

3D Object Representation
(Loop) Subdivision Surfaces

Misha Kazhdan

(601.457/657)

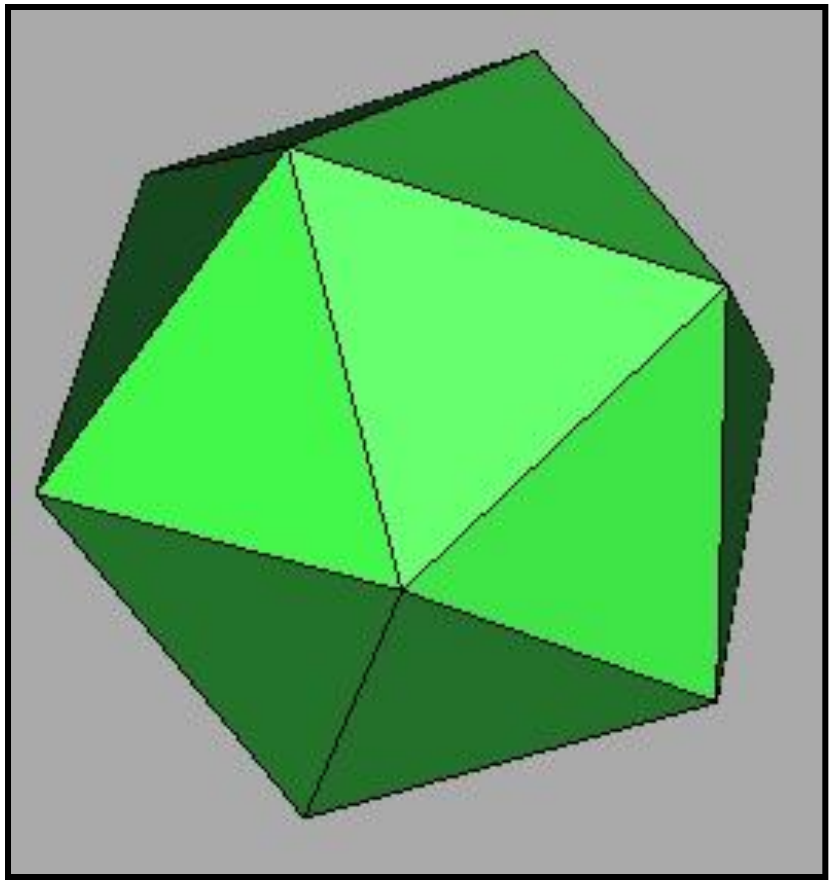
Subdivision for Modeling and Animation, Zorin and Schröder (2000)

