Eurographics Symposium on Geometry Processing 2023
P. Memari and J. Solomon
(Guest Editors)

Volume 42 (2023), Number 5

Poisson Manifold Reconstruction — Beyond Co-dimension One: Supplemental Material

M. Kohlbrenner[1] S. Lee[2] M. Alexa[1] M. Kazhdan[2]

[1]TU Berlin, Germany [2]Johns Hopkins University, USA

Here we describe how the hierarchical solver used for minimizing the bi-quadratic energy can be adapted to support fine-to-coarse relaxation. To see why this is challenging, consider the naïve approach where one defines the coarse residual constraints by restricting the exterior product of the gradients of the fine solution. Such an approach will not work because the definition of the coarse energy requires knowing which of the finer coordinate functions the individual coarse coordinate functions would correct. However, as the finer solutions are coupled through the wedge product, there is no way to "disentangle" them at the coarse resolution.

1. Fine-to-Coarse Hierarchical Solver

We recall that that our aim is to minimize the bi-quadratic energy:

$$E(\mathbf{F}) = \mathbf{W}_\mathbf{F}^\top \cdot \mathbf{M} \cdot \mathbf{W}_\mathbf{F} - 2\cdot \mathbf{W}_\mathbf{F}^\top \cdot \mathbf{b} + \langle \omega, \omega \rangle + \mathrm{Tr}\left(\mathbf{F}^\top \cdot \mathbf{R} \cdot \mathbf{F}\right), \quad (1)$$

where:

- ω is the target skew-symmetric matrix field;
- $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2) \in \mathbb{R}^{N\times 2}$ are the coefficients of the coordinate functions in the basis $\{\phi_i\}_{i=1}^N$;
- $\mathbf{W}_\mathbf{F}$ are the coefficients of the exterior product of the gradients of the coordinate functions in the basis $\{\omega_\mathbf{i}\}_{\mathbf{i}\in\mathcal{I}}$;
- $\mathbf{M} \in \mathbb{R}^{|\mathcal{I}|\times|\mathcal{I}|}$ is the mass matrix for the basis $\{\omega_\mathbf{i}\}$;
- $\mathbf{b} \in \mathbb{R}^{|\mathcal{I}|}$ is the vector obtained by integrating ω against the basis functions $\{\omega_\mathbf{i}\}$; and
- $\mathbf{R} \in \mathbb{R}^{N\times N}$ is the matrix defining the regularization (e.g. screening and Dirichlet) energy.

We describe how the definition of the energy at a coarse resolution can be adjusted to take into account the solution previously estimated at finer resolutions. As with traditional multigrid, we show that it is possible to successively restrict finer solutions to coarser levels. However, unlike traditional multigrid, the multi-quadratic nature of our problem requires that we encapsulate the restricted information from finer levels as both matrices and vectors.

We begin by considering a two-level hierarchy, defining the restricting matrices and vectors. Then, proceeding to a three-level hierarchy. we show that restriction can be performed successively, so that the adaptation of the energy at a coarse level only needs to know (1) the solution and (2) the restriction information from the *next* finer level.

1.1. Two-level hierarchies ($\widehat{N} < N$)

We start with a two-level hierarchy, with bases $\{\hat{\phi}_i\}_{i=1}^{\widehat{N}}$ and $\{\phi_i\}_{i=1}^N$, and associated prolongation matrix $\mathbf{P}^\mathcal{F}$. As in the main body of the text, these define bases for skew-symmetric matrix fields $\{\hat{\omega}_\mathbf{i}\}_{\mathbf{i}\in\widehat{\mathcal{I}}}$ and $\{\omega_\mathbf{i}\}_{\mathbf{i}\in\mathcal{I}}$, with associated prolongation matrix $\mathbf{P}^\mathcal{W}$. And again, we drop the $\mathcal{F}$ and $\mathcal{W}$ superscripts and denote by

$$\widehat{\mathbf{M}} = \mathbf{P}^\top \cdot \mathbf{M} \cdot \mathbf{P}, \qquad \widehat{\mathbf{R}} = \mathbf{P}^\top \cdot \mathbf{R} \cdot \mathbf{P}, \qquad \text{and} \qquad \hat{\mathbf{b}} = \mathbf{P}^\top \cdot \mathbf{b}$$

the restricted mass matrix, restricted regularization matrix, and restricted constraints, respectively.

