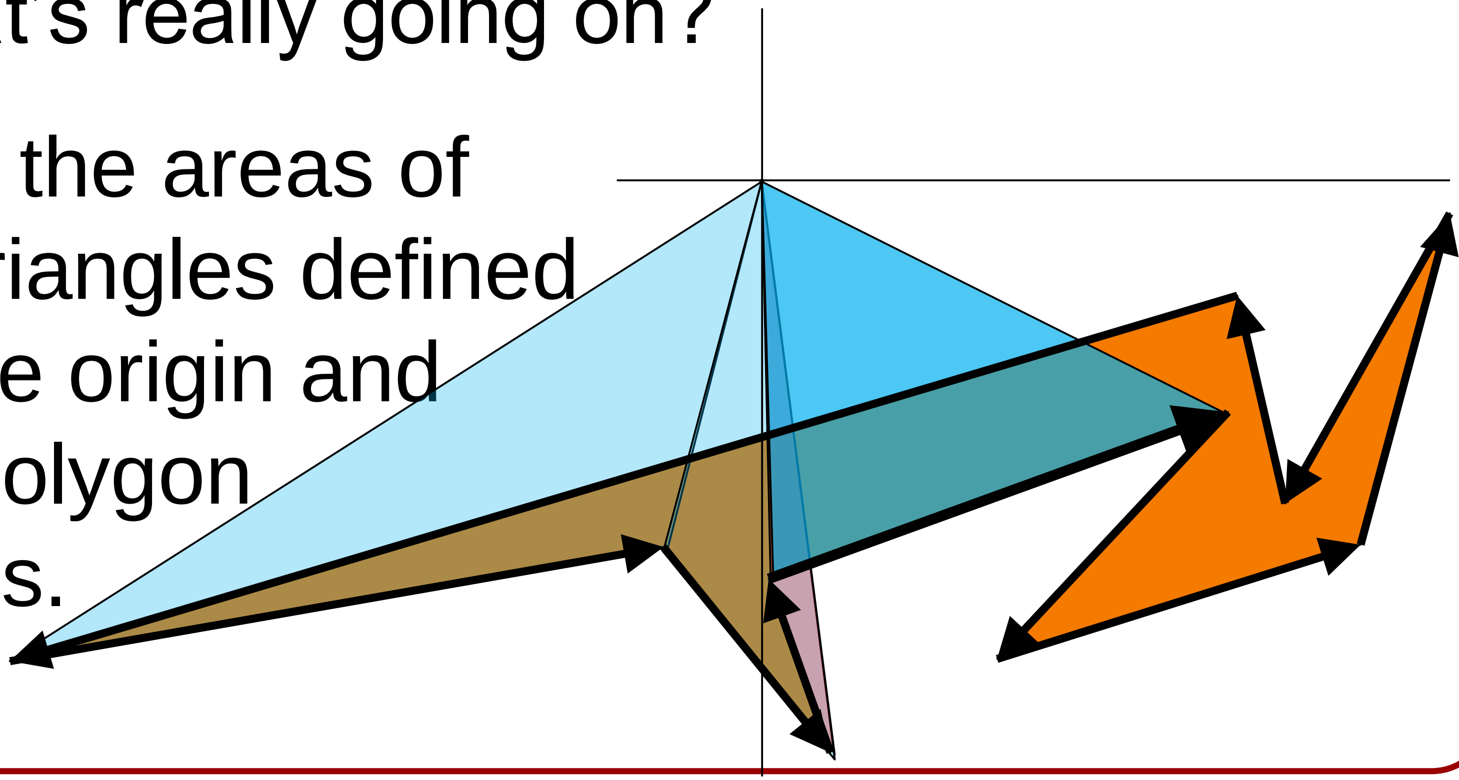

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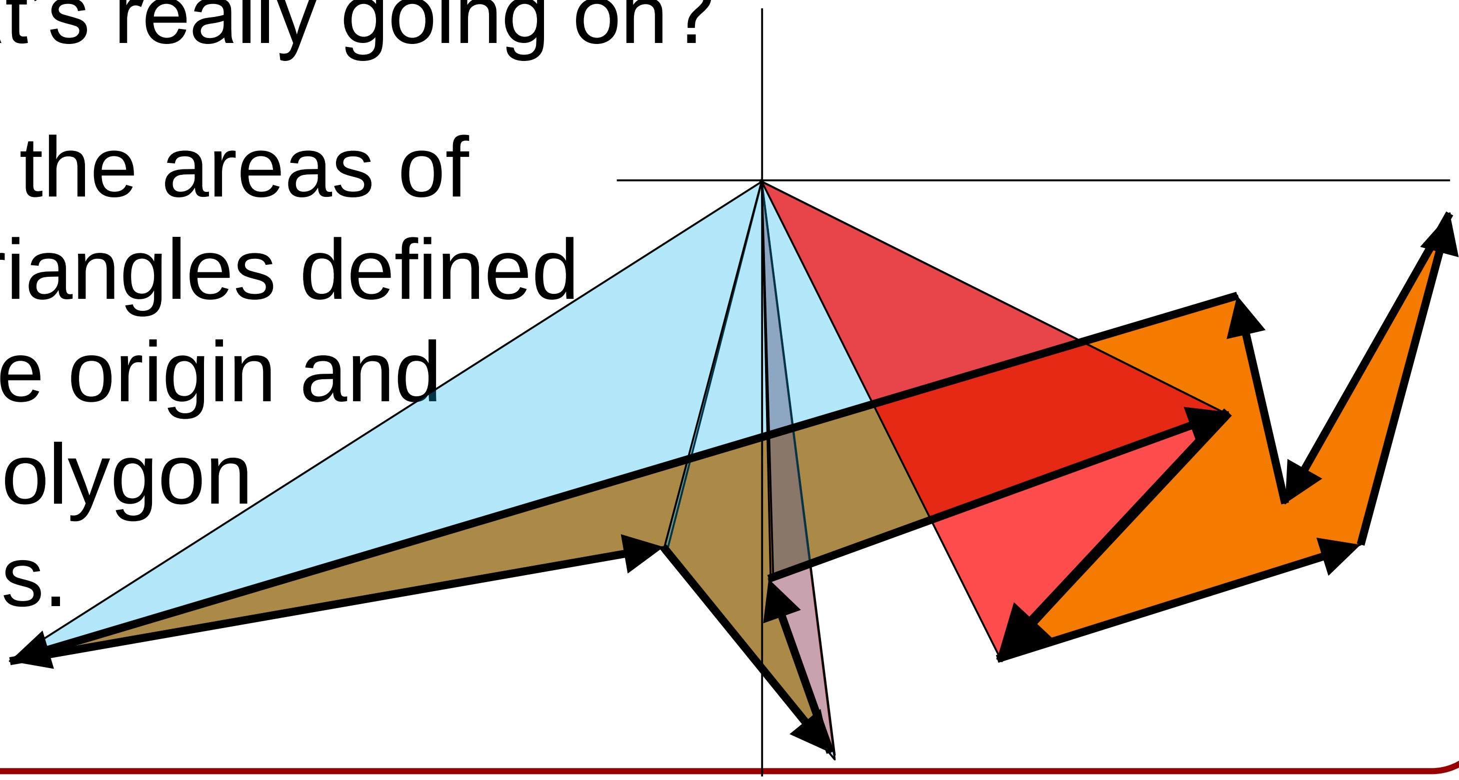