Announcements



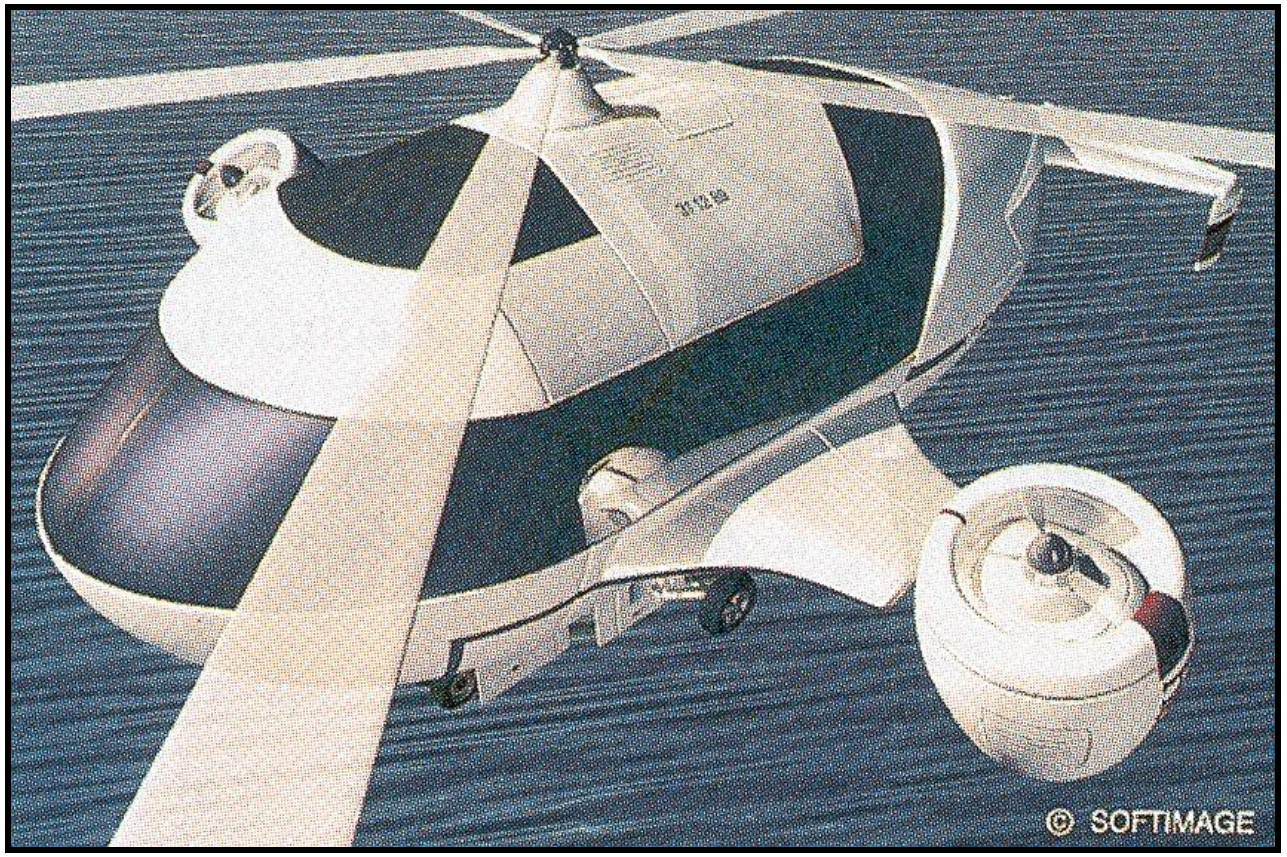
- Assignment 3 now posted

3D Objects



How can this object be represented in a computer?