Given an estimated fine solution, $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2) \in \mathbb{R}^{N\times 2}$, the energy of adding a coarse correction term $\widehat{\mathbf{F}} = (\hat{\mathbf{f}}_1, \hat{\mathbf{f}}_2) \in \mathbb{R}^{\widehat{N}\times 2}$ can be defined by prolonging to the finer resolution, combining the prolonged correction and fine solution, and evaluating the energy:

$$E_\mathbf{F}(\widehat{\mathbf{F}}) \equiv E(\mathbf{P}\cdot\hat{\mathbf{f}} + \mathbf{F}).$$

Expanding using Equation 1 we get:†

$$\begin{aligned}
E_\mathbf{F}(\widehat{\mathbf{F}}) = & \underbrace{\mathbf{W}_{\widehat{\mathbf{F}}}^\top \cdot \widehat{\mathbf{M}} \cdot \mathbf{W}_{\widehat{\mathbf{F}}}}_{(2,2)} \\
& + \underbrace{2\cdot \mathbf{W}_{\widehat{\mathbf{F}}}^\top \cdot \widehat{\mathbf{M}}_{02} \cdot \hat{\mathbf{f}}_1}_{(2,1)} + \underbrace{2\cdot \mathbf{W}_{\widehat{\mathbf{F}}}^\top \cdot \widehat{\mathbf{M}}_{01} \cdot \hat{\mathbf{f}}_2}_{(1,2)} \\
& + \underbrace{\hat{\mathbf{f}}_1^\top \cdot \left(\widehat{\mathbf{M}}_{22} + \widehat{\mathbf{R}}\right) \cdot \hat{\mathbf{f}}_1}_{(2,0)} + \underbrace{\hat{\mathbf{f}}_2^\top \cdot \left(\widehat{\mathbf{M}}_{11} + \widehat{\mathbf{R}}\right) \cdot \hat{\mathbf{f}}_2}_{(0,2)} \\
& + \underbrace{2\cdot \mathbf{W}_{\widehat{\mathbf{F}}}^\top \cdot \hat{\mathbf{w}}_0 + 2\cdot \hat{\mathbf{f}}_1^\top \cdot \widehat{\mathbf{M}}_{21} \cdot \hat{\mathbf{f}}_2 - 2\cdot \mathbf{W}_{\widehat{\mathbf{F}}}^\top \cdot \hat{\mathbf{b}}}_{(1,1)} \\
& + \underbrace{2\cdot \hat{\mathbf{f}}_1^\top \cdot \left(\hat{\mathbf{w}}_2 - \hat{\mathbf{b}}_2 + \hat{\mathbf{r}}_2\right)}_{(1,0)} + \underbrace{2\cdot \hat{\mathbf{f}}_2^\top \cdot \left(\hat{\mathbf{w}}_1 - \hat{\mathbf{b}}_1 + \hat{\mathbf{r}}_1\right)}_{(0,1)} \\
& + \underbrace{\hat{c}}_{(0,0)}
\end{aligned}$$

† The pair of integers under the different terms gives the degree of the resulting polynomial in the coordinate functions $\hat{\mathbf{f}}_1$ and $\hat{\mathbf{f}}_2$.

with the matrices $\widehat{\mathbf{M}}_{ij}$, the vectors $\hat{\mathbf{b}}_i$, $\hat{\mathbf{w}}_i$, and $\hat{\mathbf{r}}_i$, and the scalar $\hat{c}$ defined as follows:

- The matrices $\widehat{\mathbf{M}}_{ij}$ satisfy $\widehat{\mathbf{M}}_{ji} = \widehat{\mathbf{M}}_{ij}^\top$ and are defined by:

$$\begin{aligned}
\widehat{\mathbf{M}}_{01} &= \mathbf{P}^\top \cdot \mathbf{M} \cdot [\mathbf{F}]_1 \cdot \mathbf{P} \in \mathbb{R}^{|\widehat{\mathcal{I}}| \times \widehat{N}} \\
\widehat{\mathbf{M}}_{02} &= \mathbf{P}^\top \cdot \mathbf{M} \cdot [\mathbf{F}]_2 \cdot \mathbf{P} \in \mathbb{R}^{|\widehat{\mathcal{I}}| \times \widehat{N}} \\
\widehat{\mathbf{M}}_{12} &= \mathbf{P}^\top \cdot [\mathbf{F}]_1^\top \cdot \mathbf{M} \cdot [\mathbf{F}]_2 \cdot \mathbf{P} \in \mathbb{R}^{\widehat{N} \times \widehat{N}} \\
\widehat{\mathbf{M}}_{11} &= \mathbf{P}^\top \cdot [\mathbf{F}]_1^\top \cdot \mathbf{M} \cdot [\mathbf{F}]_1 \cdot \mathbf{P} \in \mathbb{R}^{\widehat{N} \times \widehat{N}} \\
\widehat{\mathbf{M}}_{22} &= \mathbf{P}^\top \cdot [\mathbf{F}]_2^\top \cdot \mathbf{M} \cdot [\mathbf{F}]_2 \cdot \mathbf{P} \in \mathbb{R}^{\widehat{N} \times \widehat{N}}
\end{aligned}$$

- The vectors $\hat{\mathbf{b}}_i$, $\hat{\mathbf{w}}_i$, and $\hat{\mathbf{r}}_i$ are defined by:

$$\begin{aligned}
\hat{\mathbf{b}}_1 &= \mathbf{P}^\top \cdot [\mathbf{F}]_1^\top \cdot \mathbf{b} \in \mathbb{R}^{\widehat{N}} \\
\hat{\mathbf{b}}_2 &= \mathbf{P}^\top \cdot [\mathbf{F}]_2^\top \cdot \mathbf{b} \in \mathbb{R}^{\widehat{N}} \\
\hat{\mathbf{w}}_0 &= \mathbf{P}^\top \cdot \mathbf{M} \cdot \mathbf{W}_\mathbf{F} \in \mathbb{R}^{|\widehat{\mathcal{I}}|} \\
\hat{\mathbf{w}}_1 &= \mathbf{P}^\top \cdot [\mathbf{F}]_2^\top \cdot \mathbf{M} \cdot \mathbf{W}_\mathbf{F} \in \mathbb{R}^{\widehat{N}} \\
\hat{\mathbf{w}}_2 &= \mathbf{P}^\top \cdot [\mathbf{F}]_2^\top \cdot \mathbf{M} \cdot \mathbf{W}_\mathbf{F} \in \mathbb{R}^{\widehat{N}} \\
\hat{\mathbf{r}}_1 &= \mathbf{P}^\top \cdot \widehat{\mathbf{R}} \cdot \mathbf{f}_2 \in \mathbb{R}^{\widehat{N}} \\
\hat{\mathbf{r}}_2 &= \mathbf{P}^\top \cdot \widehat{\mathbf{R}} \cdot \mathbf{f}_1 \in \mathbb{R}^{\widehat{N}}
\end{aligned}$$

- The scalar $\hat{c}$ is defined by:

$$\hat{c} = \langle \omega, \omega \rangle + \mathbf{W}_\mathbf{F}^\top \cdot \mathbf{M} \cdot \mathbf{W}_\mathbf{F} - 2 \cdot \mathbf{W}_\mathbf{F}^\top \cdot \mathbf{b} + \mathrm{Tr}(\mathbf{F}^\top \cdot \mathbf{R} \cdot \mathbf{F}) \in \mathbb{R}$$

1.2. Three-level hierarchies ($\widehat{\widehat{N}} < \widehat{N} < N$)

We follow the notation above, using a double hat to denote quantities at the coarse resolution, a single hat to denote quantities at the middle resolution, and no hat to denote quantities at the finest resolution. We also denote by $\widehat{\mathbf{P}}$ the prolongation matrix from the coarse resolution to the middle resolution and by $\mathbf{P}$ the prolongation matrix from the middle resolution to the fine resolution.

