FFTs in Graphics and Vision

Spherical Harmonics

Outline

Math Stuff

Review

Finding the Spherical Harmonics

Homogenous Polynomials

A homogenous polynomial of degree d in n variables can be expressed as:

$$
\begin{aligned}
p_d(x_1,\cdots,x_n) &= \sum_{j_1+\cdots+j_n=d} a_{j_1\cdots j_n}\cdot x_1^{j_1}\cdots x_n^{j_n}\\
&= \sum_{j_1=0}^{d} x_1^{j_1}\left(\sum_{j_2+\cdots+j_n=d-j_1} a_{j_1\cdots j_n}\cdot x_2^{j_2}\cdots x_n^{j_n}\right)
\end{aligned}
$$

Homogenous Polynomials

$$p_d(x_1, \cdots, x_n) = \sum_{j_1=0}^{d} x_1^{j_1} \left(\sum_{j_2+\cdots+j_n=d-j_1} a_{j_1\cdots j_n} \cdot x_2^{j_2} \cdots x_n^{j_n} \right)$$

If we fix the value of the first coefficient at $x_1 = \zeta$, we get a new polynomial in $n - 1$ variables:

$$q_d(x_2, \cdots, x_n) = p_d(\zeta, x_2, \cdots, x_n)$$

This gives:

$$q_d(x_2, \cdots, x_n) = \sum_{j_1=0}^{d} \zeta^{j_1} \left(\sum_{j_2+\cdots+j_n=d-j_1} a_{j_1\cdots j_n} \cdot x_2^{j_2} \cdots x_n^{j_n} \right)$$

Homogenous Polynomials

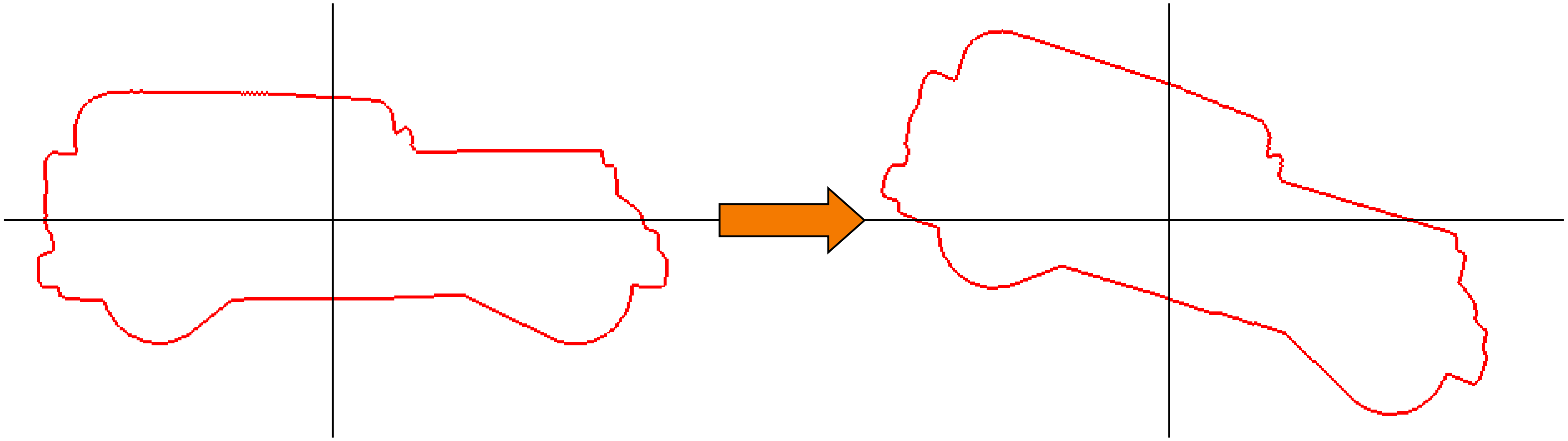
$$q_d(x_2, \cdots, x_n) = \sum_{j_1=0}^{d} \zeta^{j_1} \left(\sum_{j_2+\cdots+j_n=d-j_1} a_{j_1\cdots j_n} \cdot x_2^{j_2} \cdots x_n^{j_n} \right)$$

So the new polynomial, obtained by fixing the value of the first variable is a polynomial of degree at most d in $n-1$ variables.

Review

So far, we have considered the representation of the 2D group of rotations, acting on the space of (complex-valued) functions on the unit circle:

$$(\rho_R f)(p) = f(R^{-1}p)$$



Review

Since the group of 2D rotations is commutative, Schur's lemma tells us that the space of functions can be expressed as the sum of irreducible representations:

$$F = \bigoplus F_l$$

where each F_l is a one-dimensional.

Review

In the 2D case, we know that the F_l are spanned by the complex exponentials of degree l:

$$F_l = \left\{ a \cdot e^{il\theta} \middle| a \in \mathbb{C} \right\}$$

⇒ The ability to compute the Fourier transform of an arbitrary function $f(\theta)$:

$$f(\theta) = \sum_{l=-\infty}^{\infty} \hat{f}(l) \cdot e^{il\theta}$$

has applications to operations such as smoothing and correlation that are tied to the action of the group of rotation on the space of functions.

Functions on the Sphere

What happens when we consider the space of functions on the unit sphere?

Since the group of 3D rotations doesn't commute, we don't expect to express the space of functions as the sum of irreducible representations:

$$F = \bigoplus F_l$$

where each of the F_l is one-dimensional.

Functions on the Sphere

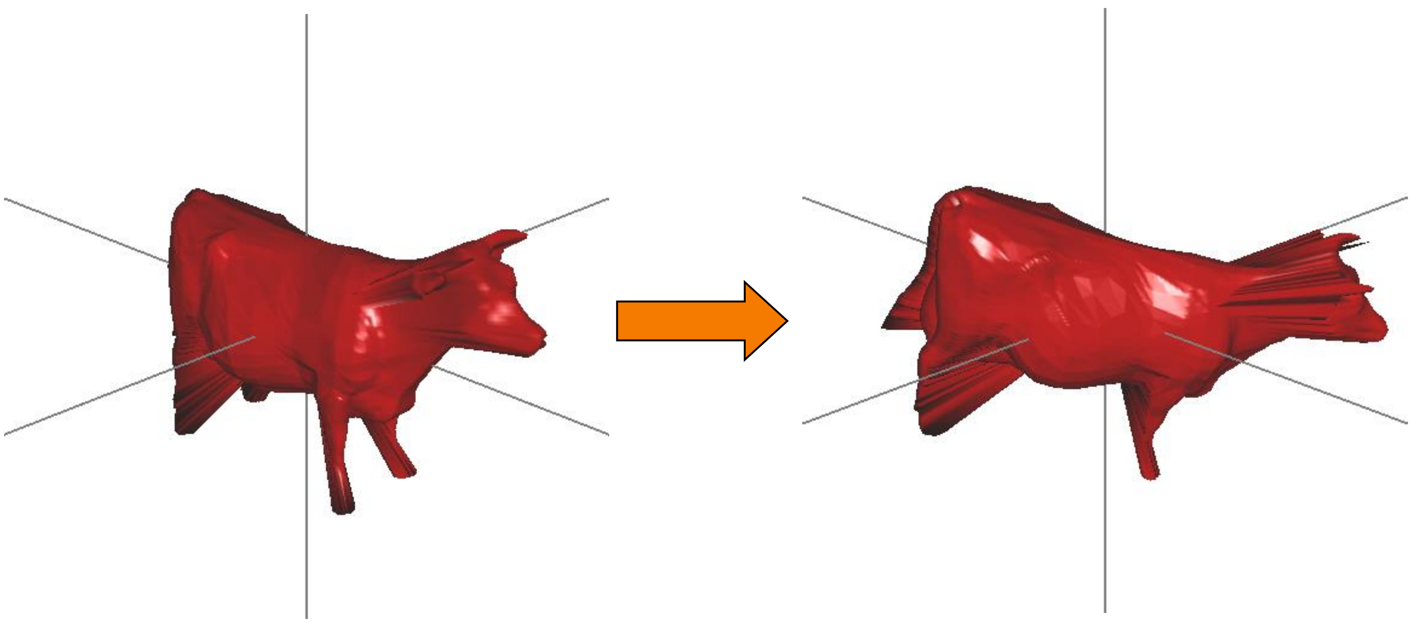
What happens when we consider the space of functions on the unit sphere?