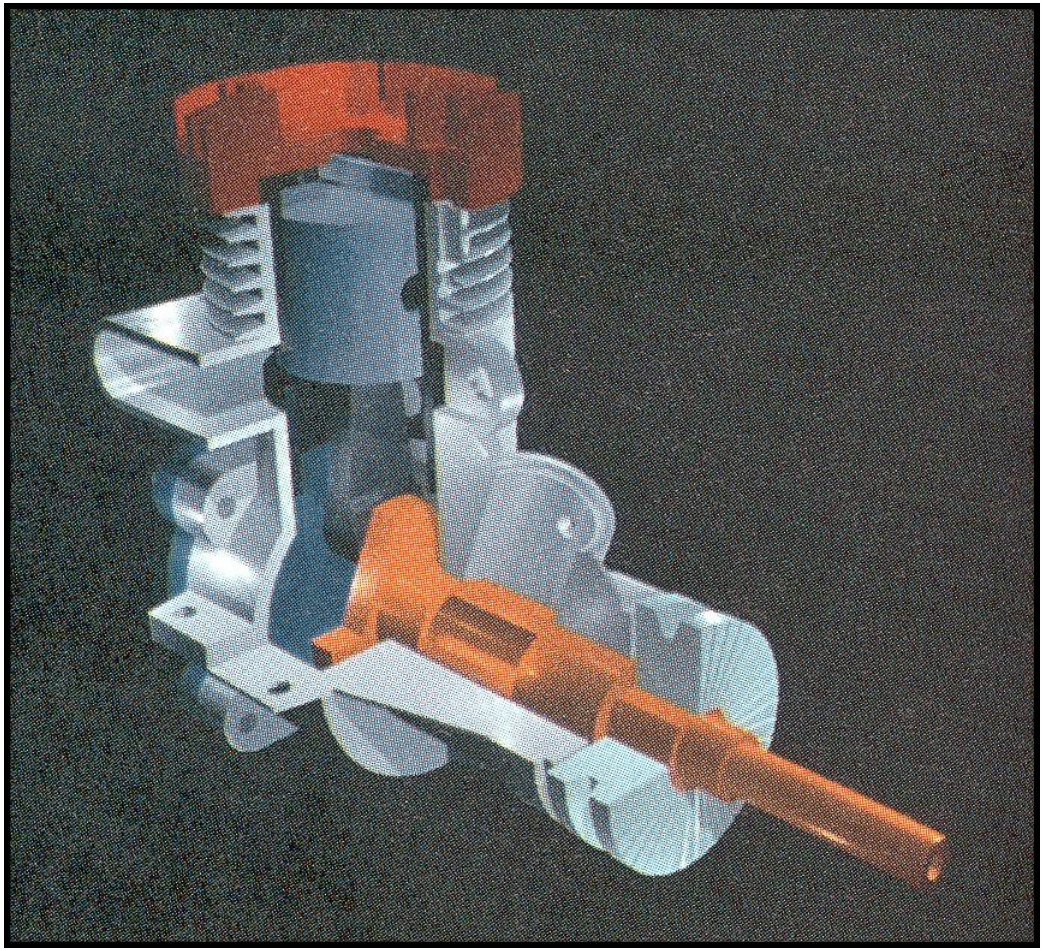
3D Objects



H&B Figure 10.46

This one?

3D Objects



H&B Figure 9.9

This one?

3D Object Representations



Raw data
- Point cloud
- Polygon soup
- Range image

Solids
- Implicit
- Voxels
- CSG

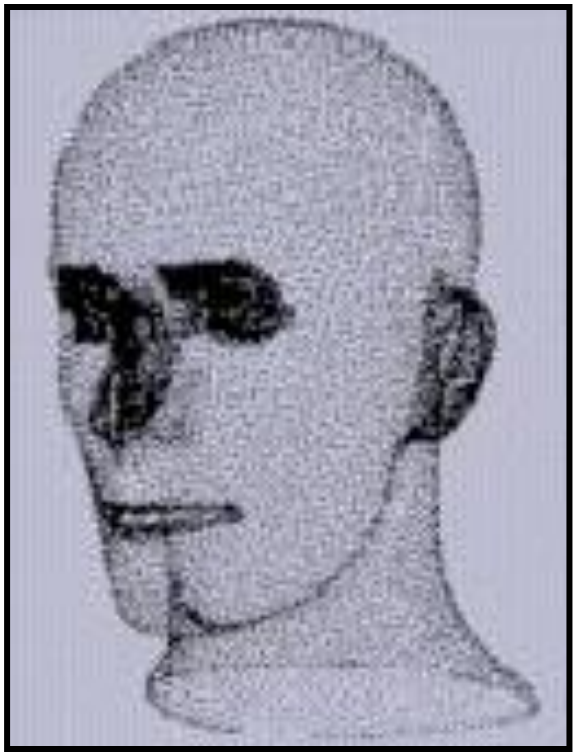
Surfaces
- Mesh
- Subdivision
- Parametric

High-level structures
- Scene graph
- Skeleton
- Application specific

Point Clouds

Unstructured set of 3D point samples

- Acquired from random sampling, particle system implementations, etc.