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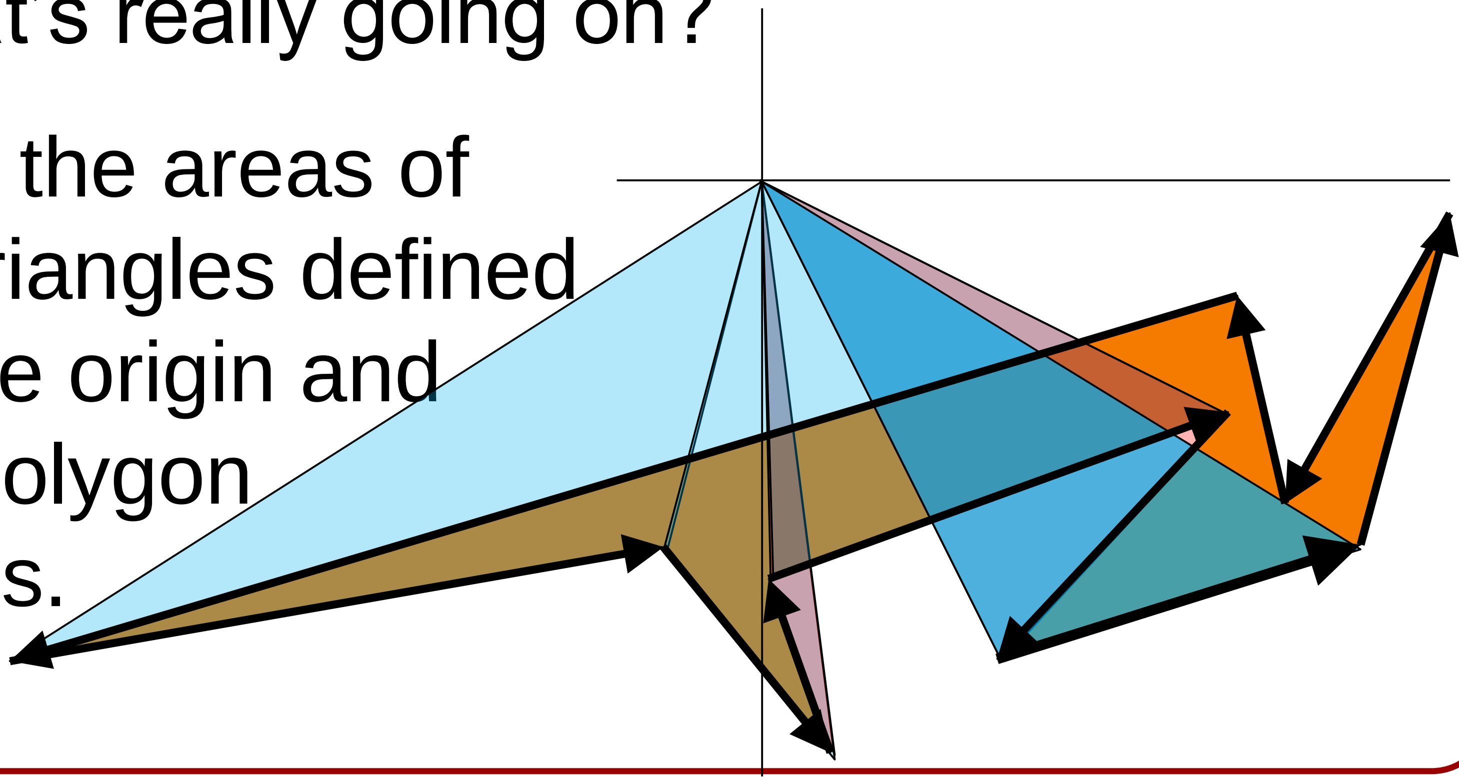

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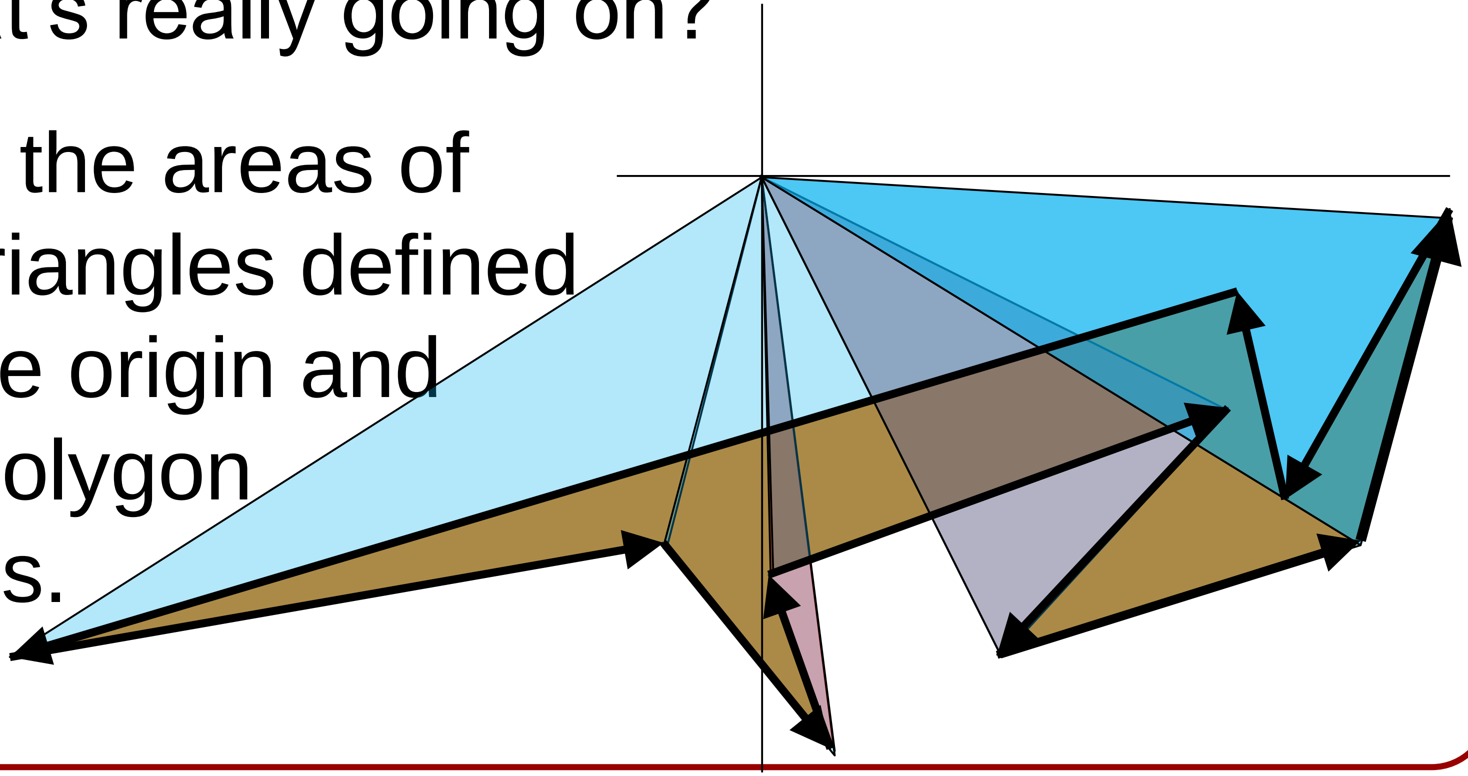

