FFTs in Graphics and Vision

Spherical Harmonics
and
Legendre Polynomials

Outline

- Math Stuff
  - Gram-Schmidt Orthogonalization
  - Completing Homogenous Polynomials
- Review
- Defining the Harmonics

Gram–Schmidt Orthogonalization

Given an inner product space V, and given a basis $\{v_1, \cdots, v_n\}$ we can define an orthonormal basis $\{w_1, \cdots, w_n\}$ for V:

$$\langle w_i, w_j \rangle = \delta_{ij}$$

Gram–Schmidt Orthogonalization

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<u>Algorithm</u>:

Start by making v_1 a unit vector:

$$w_1 = \frac{v_1}{\|v_1\|}$$

Gram–Schmidt Orthogonalization

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$$\langle w_i, w_j \rangle = \delta_{ij}$$

Algorithm:

To get the 2nd basis element, subtract from v_2 the w_1 component and then normalize:

$$w_2 = \frac{v_2 - \langle v_2, w_1 \rangle w_1}{\| v_2 - \langle v_2, w_1 \rangle w_1 \|}$$

Gram–Schmidt Orthogonalization

Given an inner product space V, and given a basis $\{v_1, \cdots, v_n\}$ we can define an orthonormal basis $\{w_1, \cdots, w_n\}$ for V:

$$\langle w_i, w_j \rangle = \delta_{ij}$$

<u>Algorithm</u>:

To get the i-th basis element, subtract off all the earlier components from v_i and then normalize:

$$w_i = \frac{v_i - \langle v_i, w_{i-1} \rangle w_{i-1} - \cdots - \langle v_i, w_1 \rangle w_1}{\| v_i - \langle v_i, w_{i-1} \rangle w_{i-1} - \cdots - \langle v_i, w_1 \rangle w_1 \|}$$

Gram–Schmidt Orthogonalization

<u>Example</u>:

Consider the space of polynomials of degree N on the interval $[-1,1]$, with the inner-product:

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x) \cdot g(x)\, dx$$

We would like to obtain an orthogonal basis:

$$\{p_0(x), \cdots, p_{N(x)}\}$$

Start with the basis of monomials:

$$\{1, x, x^2, \cdots, x^N\}$$

and perform Gram-Schmidt orthogonalization.

Gram–Schmidt Orthogonalization

Example:

Consider the space of polynomials of degree N on the interval $[-1,1]$, with the inner-product:

By induction, $p_k(x)$ is a polynomial of degree k since G.S. orthogonalization only subtracts off lower-degree basis functions.

We would like to obtain an orthogonal basis:

$$\{p_0(x), \cdots, p_{N(x)}\}$$

Start with the basis of monomials:

$$\{1, x, x^2, \cdots, x^N\}$$

and perform Gram-Schmidt orthogonalization.

Gram–Schmidt Orthogonalization

Example:

Starting with the constant term, we get:

$$p_0(x) = \frac{1}{\|1\|} = \frac{1}{\sqrt{\int_{-1}^{1} dx}} = \frac{1}{\sqrt{2}}$$

Gram–Schmidt Orthogonalization

Example:

For the linear term, we get:

$$p_1(x) = \frac{x - \langle x, p_0(x)\rangle p_0(x)}{\|x - \langle x, p_0(x)\rangle p_0(x)\|}$$

$$= \frac{x - \left(\int_{-1}^{1} x \cdot \frac{1}{\sqrt{2}}\, dx\right)\frac{1}{\sqrt{2}}}{\|x - \langle x, p_0(x)\rangle p_0(x)\|}$$

$$= \frac{x}{\sqrt{\int_{-1}^{1} x^2 dx}}$$

$$= \sqrt{\frac{3}{2}}x$$

Gram–Schmidt Orthogonalization

Example:

And for the quadratic term:

$$p_2(x) = \frac{x^2 - \langle x^2, p_1(x)\rangle p_1(x) - \langle x^2, p_0(x)\rangle p_0(x)}{\|x^2 - \langle x^2, p_1(x)\rangle p_1(x) - \langle x^2, p_0(x)\rangle p_0(x)\|}$$

These are the *Legendre Polynomials*.

Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials are all even/odd:

$$p_k(x) = a_k x^k + a_{k-1} x^{k-2} + \cdots$$

Proof by Induction:

Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials are all even/odd:

$$p_k(x) = a_k x^k + a_{k-1} x^{k-2} + \cdots$$

Proof by Induction ($k = 0$):

$$p_0(x) = \frac{1}{\sqrt{2}}$$

Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials are all even/odd:

$$p_k(x) = a_k x^k + a_{k-1} x^{k-2} + \cdots$$

Proof by Induction (assume true for $k = n$):

$$p_{n+1}(x) = \frac{x^{n+1} - \langle x^{n+1}, p_n(x)\rangle p_n(x) - \cdots - \langle x^{n+1}, p_0(x)\rangle p_0(x)}{\| x^{n+1} - \langle x^{n+1}, p_n(x)\rangle p_n(x) - \cdots - \langle x^{n+1}, p_0(x)\rangle p_0(x) \|}$$

Recall that:

$$\langle x^{n+1}, p_m(x)\rangle = \int_{-1}^{1} x^{n+1} \cdot p_m(x)\, dx$$

Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials are all even/odd:

$$p_k(x) = a_k x^k + a_{k-1} x^{k-2} + \cdots$$

Proof by Induction (assume true for $k = n$):

$$\langle x^{n+1}, p_m(x) \rangle = \int_{-1}^{1} x^{n+1} \cdot p_m(x)\, dx$$

Since $m \leq n$ we can assume that the monomials comprising $p_m(x)$ are all even/odd if m is.

Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials are all even/odd:

$$p_k(x) = a_k x^k + a_{k-1} x^{k-2} + \cdots$$

Proof by Induction (assume true for $k = n$):

So if n and m are both even/odd, then the polynomial $x^{n+1} \cdot p_m(x)$ is comprised of strictly odd-powered monomials:

$$\langle x^{n+1}, p_m(x) \rangle = \int_{-1}^{1} x^{n+1} \cdot p_m(x)\, dx = 0$$

Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials are all even/odd:

$$p_k(x) = a_k x^k + a_{k-1} x^{k-2} + \cdots$$

Proof by Induction (assume true for $k = n$):

$$p_{n+1}(x) = \frac{x^{n+1} \cancel{- \langle x^{n+1}, p_n(x) \rangle p_n(x)} - \langle x^{n+1}, p_{n-1}(x) \rangle p_{n-1}(x) - \cdots}{\left\| x^{n+1} \cancel{- \langle x^{n+1}, p_n(x) \rangle p_n(x)} - \langle x^{n+1}, p_{n-1}(x) \rangle p_{n-1}(x) - \cdots \right\|}$$

So $p_{n+1}(x)$ is obtained by starting with the monomial x^{n+1} and subtracting off monomials with the same parity.

Gram–Schmidt Orthogonalization

Example:

Consider the space of polynomials of degree N on the interval $[-1,1]$, with an inner-product giving more weight to points near the origin:

$$\langle f(x), g(x)\rangle_m = \int_{-1}^{1} (1-x^2)^m \cdot f(x) \cdot g(x)\, dx$$

We would like to obtain an orthonormal basis:

$$\{p_0^m(x), \cdots, p_N^m(x)\}$$

Gram–Schmidt Orthogonalization

Example:

$$\langle f(x), g(x)\rangle_m = \int_{-1}^{1} (1 - x^2)^m \cdot f(x) \cdot g(x)\, dx$$

We proceed as before with the new inner-product.