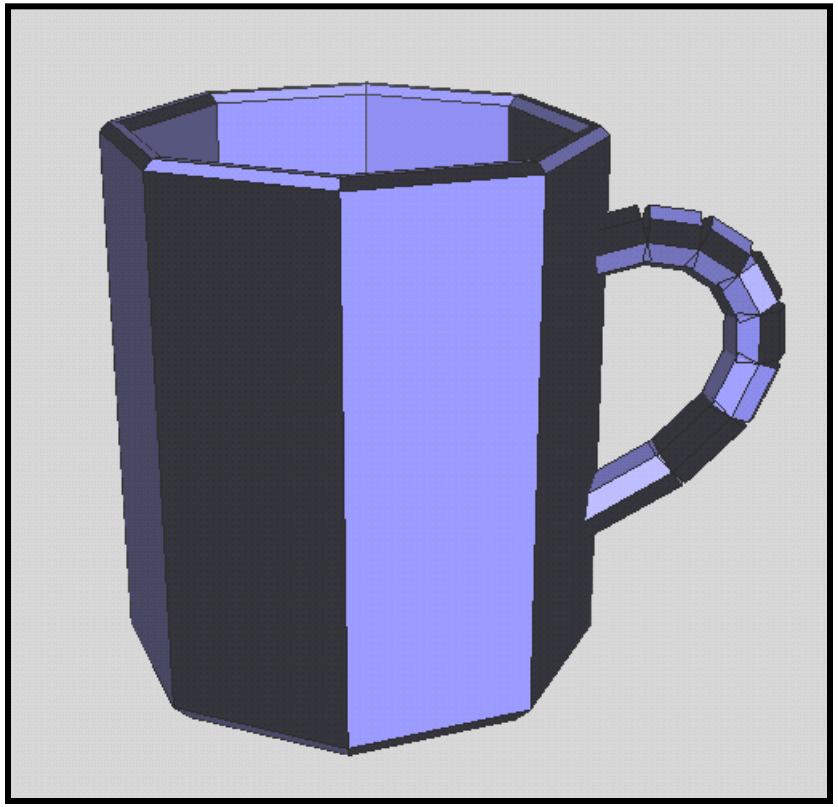


Hoppe

Polygon Soups

Unstructured set of polygons

Created with interactive modeling systems



Larson

Range/Depth Image

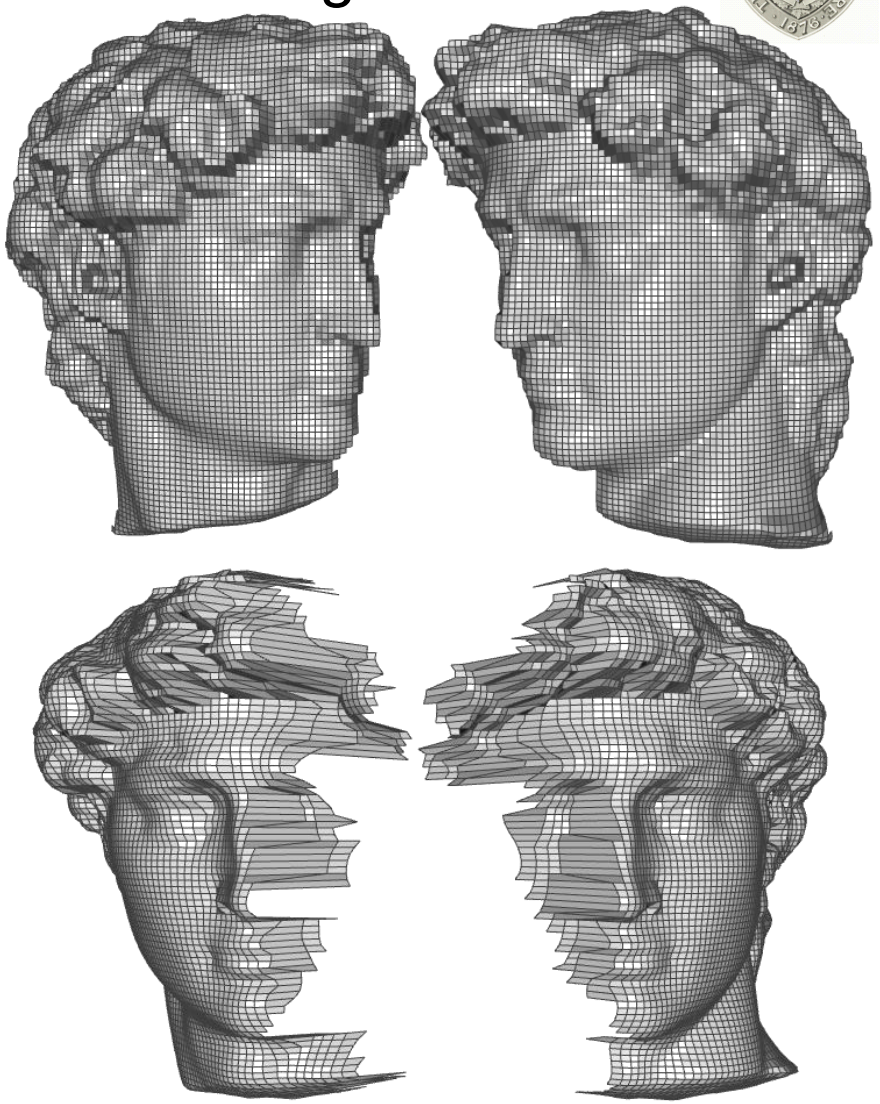
An image storing depth

Acquired from 3D scanners



Range Images

Range Surfaces