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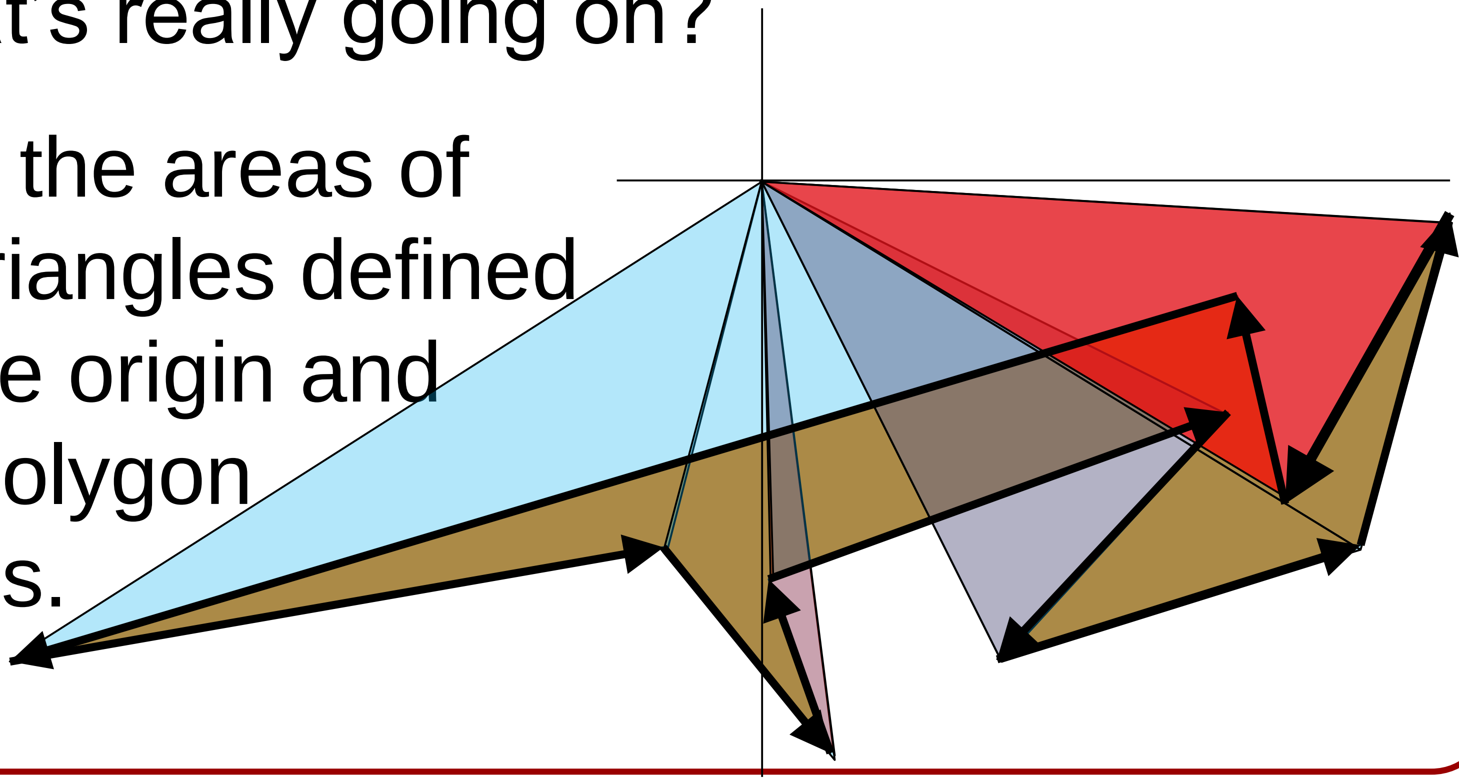

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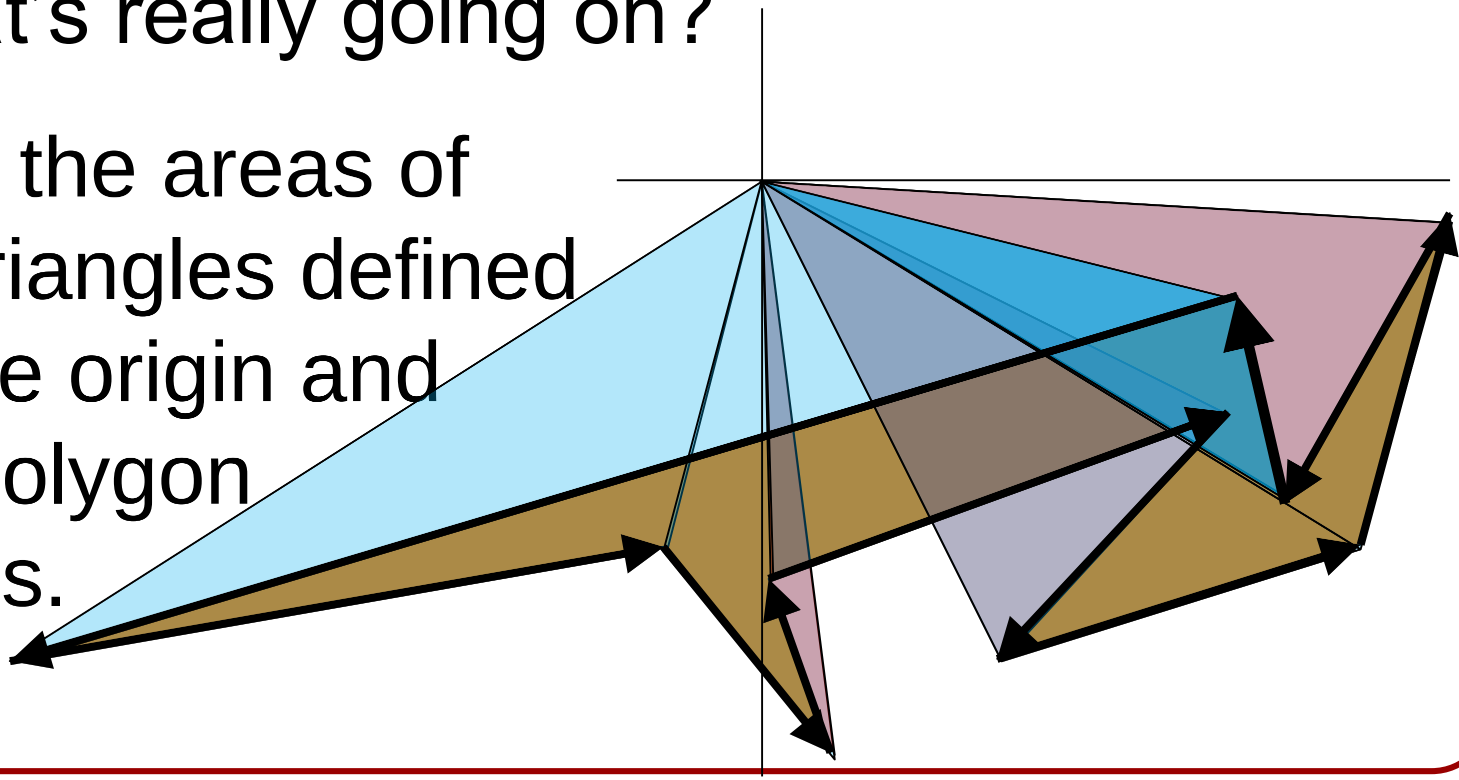