Since the weighting function is even, if f is an even function and g is an odd function (or vice-versa), the inner product must be zero:

$$\langle f(x), g(x)\rangle_m = 0$$

Thus, as before, the degree of the monomials comprising $p_l^m(x)$ must all have the same parity.

Completing Homogenous Polynomials

Given a polynomial $p(x,y,z)$ of degree d, consisting of monomials of powers d, $d-2$, …:

$$p(x,y,z) = \sum_{k=0}^{\lfloor d/2 \rfloor} \left(\sum_{l+m+n=d-2k} a_{lmn} \cdot x^l \cdot y^m \cdot z^n \right)$$

This is *not* a homogenous polynomial.

However, if we restrict it to the sphere, we can think of it as homogenous:

$$p(x,y,z) = \sum_{k=0}^{\lfloor d/2 \rfloor} (x^2+y^2+z^2)^k \left(\sum_{l+m+n=d-2k} a_{lmn} \cdot x^l \cdot y^m \cdot z^n \right)$$

Completing Homogenous Polynomials

<u>Example</u>:

$$p(x,y,z) = x^2y + y + z$$

is not a homogenous polynomial.

But on the unit-sphere, the polynomial:

$$q(x,y,z) = x^2y + (y+z)(x^2+y^2+z^2)$$

has identical values and is homogenous of degree 3.

Outline

Math Stuff

Review

- Spherical Harmonics

Defining the Harmonics

Spherical Harmonics

For each non-negative integer l, there are $2l + 1$ spherical harmonics of degree l satisfying:

1. Each spherical harmonic of degree l can be expressed as the restriction of a homogenous polynomial of degree l to the unit-sphere.
2. The different spherical harmonics are orthogonal to each other.

Spherical Harmonics

We saw that by considering just rotations about the y-axis, we could factor the spherical harmonics as:

$$\begin{aligned} Y_l^m(\theta,\phi) &= e^{im\theta} \cdot P_l^m(\phi) \\ &= (\cos\theta + i\sin\theta)^m \cdot P_l^m(\phi) \end{aligned}$$

where $|m| \leq l$.

Writing $x = \cos\phi$, the functions $P_l^m(\phi)$ are given by the associated Legendre polynomials:

$$P_l^k(\phi) = \frac{(-1)^k}{2^l l!}(1-x^2)^{k/2}\frac{d^{l+k}}{dx^{l+k}}(x^2-1)^l$$

Outline

Math Stuff

Review

Defining the Harmonics

Defining the Harmonics ($m \geq 0$)

To define the spherical harmonics, we would like to express the function:

$$Y_l^m(\theta, \phi) = (\cos\theta + i\sin\theta)^m \cdot P_l^m(\phi)$$

as the restriction of a homogenous polynomial of degree l to the unit sphere.

Defining the Harmonics ($m \geq 0$)

Using the parameterization of the unit-sphere

$$\Phi(\theta,\phi) = (\cos\theta \cdot \sin\phi\,, \cos\phi\,, \sin\theta \cdot \sin\phi)$$

we get:

$$\begin{aligned}
Y_l^m(\theta,\phi) &= (\cos\theta + i\sin\theta)^m \cdot P_l^m(\phi) \\
&= \left(\frac{x}{\sin\phi} + i\,\frac{z}{\sin\phi}\right)^m \cdot P_l^m(\phi) \\
&= (x+iz)^m \cdot \frac{P_l^m(\phi)}{\sin^m\phi} \\
&= (x+iz)^m \cdot \frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1-y^2}\right)^m}
\end{aligned}$$

Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta,\phi) = (x+iz)^m \cdot \frac{P_l^m(\cos^{-1} y)}{(1-y^2)^{m/2}}$$

This • is a homogenous polynomial of degree l.

This • is a homogenous polynomial of degree m.