In this context we are given solutions $\widehat{\mathbf{F}} \in \mathbb{R}^{\widehat{N}}$ and $\mathbf{F} \in \mathbb{R}^N$ and we would like to define matrices $\widehat{\widehat{\mathbf{M}}}_{ij}$, vectors $\hat{\hat{\mathbf{b}}}_i$, $\hat{\hat{\mathbf{w}}}_i$, and $\hat{\hat{\mathbf{r}}}_i$, and constant $\hat{\hat{c}}$ encapsulating the effects of both $\widehat{\mathbf{F}}$ and $\mathbf{F}$ on the correction energy. As above, we proceed using prolongation, this time applying it twice to move from the coarse level to the fine one.

Take for example the construction of the matrix $\widehat{\widehat{\mathbf{M}}}_{01}$. Proceeding as before, we get:

$$\widehat{\widehat{\mathbf{M}}}_{01} = \widehat{\mathbf{P}}^\top \cdot \mathbf{P}^\top \cdot \mathbf{M} \cdot [\widehat{\mathbf{P}} \cdot \widehat{\mathbf{F}} + \mathbf{F}]_1 \cdot \mathbf{P} \cdot \widehat{\mathbf{P}}.$$

Considered naïvely, this requires explicitly incorporating the fine correction term $\mathbf{F}$. However, recalling the Galerkin condition, $\widehat{\mathbf{M}} = \mathbf{P}^\top \cdot \mathbf{M} \cdot \mathbf{P}$, the fact that prolongation commutes with the exterior product of gradients, $[\mathbf{P} \cdot \hat{\mathbf{F}}]_i \cdot \mathbf{P} = \mathbf{P} \cdot [\widehat{\mathbf{F}}]_i$, and the linearity of the exterior product operator when all but one of the coordinates is fixed, $[\widehat{\mathbf{P}} \cdot \widehat{\mathbf{F}} + \mathbf{F}]_i = [\widehat{\mathbf{P}} \cdot \widehat{\mathbf{F}}]_i + [\mathbf{F}]_i$, we can expand the above to get:

$$\begin{aligned}
\widehat{\widehat{\mathbf{M}}}_{01} &= \widehat{\mathbf{P}}^\top \cdot \mathbf{P}^\top \cdot \mathbf{M} \cdot [\widehat{\mathbf{P}} \cdot \widehat{\mathbf{F}} + \mathbf{F}]_1 \cdot \mathbf{P} \cdot \widehat{\mathbf{P}} \\
&= \widehat{\mathbf{P}}^\top \cdot \left(\mathbf{P}^\top \cdot \mathbf{M} \cdot [\widehat{\mathbf{P}} \cdot \widehat{\mathbf{F}}]_1 \cdot \mathbf{P} + \mathbf{P}^\top \cdot \mathbf{M} \cdot [\mathbf{F}]_1 \cdot \mathbf{P} \right) \cdot \widehat{\mathbf{P}} \\
&= \widehat{\mathbf{P}}^\top \cdot \left(\mathbf{P}^\top \cdot \mathbf{M} \cdot \mathbf{P} \cdot [\widehat{\mathbf{F}}]_1 \cdot \mathbf{P} + \widehat{\mathbf{M}}_{01} \right) \cdot \widehat{\mathbf{P}} \\
&= \widehat{\mathbf{P}}^\top \cdot \left(\widehat{\mathbf{M}} \cdot [\widehat{\mathbf{F}}]_1 \cdot \mathbf{P} + \widehat{\mathbf{M}}_{01} \right) \cdot \widehat{\mathbf{P}}.
\end{aligned}$$