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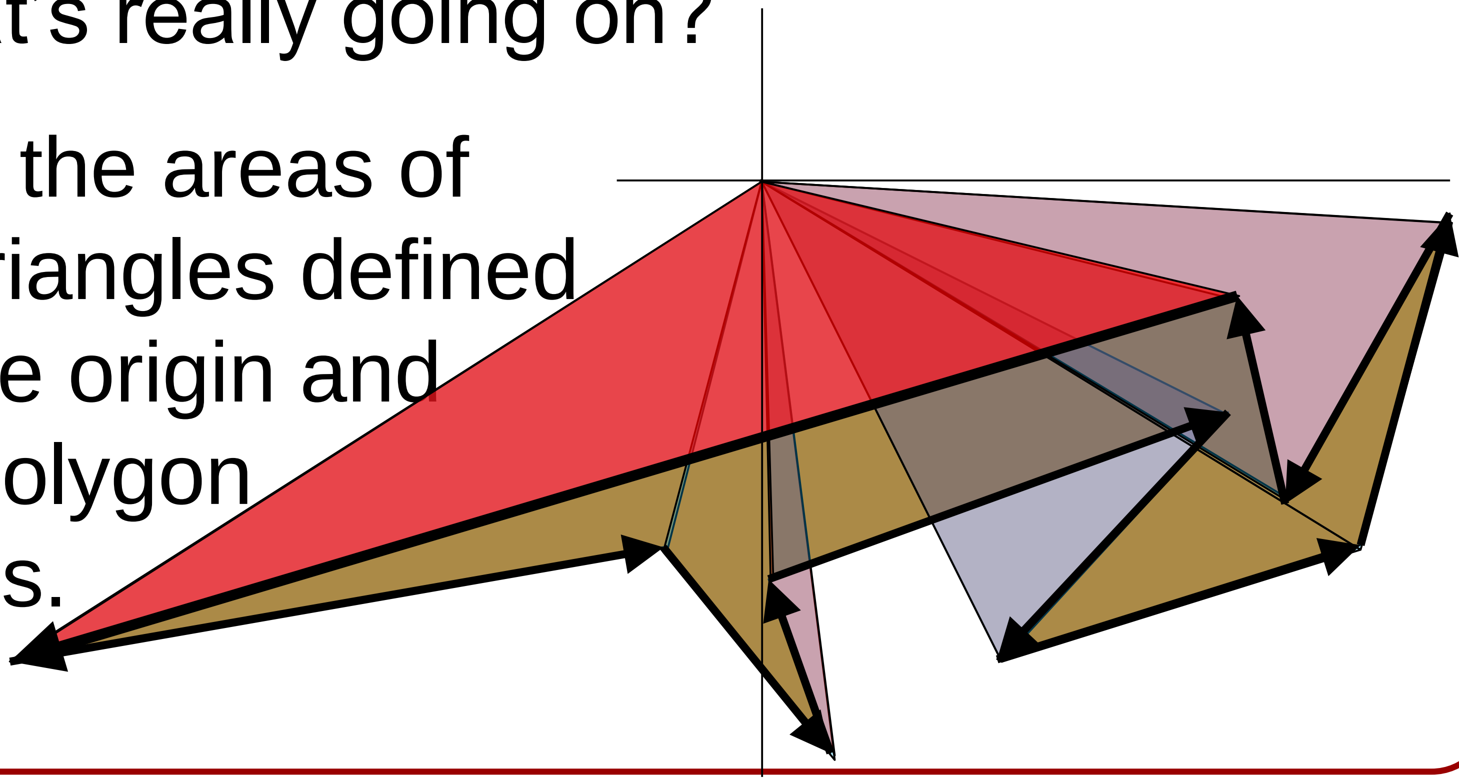
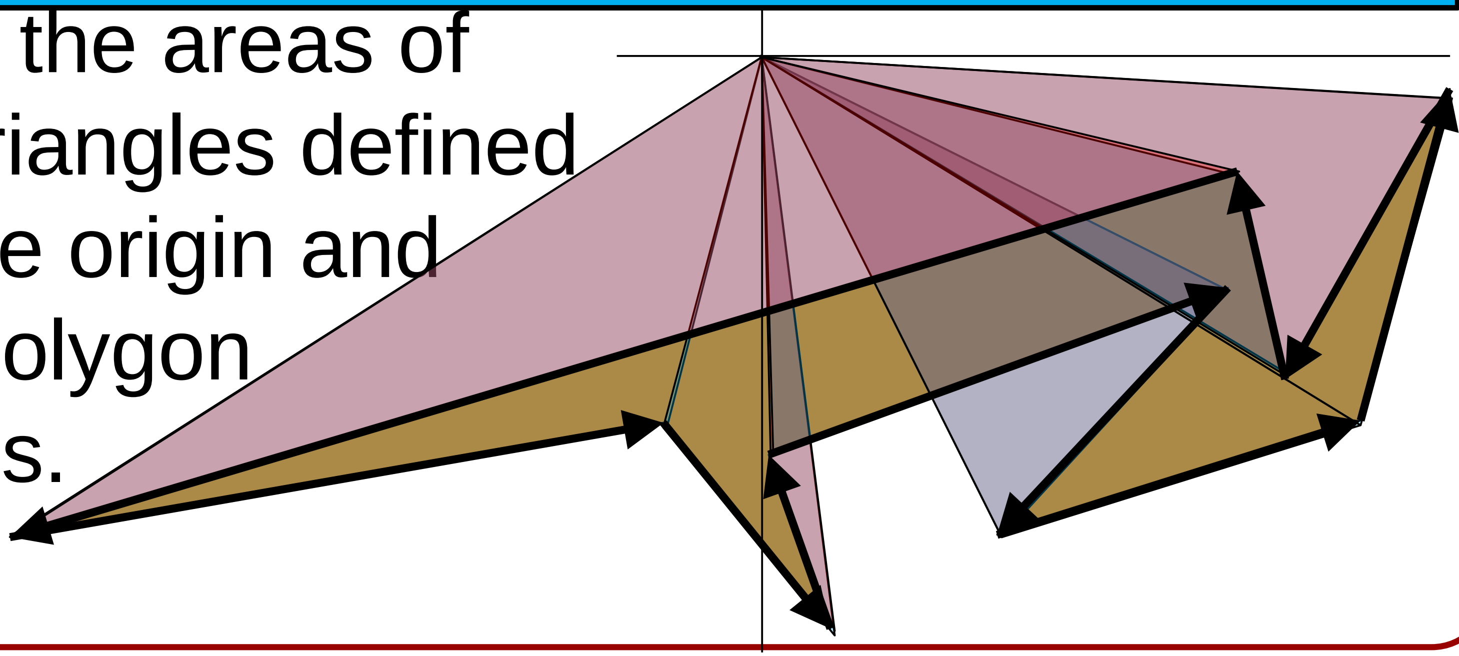


Polygon Area (Take 2)

$$2 \cdot |P| = \sum_{i=1}^{n} (x_{i+1} + x_i) \cdot (y_{i+1} - y_i)$$

In this “triangulation”, the use of signed area cancels out the unwanted contribution.

A: Sum the areas of the triangles defined by the origin and the polygon edges.



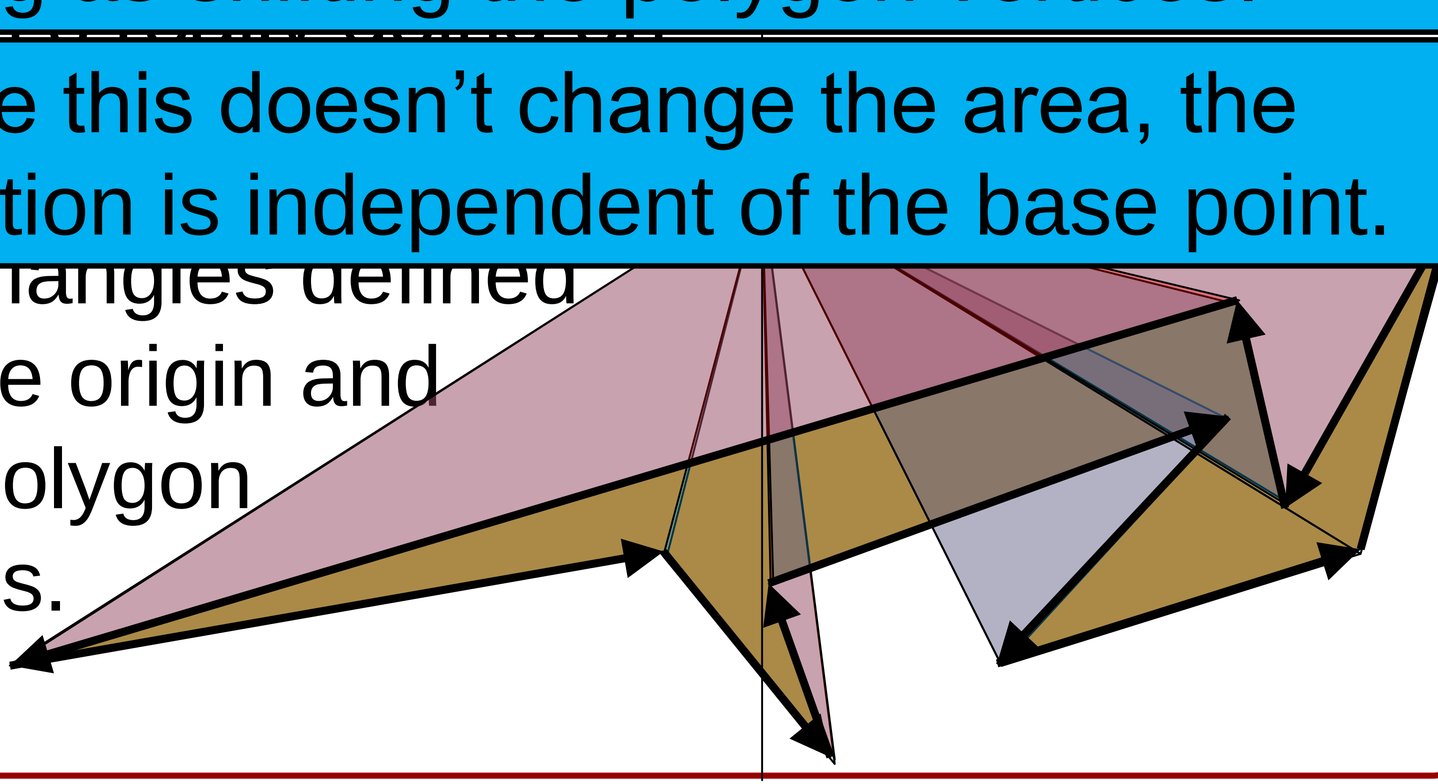
Polygon Area (Take 2)