So we want:

1. This • to complete to a homogenous polynomial of degree $l-m$.
2. The different Y_l^m to be orthogonal

Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta,\phi) \;=\; (x+iz)^m \cdot \frac{P_l^m(\cos^{-1} y)}{(1-y^2)^{m/2}}$$

<u>Homogeneous Completion</u>:

To satisfy the homogeneity constraint, we need:

$$\frac{P_l^m(\cos^{-1} y)}{(1-y^2)^{m/2}} = a_{l-m}y^{l-m} + a_{l-m-2}y^{l-m-2} + \cdots$$

Or equivalently:

$$P_l^m(\cos^{-1} y) = q_l^m(y) \cdot (1-y^2)^{m/2}$$

for some polynomial:

$$q_l^m(y) = a_{l-m}y^{l-m} + a_{l-m-2}y^{l-m-2} + \cdots$$

Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta,\phi) \ = e^{im\theta} \cdot P_l^m(\cos^{-1} y)$$

Orthogonality:

To satisfy the orthogonality constraint, we need:

$$\langle Y_l^m(\theta,\phi), Y_{l'}^{m'}(\theta,\phi)\rangle = 0 \qquad \forall\, l \neq l' \text{ or } \ m \neq m'$$

Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta,\phi) = e^{im\theta} \cdot P_l^m(\cos^{-1} y)$$

<u>Orthogonality ($m \neq m'$)</u>:

Since we have:

$$Y_l^m(\theta,\phi) = e^{im\theta} \cdot P_l^m(\phi)$$

we know that:

$$\begin{aligned}\langle Y_l^m(\theta,\phi), Y_{l'}^{m'}(\theta,\phi)\rangle &= \int_0^\pi \int_0^{2\pi} e^{im\theta} \cdot P_l^m(\phi) \cdot \overline{e^{im'\theta} \cdot P_{l'}^{m'}(\phi)}\, d\theta \sin\phi\, d\phi \\ &= \left(\int_0^\pi P_l^m(\phi) \cdot \overline{P_{l'}^{m'}(\phi)} \sin\phi\, d\phi\right) \cdot \left(\int_0^{2\pi} e^{i(m-m')\theta} d\theta\right)\end{aligned}$$

Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\cos^{-1} y)$$

Orthogonality ($m \neq m'$):

$$\langle Y_l^m(\theta,\phi), Y_{l'}^{m'}(\theta,\phi)\rangle = \left(\int_0^{\pi} P_l^m(\phi)\cdot\overline{P_{l'}^{m'}(\phi)}\sin\phi\, d\phi\right)\cdot\boxed{\left(\int_0^{2\pi} e^{i(m-m')\theta}\, d\theta\right)}$$

But this is zero whenever $m \neq m'$:

$$\int_0^{2\pi} e^{i(m-m')\theta}\, d\theta = \frac{1}{i(m-m')}\cdot e^{i(m-m')\theta}\Big|_0^{2\pi} = 0$$

Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta, \phi) \; = e^{im\theta} \cdot P_l^m(\cos^{-1} y)$$

<u>Orthogonality ($m = m'$ and $l \neq l'$)</u>:

We have to choose the function:

$$P_l^m(\cos^{-1} y) = q_l^m(y) \cdot (1 - y^2)^{m/2}$$

so that:

$$0 = \int_0^{\pi} \int_0^{2\pi} e^{im\theta} \cdot P_l^m(\phi) \cdot \overline{e^{im\theta} \cdot P_{l'}^m(\phi)} \, d\theta \sin\phi \, d\phi$$

$$\Downarrow$$

$$0 = \int_0^{\pi} P_l^m(\phi) \cdot \overline{P_{l'}^m(\phi)} \cdot \sin\phi \, d\phi$$

Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta,\phi) = e^{im\theta} \cdot P_l^m(\cos^{-1} y)$$

Orthogonality ($m = m'$ and $l \neq l'$):