Aligned Range Surfaces

3D Object Representations



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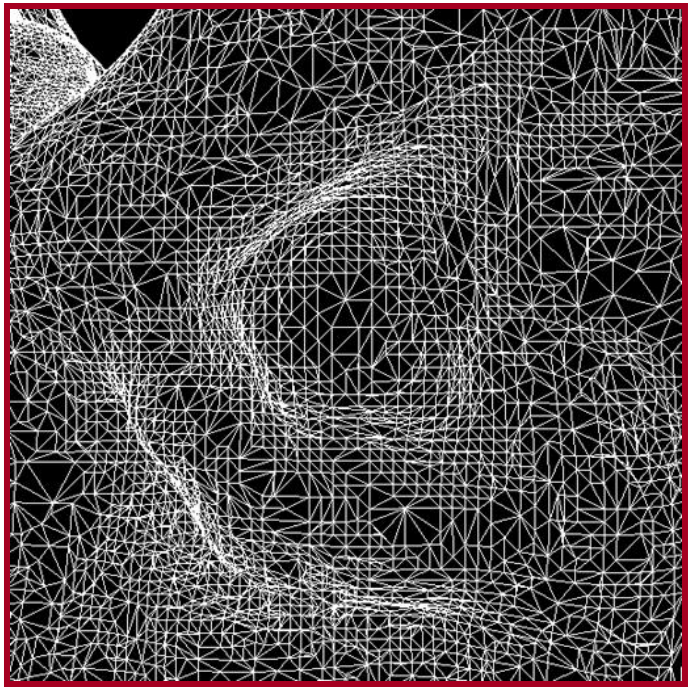
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(Manifold) Meshes

Connected set of polygons (usually triangles)

- Merging range images, etc.