In particular, this shows that we can construct the matrix $\widehat{\widehat{\mathbf{M}}}_{01}$ only using the solution, $\widehat{\mathbf{F}}$, and the restriction information, $\widehat{\mathbf{M}}_{01}$, from the middle level. In a similar fashion, we get:

$$\begin{aligned}
\widehat{\widehat{\mathbf{M}}}_{02} &= \widehat{\mathbf{P}}^\top \cdot \left(\widehat{\mathbf{M}} \cdot [\widehat{\mathbf{F}}]_2 + \widehat{\mathbf{M}}_{02} \right) \cdot \widehat{\mathbf{P}} \\
\widehat{\widehat{\mathbf{M}}}_{12} &= \widehat{\mathbf{P}}^\top \cdot \left([\widehat{\mathbf{F}}]_1^\top \cdot \widehat{\mathbf{M}} \cdot [\widehat{\mathbf{F}}]_2 + \widehat{\mathbf{M}}_{12} + [\widehat{\mathbf{F}}]_1^\top \cdot \widehat{\mathbf{M}}_{02} + \widehat{\mathbf{M}}_{10} \cdot [\widehat{\mathbf{F}}]_2 \right) \cdot \widehat{\mathbf{P}} \\
\widehat{\widehat{\mathbf{M}}}_{11} &= \widehat{\mathbf{P}}^\top \cdot \left([\widehat{\mathbf{F}}]_1^\top \cdot \widehat{\mathbf{M}} \cdot [\widehat{\mathbf{F}}]_1 + \widehat{\mathbf{M}}_{11} + [\widehat{\mathbf{F}}]_1^\top \cdot \widehat{\mathbf{M}}_{01} + \widehat{\mathbf{M}}_{10} \cdot [\widehat{\mathbf{F}}]_1 \right) \cdot \widehat{\mathbf{P}} \\
\widehat{\widehat{\mathbf{M}}}_{22} &= \widehat{\mathbf{P}}^\top \cdot \left([\widehat{\mathbf{F}}]_2^\top \cdot \widehat{\mathbf{M}} \cdot [\widehat{\mathbf{F}}]_2 + \widehat{\mathbf{M}}_{22} + [\widehat{\mathbf{F}}]_2^\top \cdot \widehat{\mathbf{M}}_{02} + \widehat{\mathbf{M}}_{20} \cdot [\widehat{\mathbf{F}}]_2 \right) \cdot \widehat{\mathbf{P}} \\
\hat{\hat{\mathbf{b}}}_1 &= \widehat{\mathbf{P}}^\top \cdot \left(\hat{\mathbf{b}}_1 + [\widehat{\mathbf{F}}]_1^\top \cdot \hat{\mathbf{b}} \right) \\
\hat{\hat{\mathbf{b}}}_2 &= \widehat{\mathbf{P}}^\top \cdot \left(\hat{\mathbf{b}}_2 + [\widehat{\mathbf{F}}]_2^\top \cdot \hat{\mathbf{b}} \right) \\
\hat{\hat{\mathbf{w}}}_0 &= \widehat{\mathbf{P}}^\top \cdot \left(\hat{\mathbf{w}}_0 + \widehat{\mathbf{M}} \cdot \mathbf{W}_{\widehat{\mathbf{F}}} + \widehat{\mathbf{M}}_{01} \cdot \hat{\mathbf{f}}_2 + \widehat{\mathbf{M}}_{02} \cdot \hat{\mathbf{f}}_1 \right) \\
\hat{\hat{\mathbf{w}}}_1 &= \widehat{\mathbf{P}}^\top \cdot \Big(\hat{\mathbf{w}}_1 + [\widehat{\mathbf{F}}]_1 \cdot \widehat{\mathbf{M}} \cdot \mathbf{W}_{\widehat{\mathbf{F}}} + [\widehat{\mathbf{F}}]_2 \cdot \left(\widehat{\mathbf{M}}_{02} \cdot \hat{\mathbf{f}}_1 + \widehat{\mathbf{M}}_{01} \cdot \hat{\mathbf{f}}_2 + \hat{\mathbf{w}}_0 \right) \\