We have to choose the function:

$$P_l^m(\cos^{-1} y) = q_l^m(y) \cdot (1 - y^2)^{m/2}$$

Using the change of variables:

$$\int_0^\pi P_l^m(\phi) \cdot \overline{P_{l'}^m(\phi)} \cdot \sin\phi \, d\phi = \int_{-1}^1 P_l^m(\cos^{-1} y) \cdot \overline{P_{l'}^m(\cos^{-1} y)} \, dy$$

$$= \int_{-1}^1 q_l^m(y) \cdot \overline{q_{l'}^m(y)} \cdot (1 - y^2)^m \, dy$$

Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta,\phi) = e^{im\theta} \cdot P_l^m(\cos^{-1} y)$$
$$P_l^m(\cos^{-1} y) = q_l^m(y) \cdot (1-y^2)^{m/2}$$

Thus, the polynomials $q_l^m(y)$ should:

1. Complete to homogeneous polynomials of degree $l-m$:
$$q_l^m(y) = a_{l-m} y^{l-m} + a_{l-m-2} y^{l-m-2} + \cdots$$

2. Satisfy the orthogonality condition:
$$0 = \int_{-1}^{1} q_l^m(y) \cdot \overline{q_{l'}^m(y)} \cdot (1-y^2)^m \, dy$$

Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta,\phi) \; = e^{im\theta} \cdot P_l^m(\cos^{-1} y)$$
$$P_l^m(\cos^{-1} y) = q_l^m(y) \cdot (1-y^2)^{m/2}$$

This is what we get with G.S. orthogonalization $\{1, y, y^2, \cdots\} \to \{p_0^m(y), p_1^m(y), p_2^m(y), \cdots\}$ relative to the inner-product:

$$\langle f(y), g(y)\rangle_m = \int_{-1}^{1} f(y) \cdot g(y) \cdot (1-y^2)^m dy$$

and set:

$$q_l^m(y) = p_{l-m}^m(y)$$

Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta,\phi) = e^{im\theta} \cdot P_l^m(\cos^{-1} y)$$
$$P_l^m(\cos^{-1} y) = q_l^m(y) \cdot (1-y^2)^{m/2}$$

In sum, we get an expression for the spherical harmonics as:

$$Y_l^m(\theta,\phi) = e^{im\theta} \cdot p_{l-m}^m(\cos\phi) \cdot \left(\sqrt{1-\cos^2\phi}\right)^m$$
$$= e^{im\theta} \cdot p_{l-m}^m(\cos\phi) \cdot \sin^m\phi$$

where $p_{l-m}^m(y)$ is a (homogeneously completable) polynomial of degree $l-m$.

Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta,\phi) = e^{im\theta} \cdot p_{l-m}^m(\cos\phi) \cdot \sin^m\phi$$

Examples ($l = 0$):

$$Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$

Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta,\phi) = e^{im\theta} \cdot p_{l-m}^m(\cos\phi) \cdot \sin^m \phi$$

Examples ($l = 1$):

$$Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}} \sin\phi \cdot e^{-i\theta}$$

$$Y_1^{0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}} \cos\phi$$

$$Y_1^{1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}} \sin\phi \cdot e^{i\theta}$$

Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta,\phi) = e^{im\theta} \cdot p_{l-m}^m(\cos\phi) \cdot \sin^m\phi$$

Examples ($l = 2$):

$$Y_2^{-2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}} \sin^2\phi \cdot e^{-i2\theta}$$

$$Y_2^{-1}(\theta,\phi) = \sqrt{\frac{15}{8\pi}} \sin\phi \cdot \cos\phi \cdot e^{-i\theta}$$

$$Y_2^{0}(\theta,\phi) = \sqrt{\frac{5}{16\pi}} (3\cos^2\phi - 1)$$

$$Y_2^{1}(\theta,\phi) = \sqrt{\frac{15}{8\pi}} \sin\phi \cdot \cos\phi \cdot e^{i\theta}$$

$$Y_2^{2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}} \sin^2\phi \cdot e^{i2\theta}$$