$$2\cdot|P| = \sum^{n} (x_{i+1}+x_i)\cdot(y_{i+1}-y_i)$$

Choosing a different base point is the same thing as shifting the polygon vertices.

Since this doesn’t change the area, the calculation is independent of the base point.

the triangles defined by the origin and the polygon edges.



Polygon Area (Take 2)

Note:

The same approach can be used to compute the volume enclosed by a triangle mesh in 3D:

- Pick a base point.
- Create tetrahedra by joining the base point to the triangles of the mesh.
- Sum the signed volumes of the tetrahedra.

Outline

- Polygon Area
- Implementation

Implementation

```
template< unsigned int D >
struct Point
{
    int c [D];
    Point ( void ){ memset( c , 0 , sizeof(int)*D ); }
    int& operator[]( int idx )        { return c[idx]; }
    int  operator[]( int idx ) const { return c[idx]; }
    static long long Integral2( const Point p[D+1] );
};
```

Implementation

```
template< >
long long Point< 2 >::Integral2( const Point< 2 > p[3] )
{
    long long a = 0;
    a += ( (long long)( p[1][0] + p[0][0] ) ) * ( p[1][1] - p[0][1] );
    a += ( (long long)( p[2][0] + p[1][0] ) ) * ( p[2][1] - p[1][1] );
    a += ( (long long)( p[0][0] + p[2][0] ) ) * ( p[0][1] - p[2][1] );
    return a;
}
template< int D >
long long Point< D >:: Integral2( const Point< D > p[D+1] )
{ printf( "[ERR] Point<%d>::Integral2 unsupported\n" , D ) ; exit(0); }
```

Implementation

```
struct PVertex
{
    Point< 2 > p;
    PVertex *prev , *next;
    PVertex( Point< 2 > _p );
    PVertex& addBefore( Point< 2 > p );
    unsigned int size( void ) const;
    long long area2( void ) const;
    static PVertex* Remove( PVertex* v );
};
```

Implementation

```
PVertex::PVertex( Point< 2 > _p ){ p=_p , prev = next = this; }


PVertex& PVertex::addBefore( Point< 2 > p )
{
    PVertex* v = new PVertex(p);
    v->prev = prev , v->next = this;
    prev = prev->next = v;
    return *v;
};
```

Implementation

```
static PVertex* PVertex::Remove( PVertex* v )
{
    PVertex* temp = v->prev;
    v->prev->next = v->next;
    v->next->prev = v->prev;
    delete v;
    return temp==v ? NULL : temp;
}
```

Implementation

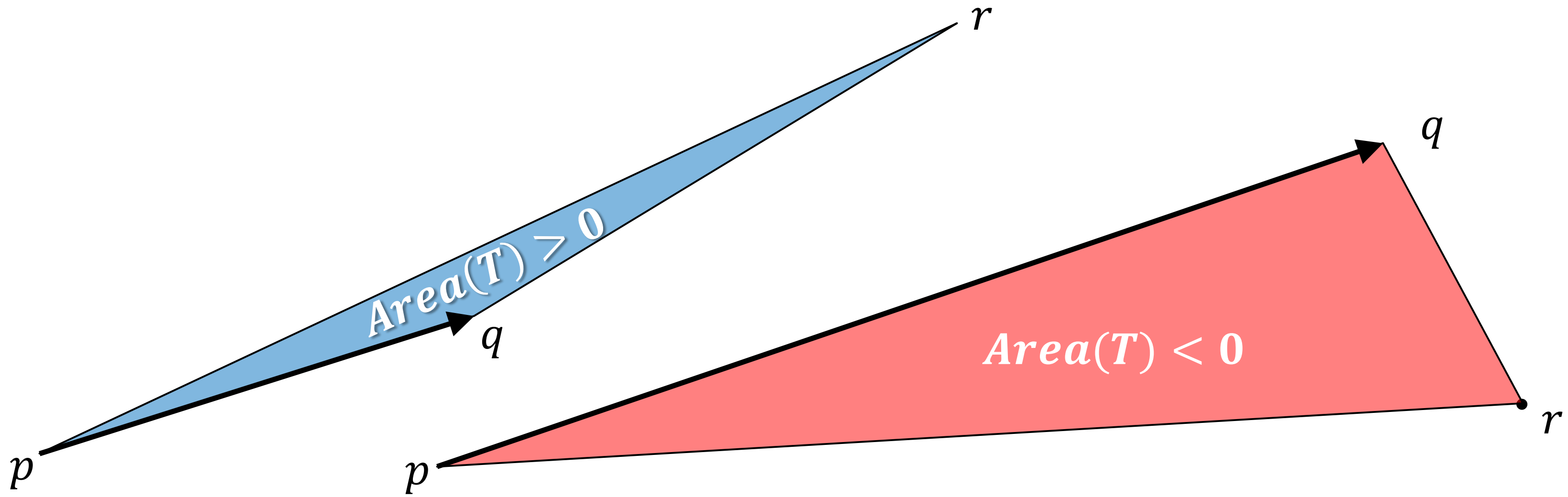
```
unsigned int PVertex::size( void ) const
{
    unsigned int s = 0;
    for( const PVertex* i=this ; ; i=i->next )
    {
        s++;
        if( i->next==this ) break;
    }
    return s;
}
```

Implementation

```
long long PVertex::area2( void ) const
{
    Point< 2 > p[3];
    long long a = 0;
    for( const PVertex* i=this ; ; i=i->next )
    {
        p[1] =i->p , p[2] = i->next->p;
        a += Point< 2 >::Integral2( p );
        if( i->next==this ) break;
    }
    return a;
}
```

Sidedness

Given a line segment, $\overrightarrow{pq}$, and a point r, we can determine if r is to the left of, on, or to the right of $\overrightarrow{pq}$ by testing the sign of the area of triangle Δpqr.