However, we would still like to compute the irreducible representations. And in particular…

Goal

Let F be the space of (complex-value) functions on the unit sphere and let ρ be the representation of the group of 3D rotations, acting on the space of functions by rotation:

$$(\rho_R f)(p) = f(R^{-1}p)$$



Goal

Let F be the space of (complex-value) functions on the unit sphere and let ρ be the representation of the group of 3D rotations, acting on the space of functions by rotation:

$$(\rho_R f)(p) = f(R^{-1}p)$$

we would like to know what the irreducible representations are.

What We Know

We know that the irreducible representations are related to the sub-spaces of homogenous polynomials of fixed degree:

- Given a homogenous polynomial of degree l, a rotation of the polynomial will still be a homogenous polynomial of degree l:

  $$\rho_R\left(HP^l(x,y,z)\right) \subset HP^l(x,y,z)$$

- If we “throw out” the homogenous polynomials whose restriction to the unit sphere can be expressed as the restriction of a homogenous polynomial of smaller degree, we get a $(2l+1)$-dimensional representation.

What We Know

Letting l index the degree of the homogenous polynomial, we can decompose the space of spherical functions as:

$$F = \bigoplus_{l=0}^{\infty} F_l$$

where:

$$\dim(F_l) = 2l + 1$$

with $f \in F_l$ expressable as the restriction of a homogenous polynomial in three variables, of degree l.

What We Want to Know

What are the functions in F_l?

That is, what are the functions forming a basis for each F_l:

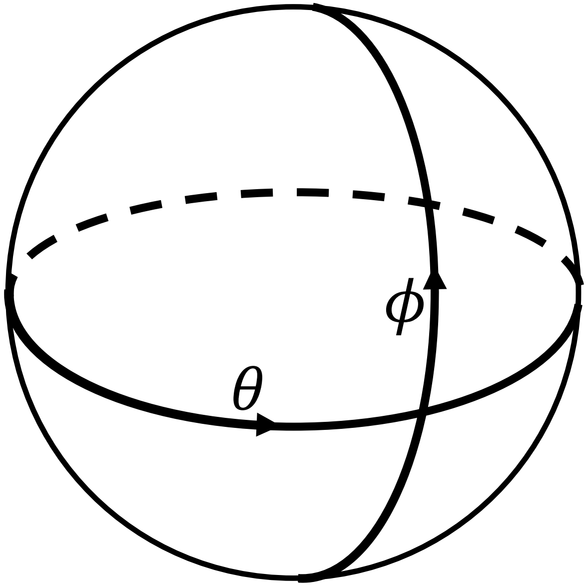
$$F_l = \mathrm{Span}\{f_l^0, \cdots, f_l^{2l}\}$$

Parameterizing Spherical Functions

A point on the unit sphere can be parameterized by its angles of longitude and latitude:

$$\Phi(\theta,\phi) = (\cos\theta \cdot \sin\phi\,, \cos\phi\,, \sin\theta \cdot \sin\phi)$$

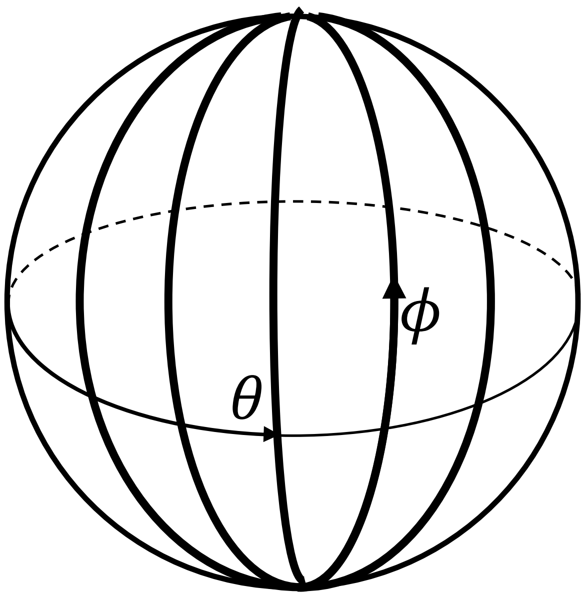
with $\theta \in [0,2\pi)$ and $\phi \in [0,\pi]$.



Parameterizing Spherical Functions

$$\Phi(\theta,\phi) = (\cos\theta \cdot \sin\phi\,, \cos\phi\,, \sin\theta \cdot \sin\phi)$$

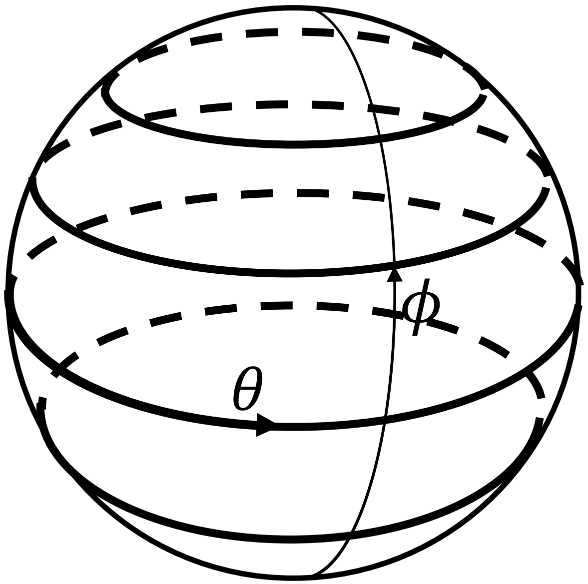
Fixing θ, we get great semi-circles (meridians) through the North and South poles.



Parameterizing Spherical Functions

$$\Phi(\theta,\phi) = (\cos\theta \cdot \sin\phi\,, \cos\phi\,, \sin\theta \cdot \sin\phi)$$

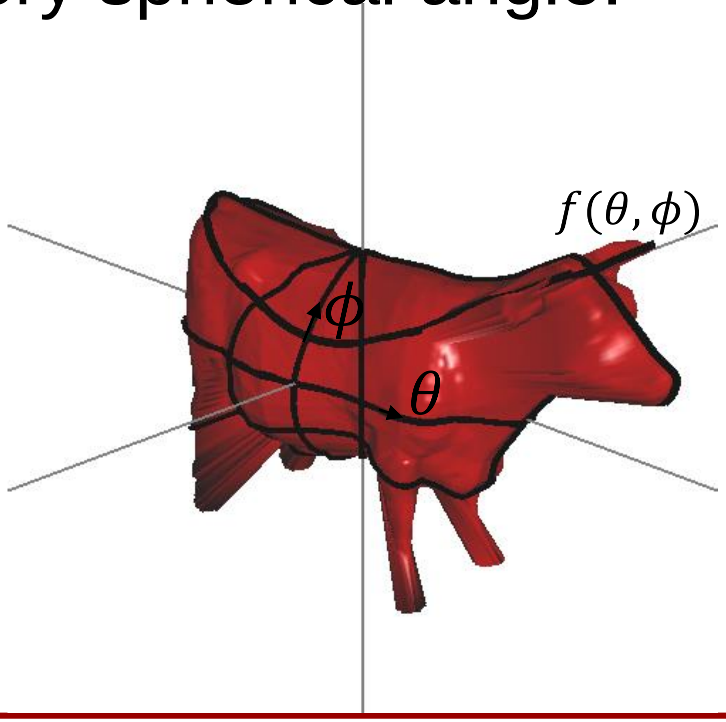
Fixing ϕ, we get parallels about the y-axis.