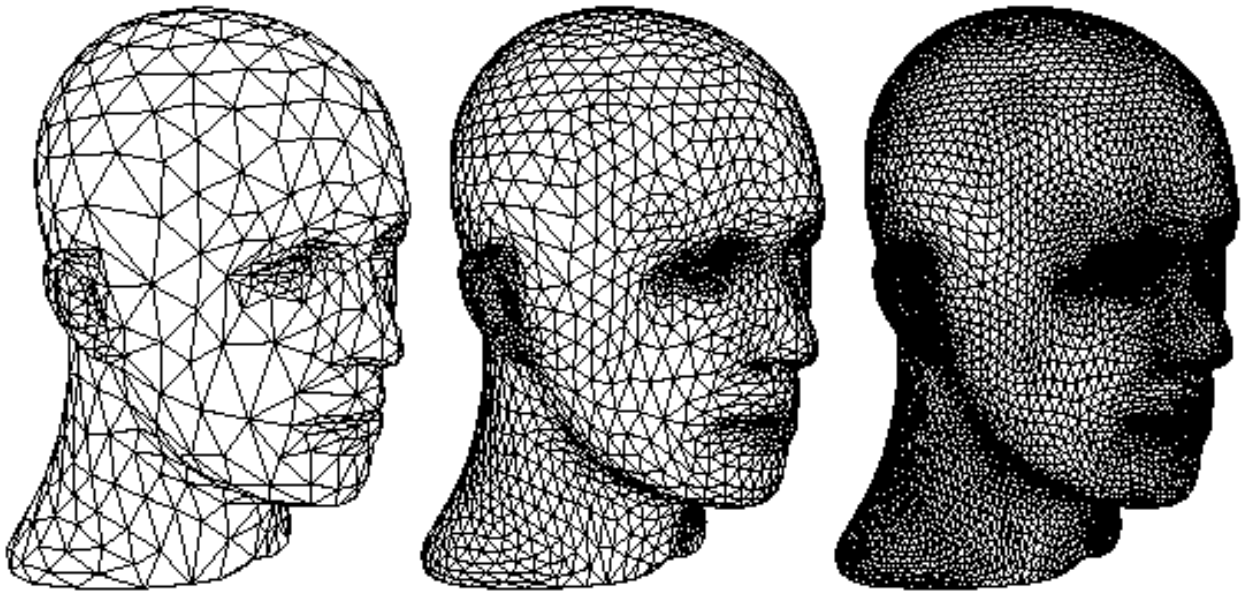


Subdivision Surfaces



Coarse mesh & subdivision rule

Define a smooth surface as limit of a hierarchical sequence of refinements

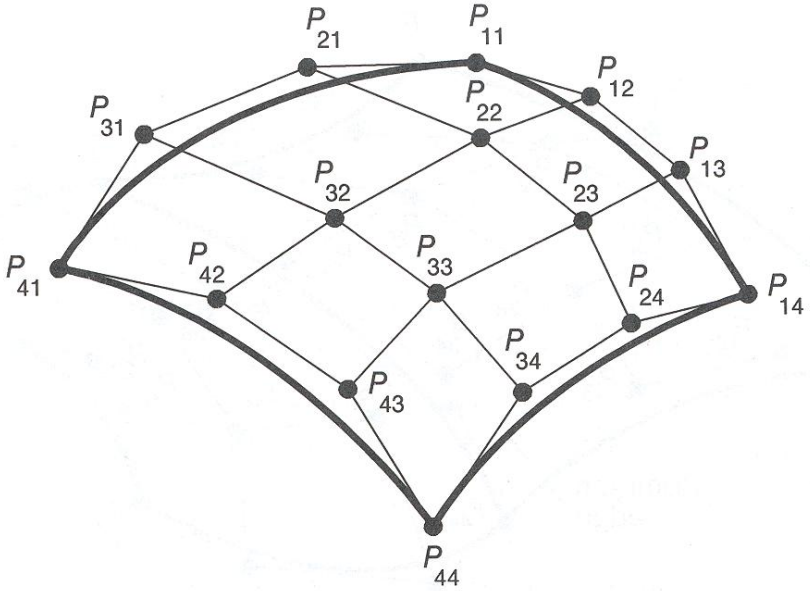


Zorin & Schroeder
SIGGRAPH 99
Course Notes

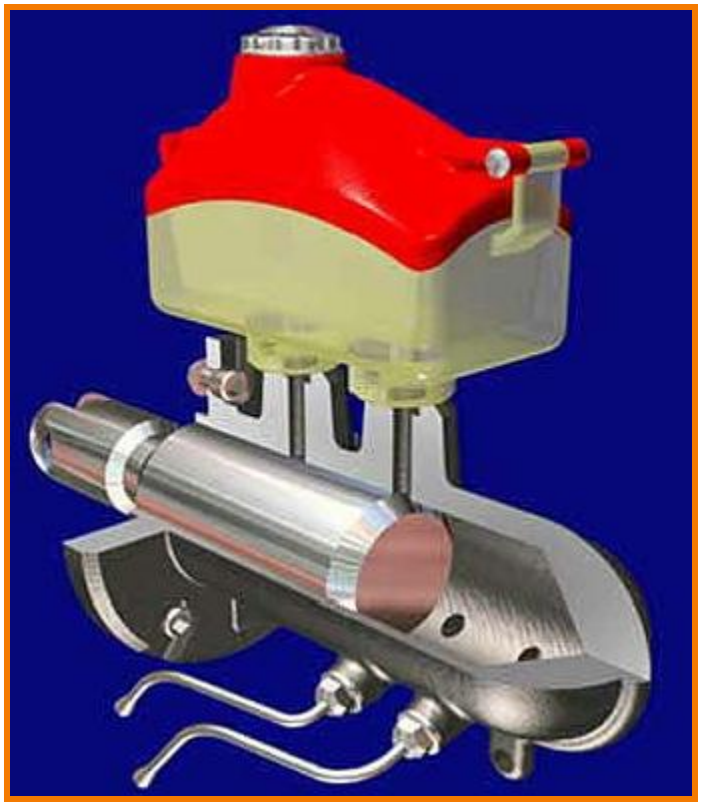