&\quad + \widehat{\mathbf{M}}_{10} \cdot \mathbf{W}_{\widehat{\mathbf{F}}} + \widehat{\mathbf{M}}_{12} \cdot \hat{\mathbf{f}}_1 + \widehat{\mathbf{M}}_{11} \cdot \hat{\mathbf{f}}_2 \Big) \\
\hat{\hat{\mathbf{w}}}_2 &= \widehat{\mathbf{P}}^\top \cdot \Big(\hat{\mathbf{w}}_2 + [\widehat{\mathbf{F}}]_1 \cdot \widehat{\mathbf{M}} \cdot \mathbf{W}_{\widehat{\mathbf{F}}} + [\widehat{\mathbf{F}}]_1 \cdot \left(\widehat{\mathbf{M}}_{02} \cdot \hat{\mathbf{f}}_1 + \widehat{\mathbf{M}}_{01} \cdot \hat{\mathbf{f}}_2 + \hat{\mathbf{w}}_0 \right) \\
&\quad + \widehat{\mathbf{M}}_{20} \cdot \mathbf{W}_{\widehat{\mathbf{F}}} + \widehat{\mathbf{M}}_{22} \cdot \hat{\mathbf{f}}_1 + \widehat{\mathbf{M}}_{21} \cdot \hat{\mathbf{f}}_2 \Big) \\
\hat{\hat{\mathbf{r}}}_1 &= \widehat{\mathbf{P}}^\top \cdot \left(\hat{\mathbf{r}}_1 + \widehat{\mathbf{R}} \cdot \hat{\mathbf{f}}_2 \right) \\
\hat{\hat{\mathbf{r}}}_2 &= \widehat{\mathbf{P}}^\top \cdot \left(\hat{\mathbf{r}}_2 + \widehat{\mathbf{R}} \cdot \hat{\mathbf{f}}_1 \right) \\
\hat{\hat{c}} &= \hat{c} + \mathbf{W}_{\widehat{\mathbf{F}}}^\top \cdot \widehat{\mathbf{M}} \cdot \mathbf{W}_{\widehat{\mathbf{F}}} - 2 \cdot \mathbf{W}_{\widehat{\mathbf{F}}}^\top \cdot \hat{\mathbf{b}}_0 \\
&\quad + \hat{\mathbf{f}}_1^\top \cdot \left(\mathbf{R} + \widehat{\mathbf{M}}_{22} \right) \cdot \hat{\mathbf{f}}_1 + \hat{\mathbf{f}}_2^\top \cdot \left(\mathbf{R} + \widehat{\mathbf{M}}_{11} \right) \cdot \hat{\mathbf{f}}_2 + 2 \cdot \hat{\mathbf{f}}_2^\top \cdot \widehat{\mathbf{M}}_{12} \cdot \hat{\mathbf{f}}_1 \\
&\quad + 2 \cdot \mathbf{W}_{\widehat{\mathbf{F}}}^\top \cdot \left(\widehat{\mathbf{M}}_{02} \cdot \hat{\mathbf{f}}_1 + \widehat{\mathbf{M}}_{01} \cdot \hat{\mathbf{f}}_2 + \hat{\mathbf{w}}_0 \right) \\
&\quad + 2 \cdot \hat{\mathbf{f}}_1^\top \cdot \left(\hat{\mathbf{w}}_2 - \hat{\mathbf{b}}_2 + \hat{\mathbf{r}}_2 \right) + 2 \cdot \hat{\mathbf{f}}_2^\top \cdot \left(\hat{\mathbf{w}}_1 - \hat{\mathbf{b}}_1 + \hat{\mathbf{r}}_1 \right).
\end{aligned}$$

As with the definition of $\widehat{\widehat{\mathbf{M}}}_{01}$, the definition of the other matrices, vectors, and scalar only depends on the solution and restriction information from the middle level.