Parameterizing Spherical Functions

$$\Phi(\theta,\phi) = (\cos\theta \cdot \sin\phi\, , \cos\phi\, , \sin\theta \cdot \sin\phi)$$

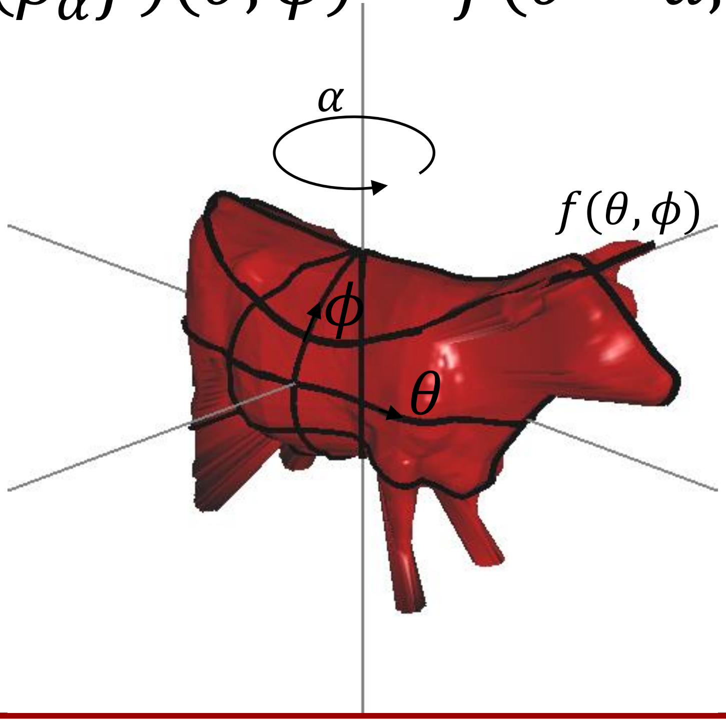
A spherical function can be represented by its values at every spherical angle.



Rotations About the *y*-Axis

Instead of considering the action of the entire group of rotations on the space of spherical functions, we can consider the subset of rotations that rotate about the y-axis:

$$(\rho_\alpha f)(\theta,\phi) = f(\theta-\alpha,\phi)$$



Rotations About the *y*-Axis

Instead of considering the entire group of rotations, we can consider the subset of rotations that rotate about the y-axis.

- This set of rotations is a group:
  - The product of two rotations about the y-axis is still a rotation about the y-axis.
  - The rotation by $-\alpha$ degrees about the y-axis is the inverse of the rotation by α degrees.

Rotations About the *y*-Axis

Instead of considering the entire group of rotations, we can consider the subset of rotations that rotate about the y-axis.

- This set of rotations is a group.
- This sub-group is commutative.

Rotations About the *y*-Axis

We know that:

- Rotations map the sub-spaces F_l back into themselves, and
- Rotations about the y-axis are a sub-group of the group of 3D rotations

$\Rightarrow$ The sub-spaces F_l are representations for the sub-group of rotations about the y-axis.

- The rotations about the y-axis is commutative

Each F_l can be expressed as the sum of one-dimensional representations that are fixed by rotations about the y-axis.

Rotations About the *y*-Axis

Thus, for each l, there must exist a basis of orthogonal functions $\left\{f_l^{\,0}(\theta,\phi),\cdots,f_l^{\,2l}(\theta,\phi)\right\}$ such that a rotation by α degrees about the y-axis becomes multiplication by a complex number:

$$\rho_R\left(f_l^k\right)=\lambda_l^k(\alpha)\cdot f_l^k$$

Rotations About the *y*-Axis

$$\rho_R\left(f_l^k\right) = \lambda_l^k(\alpha) \cdot f_l^k$$

Since the representation is unitary, we know that for any angle of rotation α, we must have:

$$\left\|\lambda_l^k(\alpha)\right\| = 1$$

Since we know that representations preserve the group structure, and since rotating by α degrees and then by β degrees is equivalent to rotating by $(\alpha + \beta)$ degrees, we have:

$$\lambda_l^k(\alpha + \beta) = \lambda_l^k(\alpha) \cdot \lambda_l^k(\beta)$$

Rotations About the y-Axis

$$\rho_R\left(f_l^k\right) = \lambda_l^k(\alpha) \cdot f_l^k$$

The only functions satisfying these properties are:

$$\lambda_l^k(\alpha) = e^{-im_{k,l}\alpha}$$

Moreover, since we know that rotations by $\alpha = 2\pi$ degrees about the y-axis do not change a function, the powers $m_{k,l}$ must be integers.

Rotations About the *y*-Axis

$$\rho_R\left(f_l^k\right) = \lambda_l^k(\alpha) \cdot f_l^k$$

Consider the function:

$$\tilde{f}_l^k(\theta, \phi) = \frac{f_l^k(\theta, \phi)}{e^{i m_{k,l} \theta}}$$

Rotations About the *y*-Axis

$$\tilde{f}_l^k(\theta,\phi) = \frac{f_l^k(\theta,\phi)}{e^{im_{k,l}\theta}}$$

When we rotate by α degrees about the y-axis:

$$\begin{aligned}
\left(\rho_\alpha \tilde{f}_l^k\right) &= \tilde{f}_l^k(\theta-\alpha,\phi) \\
&= \frac{f_l^k(\theta-\alpha,\phi)}{e^{im_{k,l}(\theta-\alpha)}} \\
&= \frac{e^{-im_{k,l}\alpha}\cdot f_l^k(\theta,\phi)}{e^{-im_{k,l}\alpha}\cdot e^{im_{k,l}\theta}} \\
&= \tilde{f}_l^k(\theta,\phi)
\end{aligned}$$

Rotations About the *y*-Axis

$$\tilde{f}_l^k(\theta,\phi) = \frac{f_l^k(\theta,\phi)}{e^{im_{k,l}\theta}}$$

When we rotate by α degrees about the y-axis:

$$\left(\rho_\alpha \tilde{f}_l^k\right) = \tilde{f}_l^k(\theta,\phi)$$

Since these functions are unchanged by rotations about the y-axis, this must imply that they are only functions of ϕ:

$$\tilde{f}_l^k(\theta,\phi) = p_l^k(\phi)$$

So we have:

$$f_l^k(\theta,\phi) = e^{im_{k,l}\theta} \cdot p_l^k(\phi)$$

Factoring the Spherical Harmonics

$$f_l^k(\theta,\phi) = e^{im_{k,l}\theta} \cdot p_l^k(\phi)$$

What can we say about the integers $m_{k,l}$?