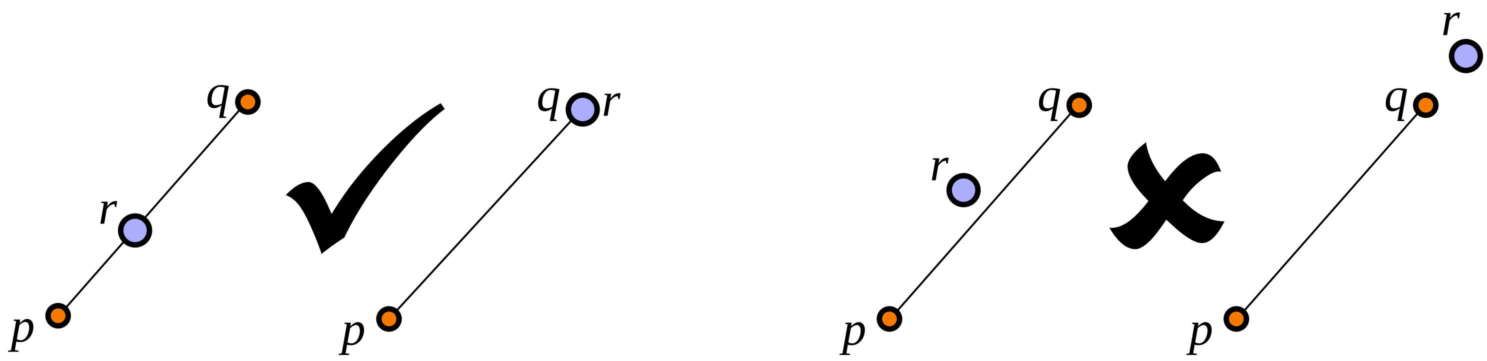
Implementation

```
bool Left( Point<2 > p , Point< 2 >  q , Point< 2 > r )
{ return Point< 2 >::Area2( p , q , r ) > 0; }

bool LeftOn( Point<2 > p , Point< 2 >  q , Point< 2 > r )
{ return Point< 2 >::Area2( p , q , r ) >= 0; }

bool Collinear( Point<2 > p , Point< 2 >  q , Point< 2 > r )
{ return Point< 2 >::Area2( p , q , r ) == 0; }

bool Right( Point<2 > p , Point< 2 >  q , Point< 2 > r )
{ return Point< 2 >::Area2( p , q , r ) < 0; }

bool RightOn( Point<2 > p , Point< 2 >  q , Point< 2 > r )
{ return Point< 2 >::Area2( p , q , r ) <= 0; }
```

Point on Line Segment

Given a line segment, $\overline{pq}$, a point r is between p and q if:

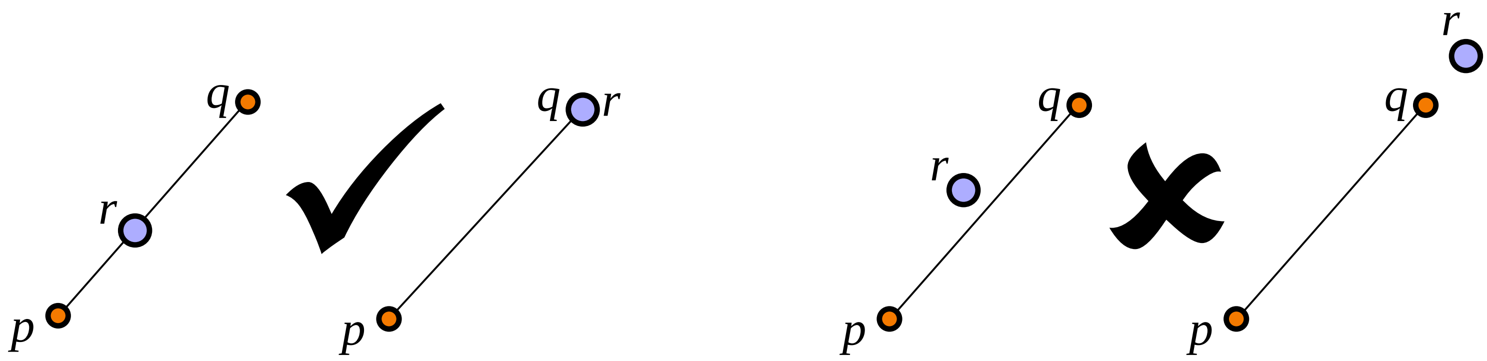
- r is on the line between p and q, and
- the x-coordinate of r is between the x-coordinates of p and q



Point on Line Segment

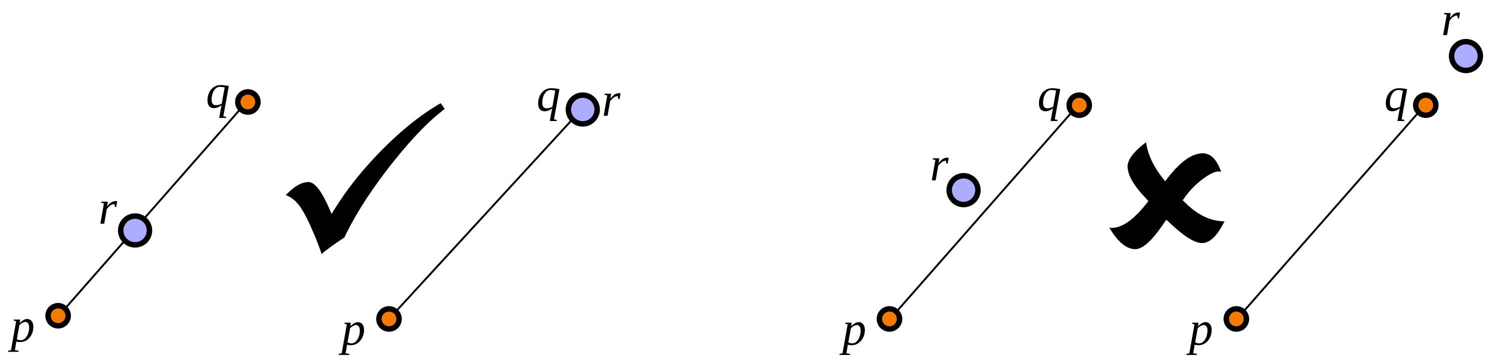
Given a line segment, $\overline{pq}$, a point r is between p and q if:

- r is on the line between p and q, and
- the x-coordinate of r is between the x-coordinates of p and q (if $\overline{pq}$ is not vertical)
- the y-coordinate of r is between the y-coordinates of p and q (if $\overline{pq}$ is vertical)



Implementation

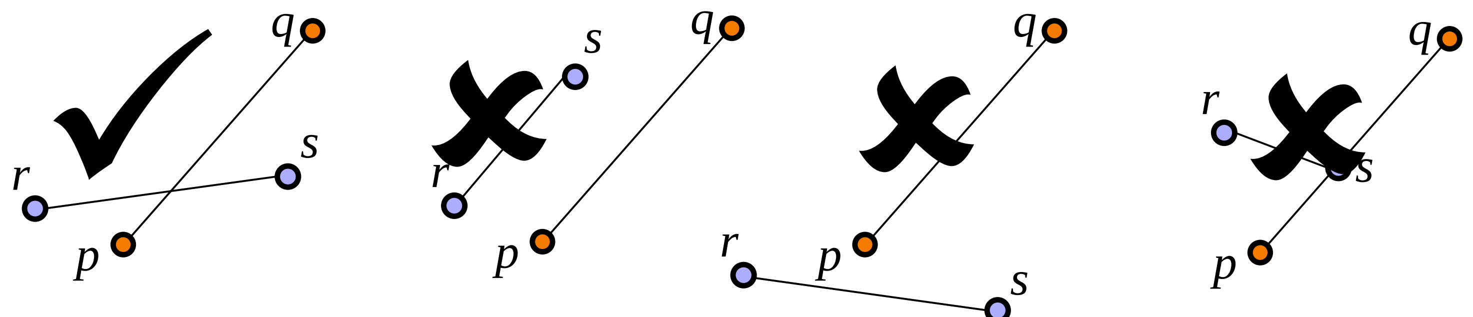
```
bool Between( Point<2 > p , Point< 2 >  q , Point< 2 > r )
{
    if( !Collinear( p , q , r ) ) return false;
    unsigned int dir = p[0]!=q[0] ? 0 : 1;
    return
        ( p[dir] <= r[dir] && r[dir] <= q[dir] ) ||
        ( q[dir] <= r[dir] && r[dir] <= p[dir] );
}
```