Parametric Surfaces

Tensor product spline patches

Used for real-world simulation



FvDFH Figure 11.44



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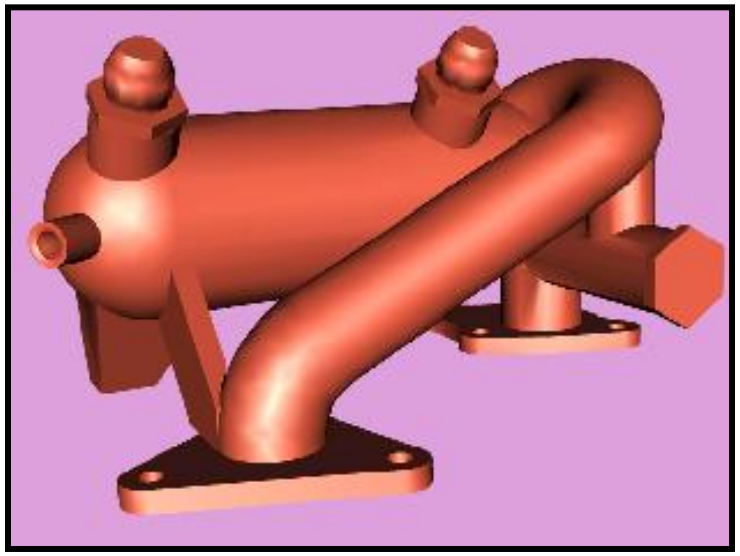
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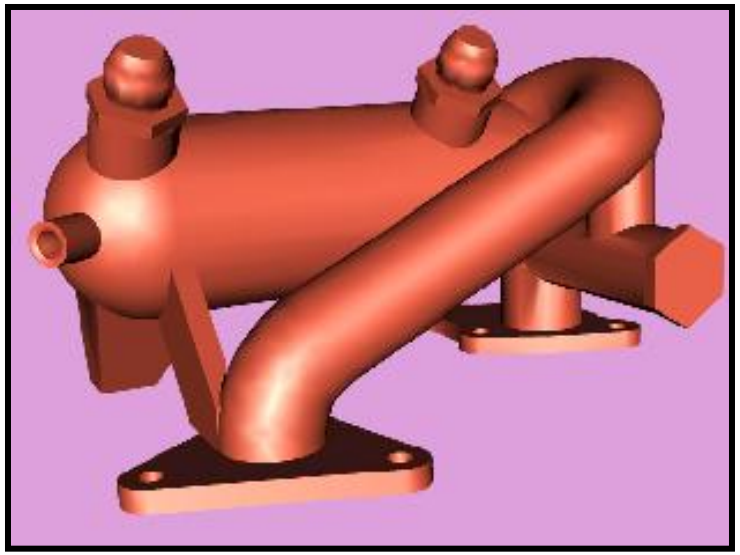
Implicit Surfaces