The (x, y, z) coordinates of a point on the sphere are defined by:

$$\Phi(\theta,\phi) = (\cos\theta \cdot \sin\phi\,, \cos\phi\,, \sin\theta \cdot \sin\phi)$$

$$\Downarrow$$

$$\cos\theta = \frac{x}{\sin\phi} \quad \cos\phi = y \quad \sin\theta = \frac{z}{\sin\phi}$$

Factoring the Spherical Harmonics

$$f_l^k(\theta,\phi) = e^{i m_{k,l}\theta} \cdot p_l^k(\phi)$$

What can we say about the integers $m_{k,l}$?

Fixing the angle of latitude, $\phi = \phi_0$, gives:

$$\begin{aligned} f_l^k(\theta,\phi_0) &= (\cos\theta + i\sin\theta)^{m_{k,l}} \cdot p_l^k(\phi_0) \\ &= \left(\frac{x}{\sin\phi_0} + i\,\frac{z}{\sin\phi_0}\right)^{m_{k,l}} \cdot p_l^k(\phi_0) \\ &= (x+iz)^{m_{k,l}} \cdot \frac{p_l^k(\phi_0)}{\sin^{k_l}\phi_0} \end{aligned}$$

Factoring the Spherical Harmonics

$$f_l^k(\theta,\phi) = e^{i m_{k,l}\theta} \cdot p_l^k(\phi)$$

What can we say about the integers $m_{k,l}$?

Fixing the angle of latitude, $\phi = \phi_0$, gives:

$$f_l^k(\theta,\phi_0) = (x+iz)^{m_{k,l}} \cdot \frac{p_l^k(\phi_0)}{\sin^{k_l}\phi_0}$$

But $f_l^k(\theta,\phi)$ is the restriction of a homogenous polynomial of degree l to the unit sphere.

So fixing $y = \cos\phi_0$, we get a polynomial of degree at most l:

$$-l \le m_{k,l} \le l$$

The Spherical Harmonics

In sum, we know that the space of spherical functions F can be expressed as the sum of sub-representations:

$$F = \bigoplus F_l$$

where the functions in F_l are obtained by considering the restrictions of homogenous polynomials of degree l to the unit sphere.

The Spherical Harmonics

Each F_l is a $(2l+1)$-dimensional space of functions, spanned by an orthogonal basis $\{f_l^0(\theta,\phi),\cdots,f_l^{2l}(\theta,\phi)\}$ where the k-th basis function can be expressed as:

$$f_l^k(\theta,\phi) = e^{im_{k,l}\theta} \cdot p_l^k(\phi)$$

where $m_{k,l}$ is an integer in the range $[-l,l]$.

The Spherical Harmonics

It turns out that for every value of $-l \leq k \leq l$ there is exactly one basis function:

$$Y_l^k(\theta,\phi) = e^{ik\theta} \cdot p_l^k(\phi)$$

These are the <u>spherical harmonics</u> of degree l.

Aside

To evaluate the spherical harmonics, we need to know what the functions $p_l^k(\phi)$ are.

These are defined by setting:

$$p_l^k(\phi) = P_l^k(\cos\phi)$$

where the P_l^k are the associated Legendre polynomials, defined by:

$$P_l^k(x) = \frac{(-1)^k}{2^l l!}(1-x^2)^{\frac{k}{2}}\frac{d^{l+k}}{dx^{l+k}}(x^2-1)^l$$

$$P_l^{-k}(x) = (-1)^k\frac{(l-k)!}{(l+k)!}P_l^k(x)$$

for $k \geq 0$.

The Spherical Harmonics

Examples ($l = 0$):

$$Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}}$$

The Spherical Harmonics

Examples ($l = 1$):

$$Y_1^{-1}(\theta, \phi) = \sqrt{\frac{3}{8\pi}} \sin\phi \cdot e^{-i\theta}$$

$$Y_1^{0}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\phi$$

$$Y_1^{1}(\theta, \phi) = \sqrt{\frac{3}{8\pi}} \sin\phi \cdot e^{i\theta}$$

The Spherical Harmonics

Examples ($l = 2$):

$$Y_2^{-2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}} \sin^2\phi \cdot e^{-i2\theta}$$

$$Y_2^{-1}(\theta,\phi) = \sqrt{\frac{15}{8\pi}} \sin\phi \cdot \cos\phi \cdot e^{-i\theta}$$

$$Y_2^{0}(\theta,\phi) = \sqrt{\frac{5}{16\pi}} (3\cos^2\phi - 1)$$

$$Y_2^{1}(\theta,\phi) = \sqrt{\frac{15}{8\pi}} \sin\phi \cdot \cos\phi \cdot e^{i\theta}$$

$$Y_2^{2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}} \sin^2\phi \cdot e^{i2\theta}$$

The Spherical Harmonics

$$\cos\theta = \frac{x}{\sin\phi} \quad \cos\phi = y \quad \sin\theta = \frac{z}{\sin\phi}$$

$$Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$

$$\Downarrow$$

$$Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$

The Spherical Harmonics

$$\cos\theta = \frac{x}{\sin\phi} \quad \cos\phi = y \quad \sin\theta = \frac{z}{\sin\phi}$$

$$Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\phi \cdot e^{-i\theta}$$

$$\Downarrow$$

$$Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\,(x - iz)$$

The Spherical Harmonics

$$\cos\theta = \frac{x}{\sin\phi} \quad \cos\phi = y \quad \sin\theta = \frac{z}{\sin\phi}$$

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\phi$$

$$\Downarrow$$

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\,y$$

The Spherical Harmonics

$$\cos\theta = \frac{x}{\sin\phi} \quad \cos\phi = y \quad \sin\theta = \frac{z}{\sin\phi}$$

$$Y_1^1(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\phi \cdot e^{i\theta}$$

$$\Downarrow$$

$$Y_1^1(\theta,\phi) = \sqrt{\frac{3}{8\pi}}(x + iz)$$

The Spherical Harmonics

$$\cos\theta = \frac{x}{\sin\phi} \quad \cos\phi = y \quad \sin\theta = \frac{z}{\sin\phi}$$

$$Y_2^{-2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2\phi \cdot e^{-i2\theta}$$

$$\Downarrow$$

$$Y_2^{-2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}(x - iz)^2$$

The Spherical Harmonics

$$\cos\theta = \frac{x}{\sin\phi} \quad \cos\phi = y \quad \sin\theta = \frac{z}{\sin\phi}$$

$$Y_2^{-1}(\theta,\phi) = \sqrt{\frac{15}{8\pi}}\sin\phi\cdot\cos\phi\cdot e^{-i\theta}$$

$$\Downarrow$$

$$Y_2^{-1}(\theta,\phi) = \sqrt{\frac{15}{8\pi}}y\cdot(x-iz)$$

The Spherical Harmonics

$$\cos\theta = \frac{x}{\sin\phi} \quad \cos\phi = y \quad \sin\theta = \frac{z}{\sin\phi}$$

$$Y_2^0(\theta,\phi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\phi - 1)$$

$$\Downarrow$$

$$Y_2^0(\theta,\phi) = \sqrt{\frac{5}{16\pi}}(3y^2 - 1)$$

$$= \sqrt{\frac{5}{16\pi}}(2y^2 - x^2 - z^2)$$

The Spherical Harmonics

$$\cos\theta = \frac{x}{\sin\phi} \quad \cos\phi = y \quad \sin\theta = \frac{z}{\sin\phi}$$

$$Y_2^1(\theta,\phi) = \sqrt{\frac{15}{8\pi}}\sin\phi\cdot\cos\phi\cdot e^{i\theta}$$

$$\Downarrow$$

$$Y_2^1(\theta,\phi) = \sqrt{\frac{15}{8\pi}}y\cdot(x+iz)$$

The Spherical Harmonics

$$\cos\theta = \frac{x}{\sin\phi} \quad \cos\phi = y \quad \sin\theta = \frac{z}{\sin\phi}$$

$$Y_2^2(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2\phi \cdot e^{i2\theta}$$

$$\Downarrow$$

$$Y_2^2(\theta,\phi) = \sqrt{\frac{15}{32\pi}}(x+iz)^2$$

Implications

For any spherical function $f(\theta, \phi)$, we can express f as the sum of functions in F_k:

$$f(\theta,\phi) = \sum_{l=0}^{\infty} f_l(\theta,\phi)$$

Each f_l is expressable as the sum of harmonics:

$$f_l(\theta,\phi) = \sum_{k=-l}^{l} \hat{f}(l,k) \cdot Y_l^k(\theta,\phi)$$

Which gives:

$$f(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{k=-l}^{l} \hat{f}(l,k) \cdot Y_l^k(\theta,\phi)$$

Implications

When the function f is real-valued, we may want to express it as the sum of real-valued functions.