Proper Intersection

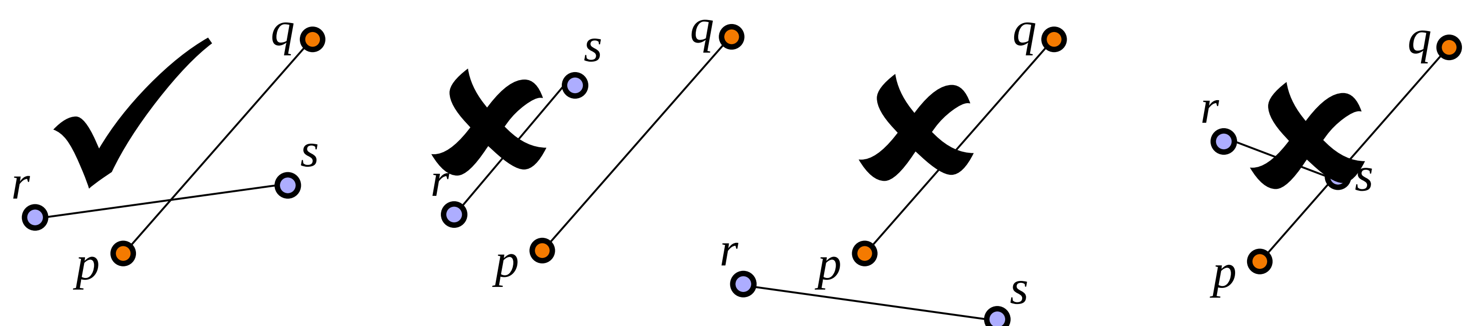
Line segments $\overline{pq}$ and $\overline{rs}$, *intersect properly* if they intersect in their interior:

- Neither r nor s is on the segment $\overline{pq}$.
- Neither p nor q is on the segment $\overline{rs}$.
- Either p and q are on different sides of $\overline{rs}$, or r and s are on different sides of $\overline{pq}$.



Implementation

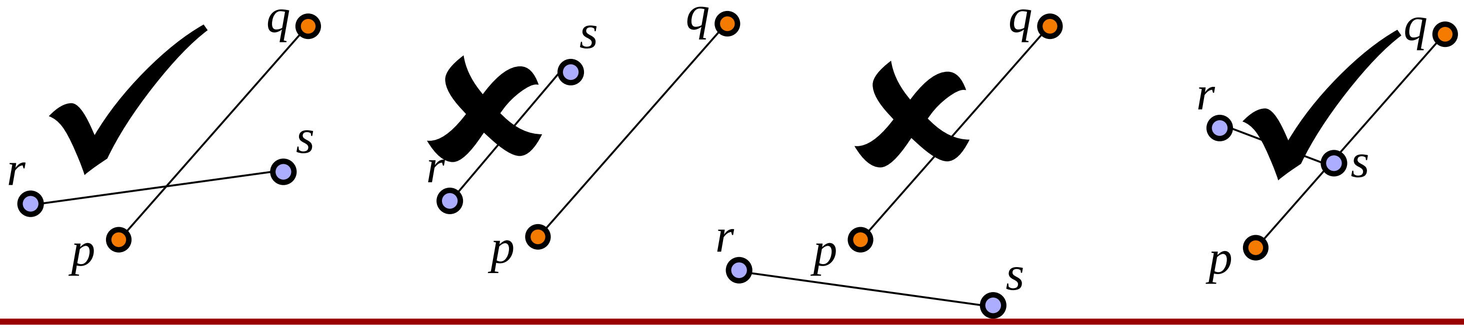
```
bool IsectProper( Point<2 > p , Point< 2 >  q , Point< 2 > r , Point< 2 > s )
{
    if( Collinear( p , q , r ) || Collinear( p , q , s ) ) return false;
    if( Collinear( r , s , p ) || Collinear( r , s , q ) ) return false;
    if( Left( p , q , r ) == Left( p , q , s ) ) return false;
    if( Left( r , s , p ) == Left( r , s , q ) ) return false;
    return true;
}
```



Intersection

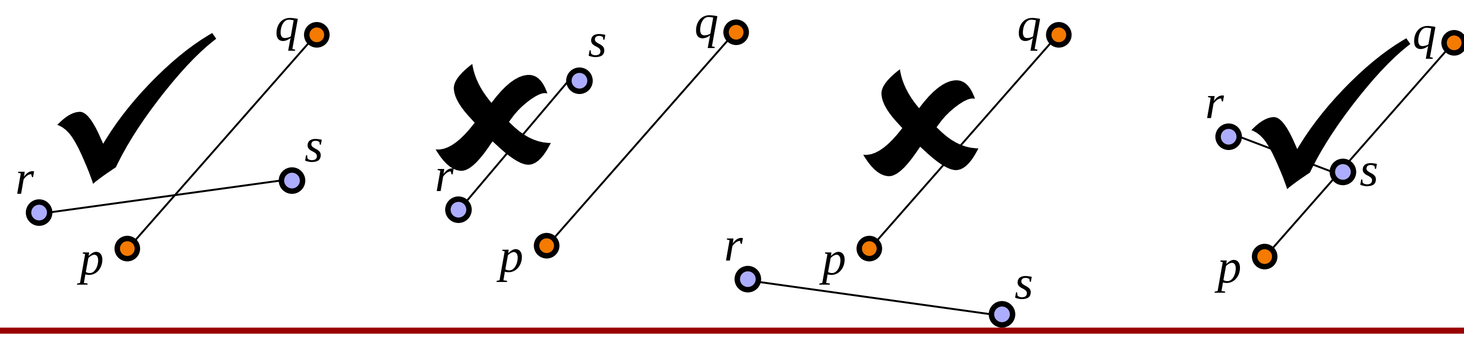
Line segments $\overline{pq}$ and $\overline{rs}$, *intersect* if:

- p is between r and s, or
- q is between r and s, or
- r is between p and q, or
- s is between p and q, or
- they intersect properly.



Implementation

```
bool Isect ( Point<2 > p , Point< 2 >  q , Point< 2 > r , Point< 2 > s )
{
    return
        IsectProper( p , q , r , s ) ||
        Between( p , q , r ) || Between( p , q , s ) ||
        Between( r , s , p ) || Between( r , s , q );
}
```



Diagonal

Property:

Given a polygon, $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^2$, an edge $\overline{p_i p_j}$ is a *diagonal* if:

1. $\forall p_k, p_l \in P$ w/ $k, l \notin \{i, j\}$: $\overline{p_i p_j} \cap \overline{p_k p_l} = \emptyset$
2. $\overline{p_i p_j}$ is internal to P around p_i and p_j

Edge Intersection

To test the first property:

1. $\forall p_k, p_l \in P$ w/ $k, l \notin \{i, j\}$: $\overline{p_i p_j} \cap \overline{p_k p_l} = \emptyset$

we check for the intersection of $\overline{p_i p_j}$ with all other edges.