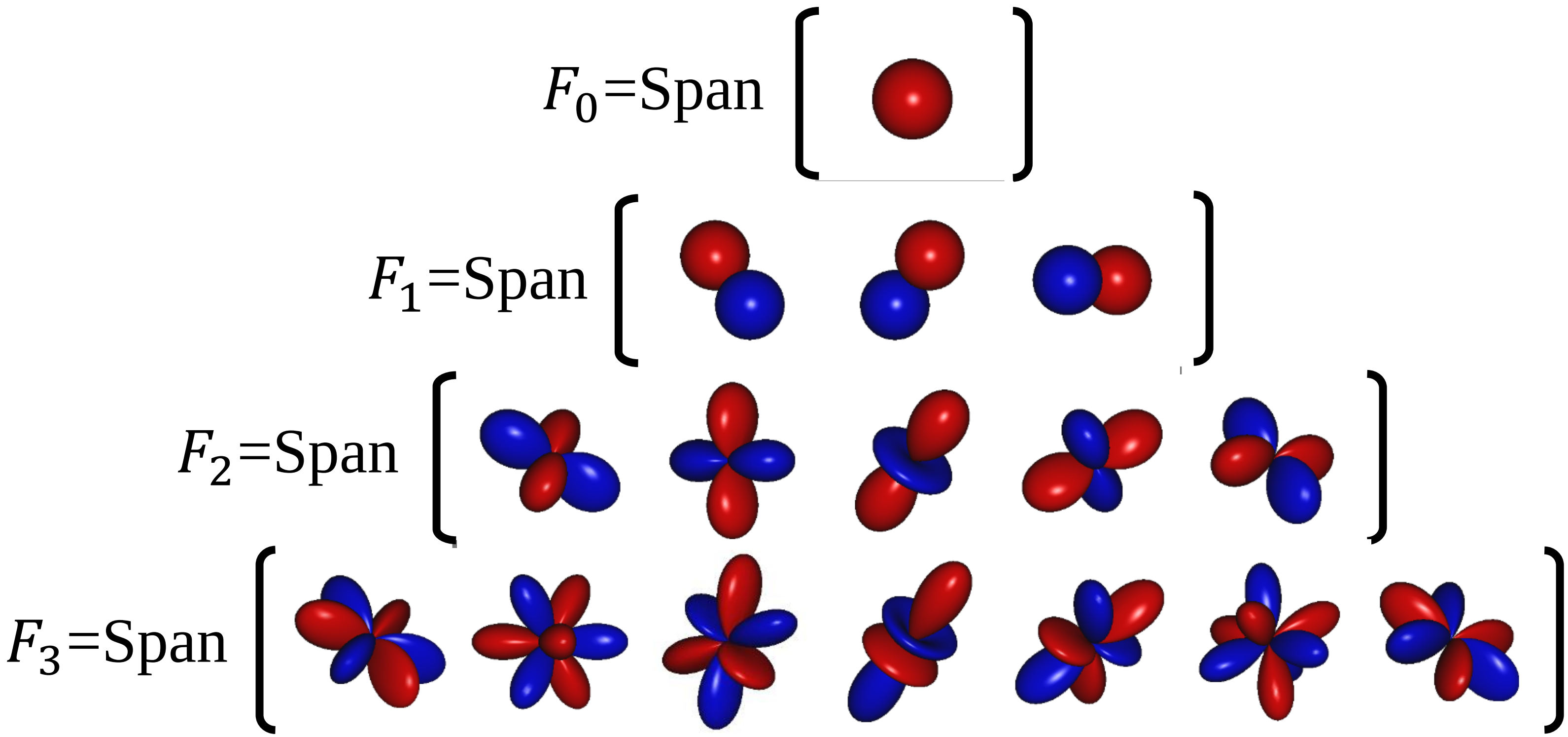
We can do this by considering the real and imaginary parts of the harmonics independently:

$$\mathrm{Re}\left(Y_l^k(\theta,\phi)\right) = (-1)^k \mathrm{Re}\left(Y_l^{-k}(\theta,\phi)\right) = \cos(k\theta)\cdot p_l^k(\phi)$$

$$\mathrm{Im}\left(Y_l^k(\theta,\phi)\right) = (-1)^{k+1} \mathrm{Im}\left(Y_l^{-k}(\theta,\phi)\right) = \sin(k\theta)\cdot p_l^k(\phi)$$

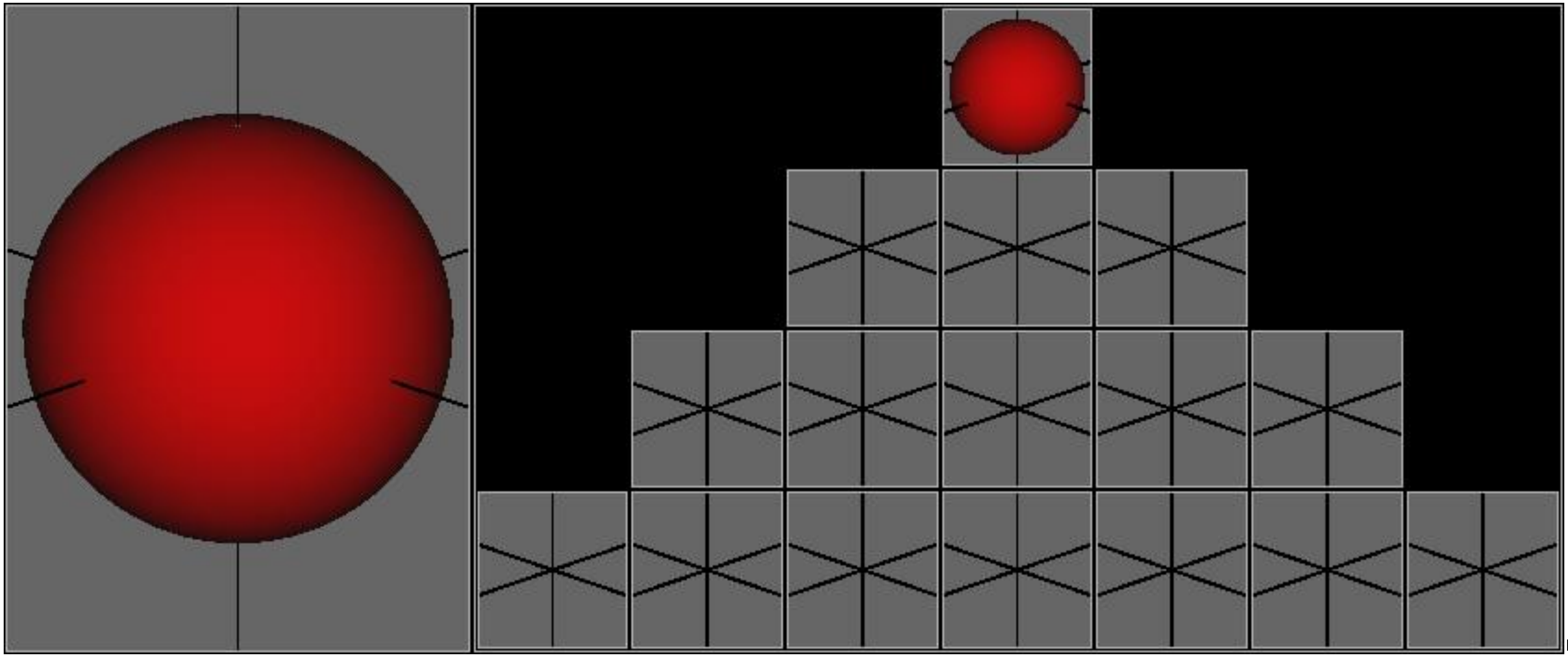
The Spherical Harmonics

When the function f is real-valued, we may want to express it as the sum of real-valued functions.



Implications

Since the spaces F_l are sub-representations, rotating a function that is the sum of the l-th spherical harmonics, will give a function that is the sum of the l-th spherical harmonics.



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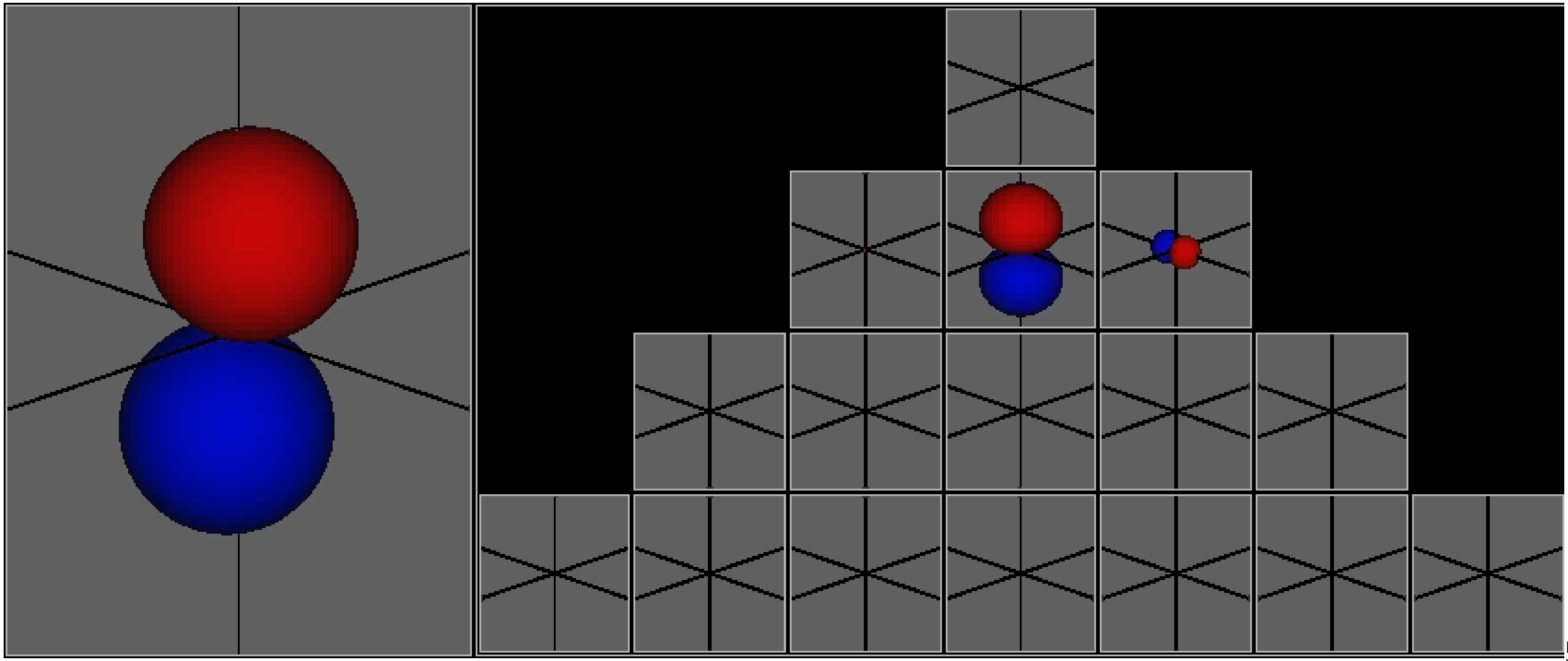

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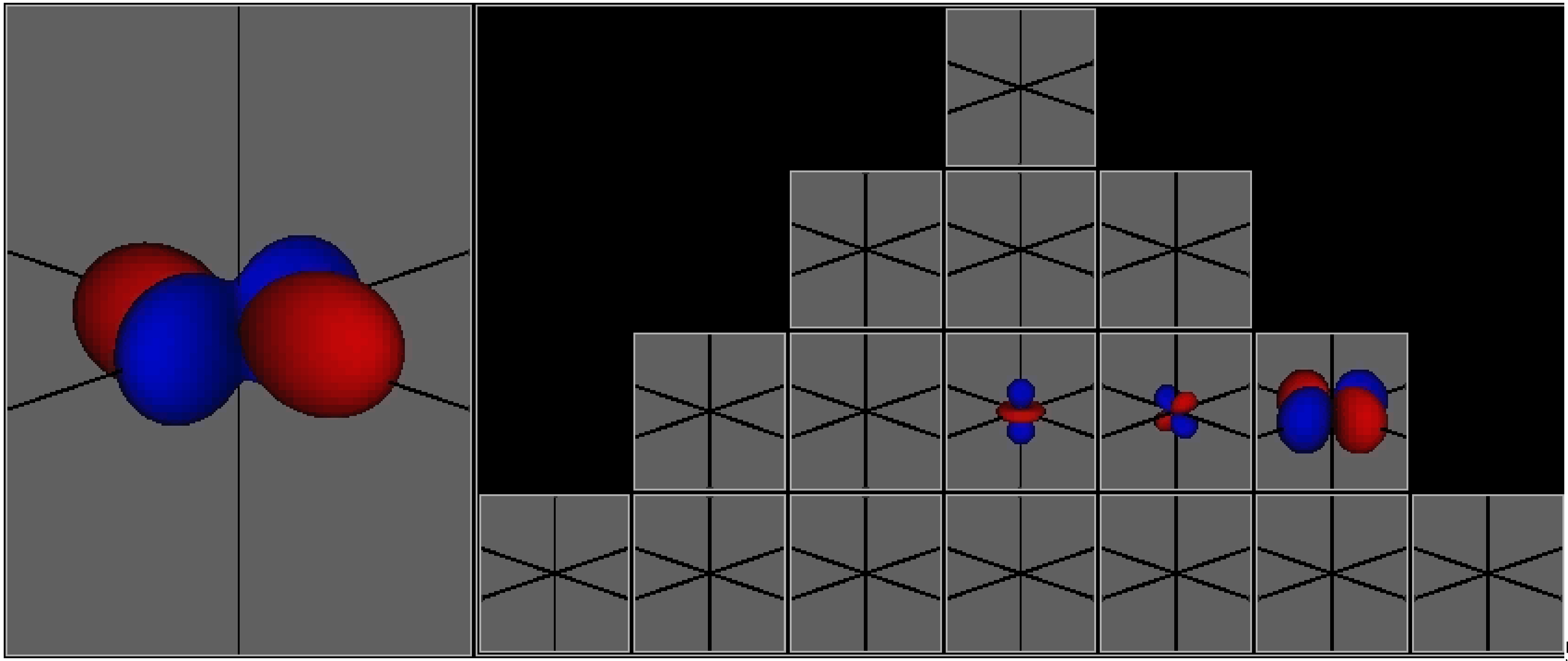