Implementation

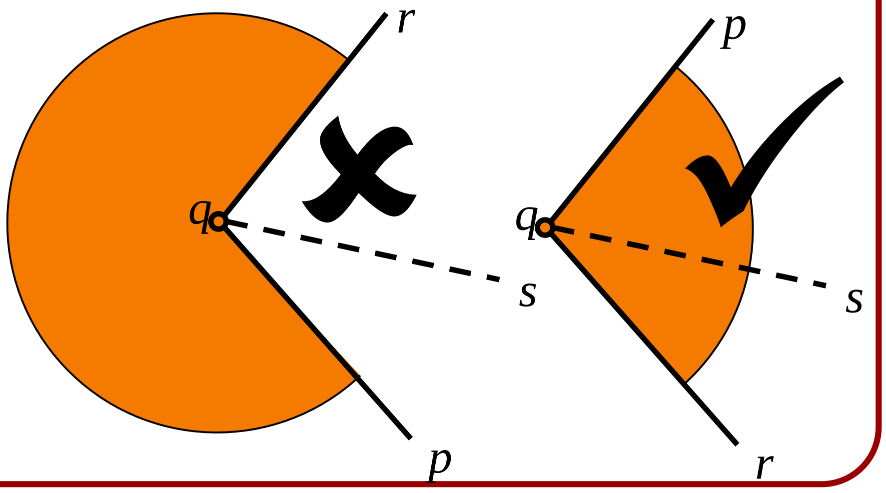
```
bool DiagonalIsect( const PVertex< 2 >* r , const PVertex< 2  >* s )
{
    for( const PVertex< 2 >* i=r ;  ; i=i->next )
    {
        if( i->prev!=r && i->prev!=s && i!=r && i!=s )
            if( Isect( r->p , s->p , i->prev->p , i->p ) ) return true;
        if( i->next==r ) break;
    }
    return false;
}
```

Complexity: O(n)

Cone Interior

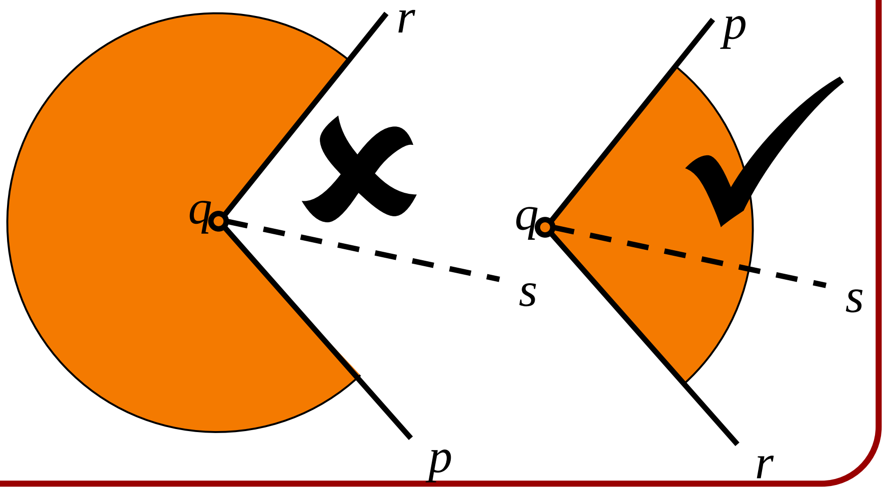
Given points p, q, and r, a line segment $\overline{qs}$ is *in the cone of* pqr if $\overline{qs}$ is strictly interior to the region swept out CW from $\overrightarrow{qp}$ to $\overrightarrow{qr}$.

- If $\angle pqr$ is a left turn:
  s must be to the left of both $\overrightarrow{pq}$ and $\overrightarrow{qr}$.
- Otherwise:
  s cannot be to the left of both $\overrightarrow{rq}$ and the right of $\overrightarrow{qp}$.



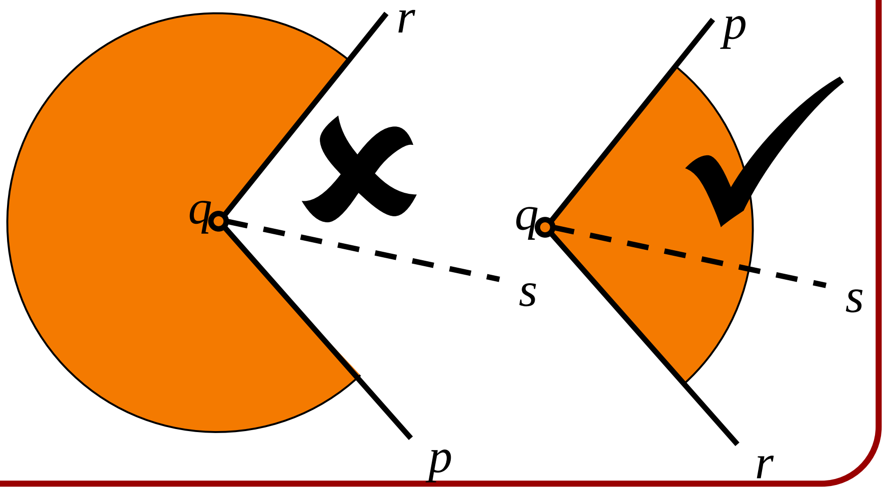
Implementation

```
bool InCone( Point<2 > p , Point< 2 >  q , Point< 2 > r , Point< 2 > s )
{
    if( Left( p , q , r ) )
        return  ( Left   ( p , q , s ) && Left    ( q , r , s ) );
    else
        return !( LeftOn( r , q , s ) && LeftOn( q , p , s ) );
}
```



Implementation

```
bool InCones( const PVertex< 2 >* q , const PVertex< 2  >* s )
{
    return
        InCone( q->prev->p , q->p , q->next->p , s->p ) &&
        InCone( s->prev->p , s->p , s->next->p , q->p );
}
```



Implementation

```
bool IsDiagonal( const PVertex< 2 >* r , const PVertex< 2  >* s )
{
	return InCones( r , s ) && !DiagonalIsect( r , s );
}
```

Trangulation (Naïve)

Recursively:

1. Find/output a diagonal.
2. Split the polygon in two.

Implementation

```
void OutputTriangulationDiagonals( PVertex< 2 >* poly )
{
    if( poly->size()>3 )
    {
        PVertex< 2 > *r , *s , *poly1 , *poly2;
        GetDiagonal( poly , r , s )
        Output( r , s );
        SplitOnDiagonal( poly , r , s , poly1 , poly2 );
        OutputTriangulationDiagonals( poly1 );
        OutputTriangulationDiagonals( poly2 );
    }
}
```

Complexity: $O(n^4)$

Triangulation (Ear Removal)

While there are more than three vertices:

1. Find an ear p_i.
2. Output the diagonal $\overline{p_{i-1}p_{i+1}}$.
3. Remove p_i from the polygon.

<u>Note</u>:

The ear status can only change for the diagonal edges p_{i-1} and p_{i+1}.

Triangulation (Ear Removal)

Initialize the ear status of all vertices.

While there are more than three vertices:

1. Find an ear p_i.
2. Output the diagonal $\overline{p_{i-1}p_{i+1}}$.
3. Remove p_i from the polygon.
4. Update the ear status of p_{i-1} and p_{i+1}.

Implementation

```
// Assumes member:
//        bool PVertex< 2 >::isEar
bool InitEars( PVertex< 2 >* poly )
{
    for( PVertex< 2 >* i=poly ; ; i=i->next )
    {
        i->isEar = IsDiagonal( poly , i->prev , i->next );
        if( i->next==poly ) break;
    }
}
```

Complexity:
$O(n^2)$

Implementation

```
PVertex< 2 >* ProcessEar( PVertex< 2 >* e )
{
    Output( ear->prev , ear , ear->next );
    e->prev->isEar = IsDiagonal( e->prev->prev , e->next );
    e->next->isEar = IsDiagonal( e->prev , e->next->next );
    return PVertex< 2 >::Remove( e );
}
```

Complexity:
$O(n)$

Implementation

```
void OutputTriangulationDiagonals( PVertex< 2 >* poly )
{
    InitEars( poly );
    unsigned int sz = poly->size();
    while( sz>3 )
        for( PVertex< 2 >* i=poly ; ; i=i->next )
        {
            if( i->isEar ){ poly = ProcessEar( i ) ; sz-- ; break; }
            if( i->next==poly ) break;
        }
    }
}
```

Complexity: $O(n^2)$