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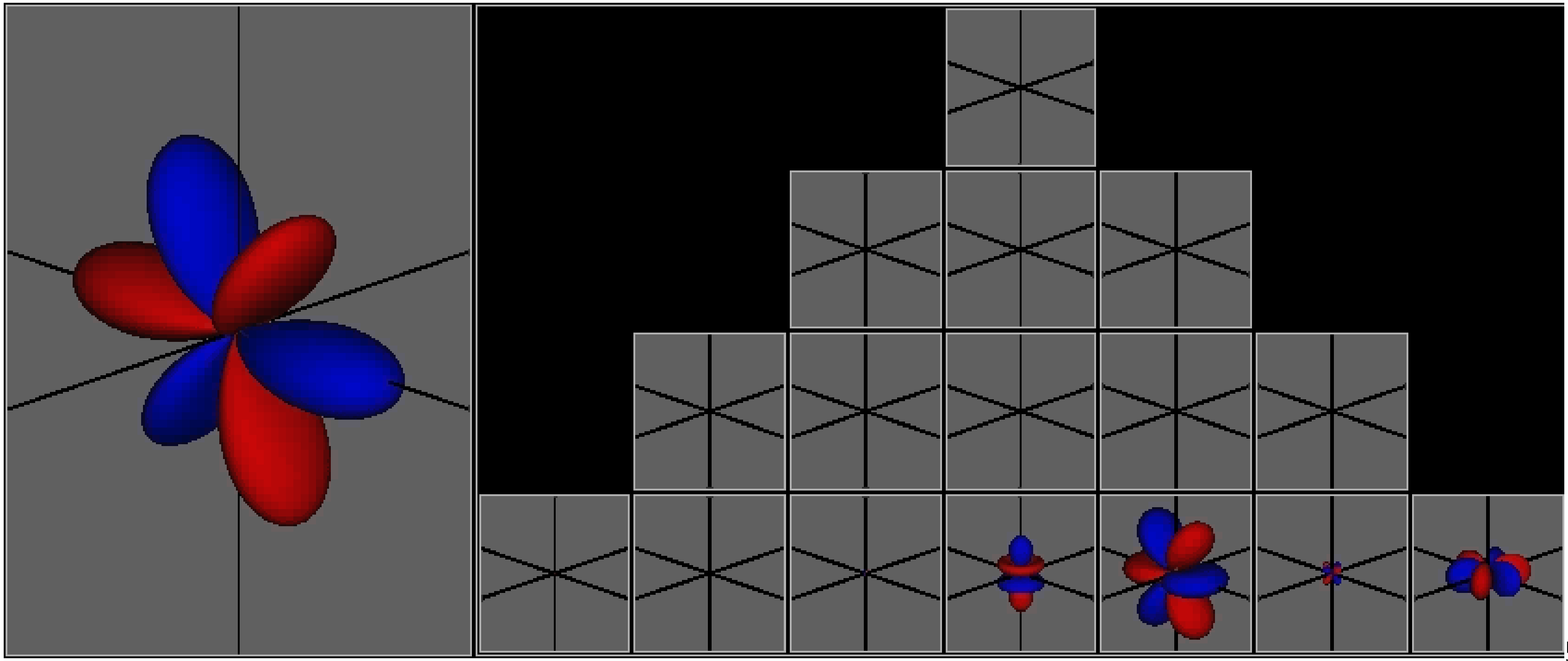
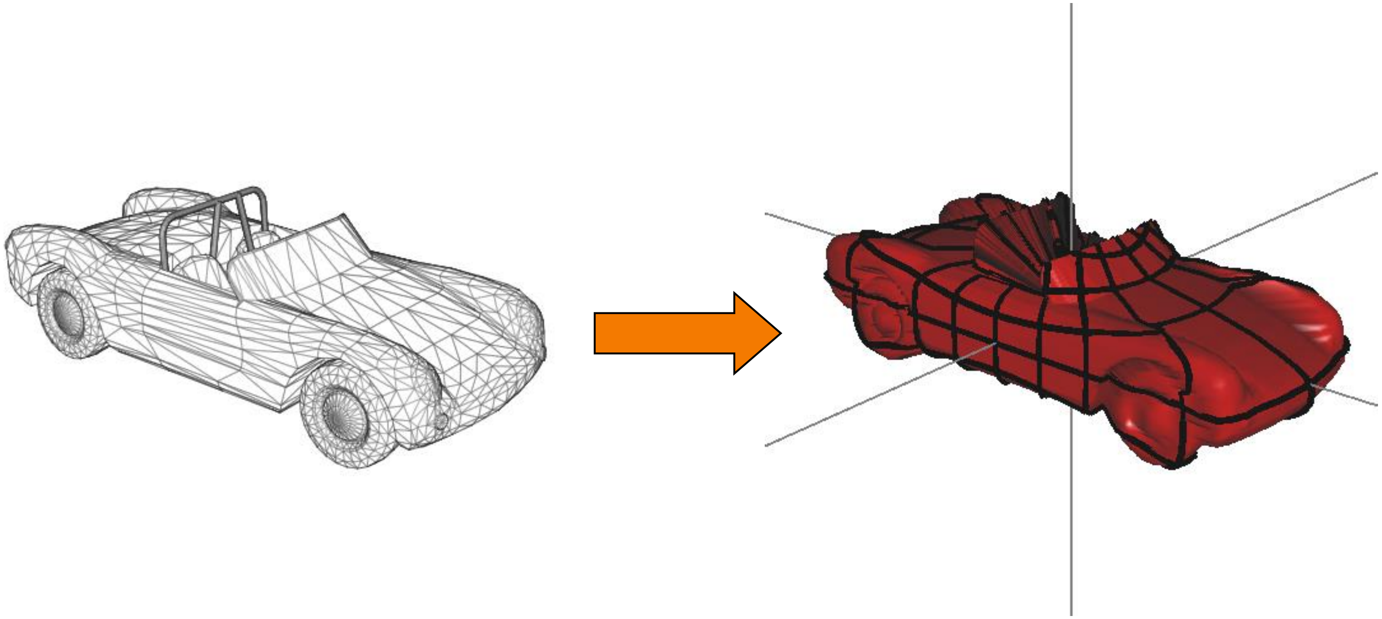


Application (Shape Matching)

Goal:

Given a spherical function representing the surface of a 3D model by a spherical function:



we would like a rotation invariant representation.

Application (Shape Matching)

<u>Approach</u>:

We use the facts that:

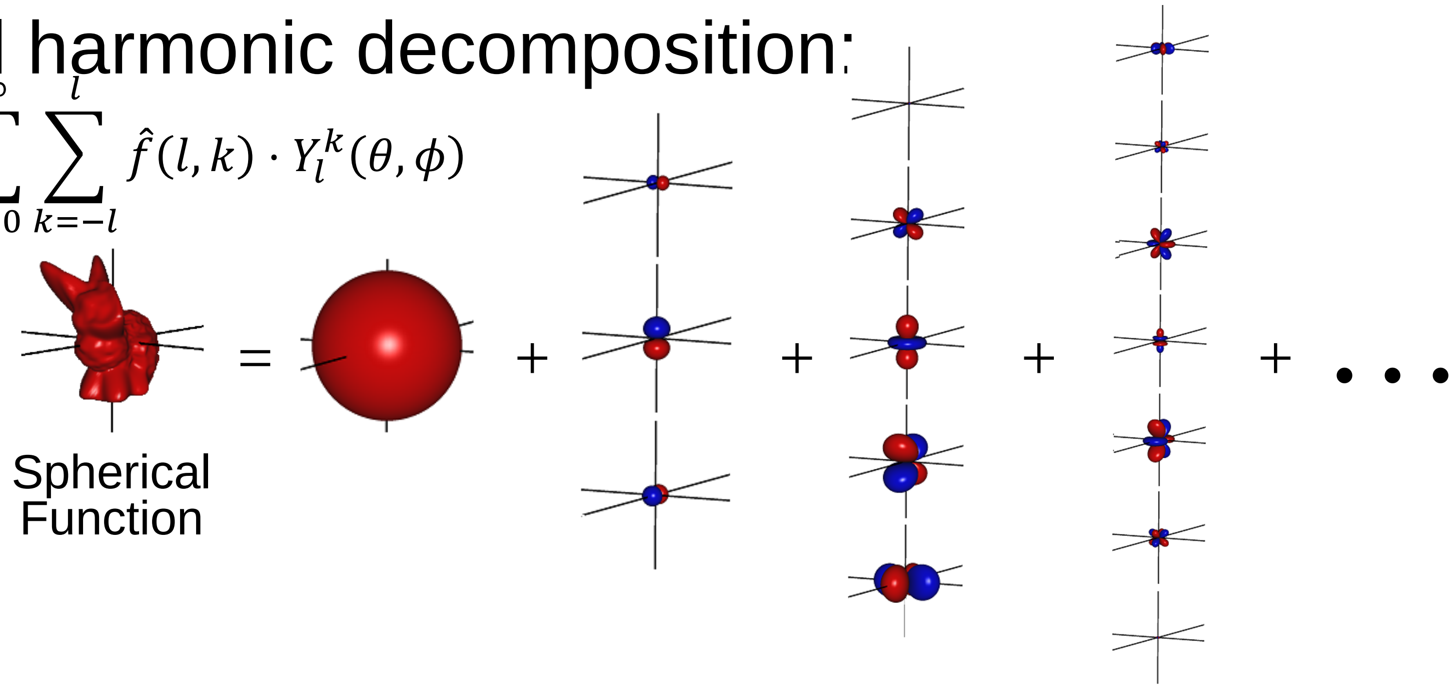
- Rotations are unitary.
- The spherical harmonics of degree l are representations of the group of rotation.

Application (Shape Matching)

Approach:

Specifically, given a spherical function, we obtain its spherical harmonic decomposition:

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{k=-l}^{l}\hat{f}(l,k)\cdot Y_l^k(\theta,\phi)$$

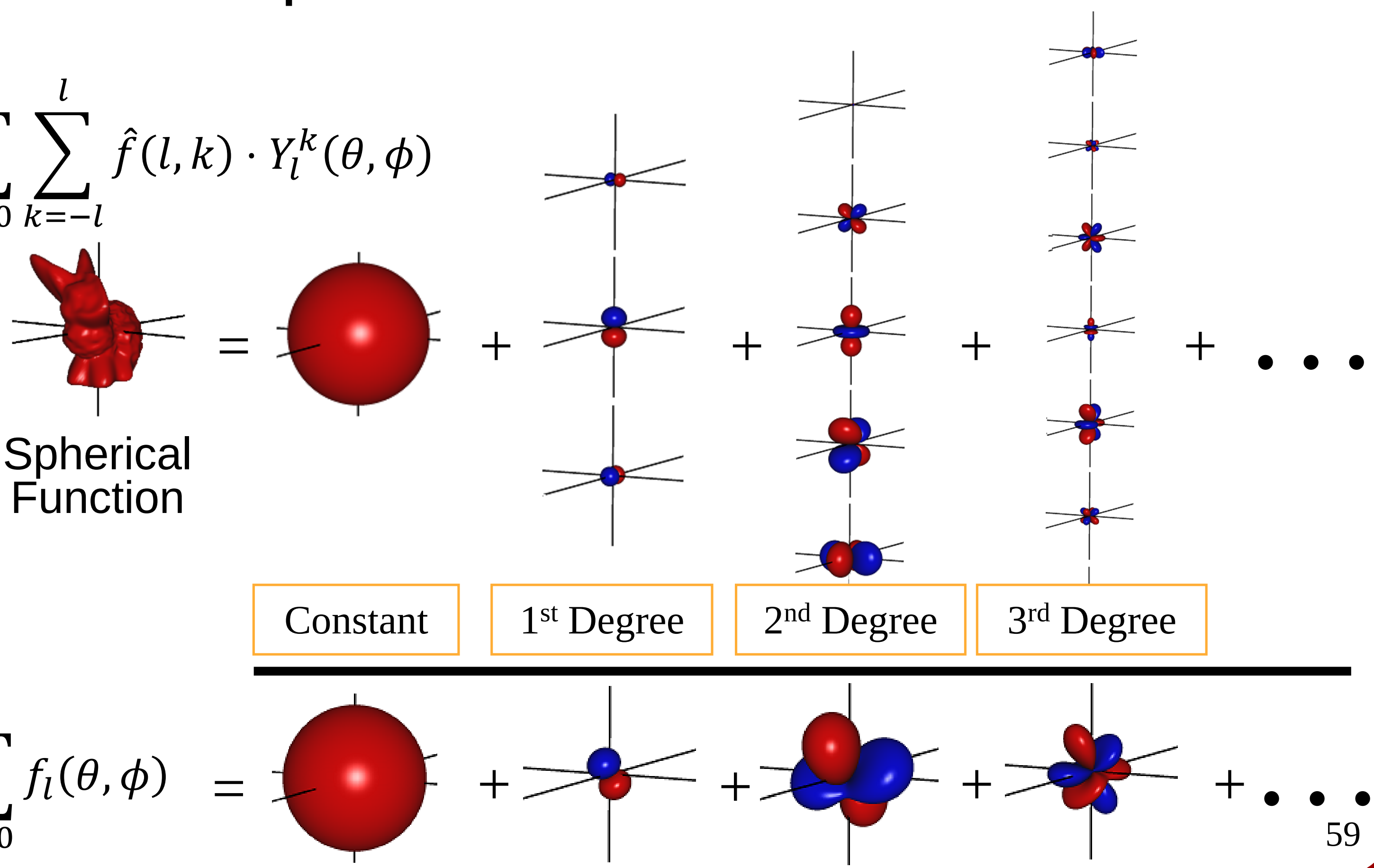


Application (Shape Matching)

Approach:

We combine the spherical harmonics of the same degree…

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{k=-l}^{l}\hat{f}(l,k)\cdot Y_l^k(\theta,\phi)$$



$$f(\theta,\phi)=\sum_{l=0}^{\infty}f_l(\theta,\phi)$$

Application (Shape Matching)

Approach:

Storing the norms:

$$\{\|f_0\|, \|f_1\|, \cdots\}$$

we obtain a rotation invariant shape descriptor.

Norms Invariant to Rotation

Constant | 1st Degree | 2nd Degree | 3rd Degree

$$f(\theta,\phi) = \sum_{l=0}^{\infty} f_l(\theta,\phi) =$$