Points satisfying: $F(x, y, z) = 0$



Polygonal Model



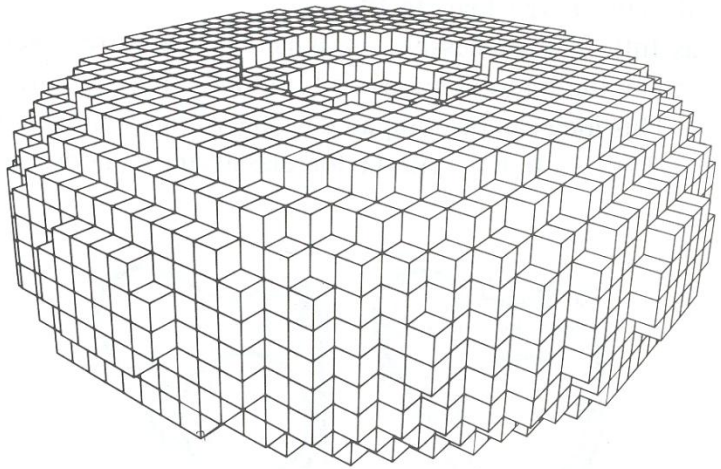
Implicit Model

Bill Lorensen
SIGGRAPH 99
Course #4 Notes

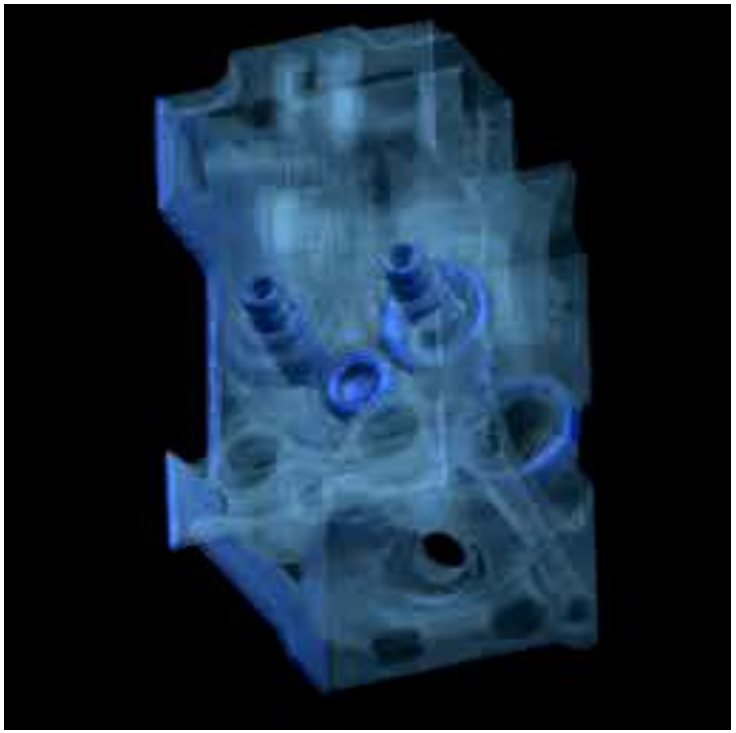
Voxels

Uniform grid of volumetric samples

Acquired from CT, MRI, etc.



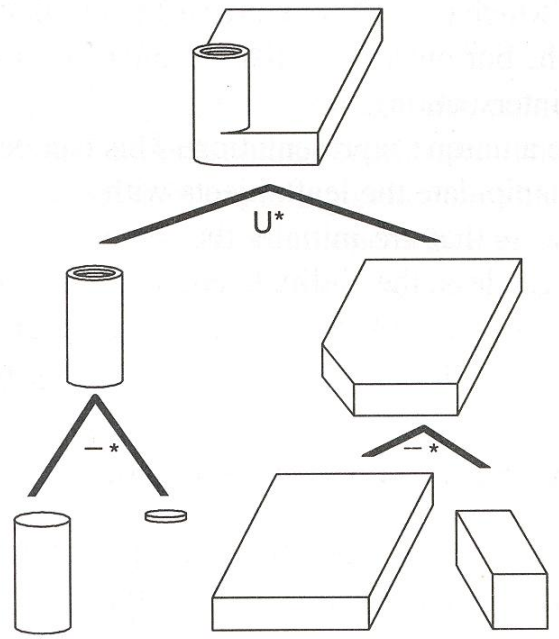
FvDFH Figure 12.20



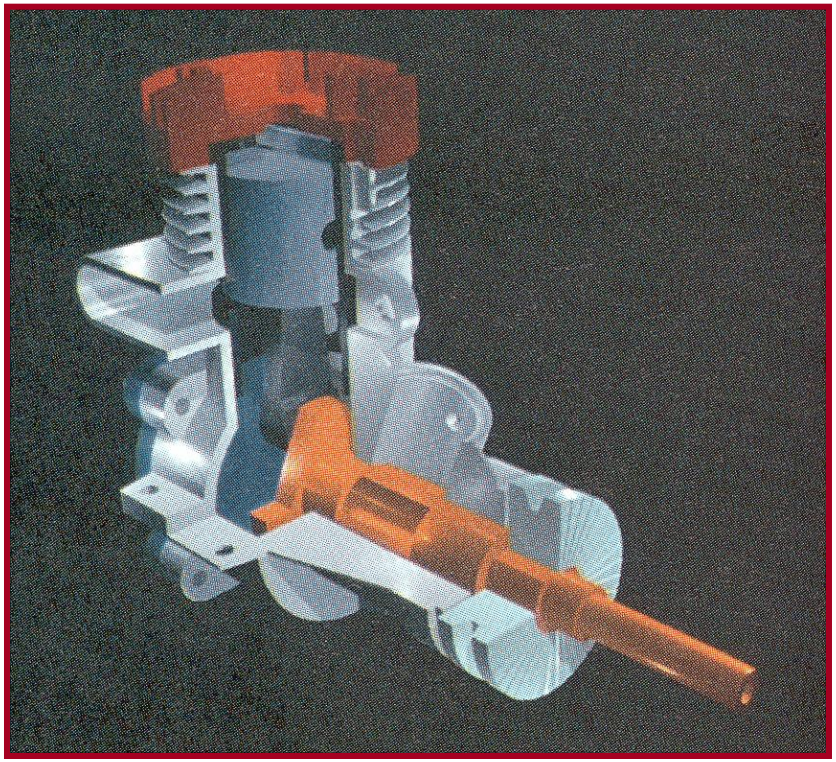
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Constructive Solid Geometry (CSG)

Hierarchies of boolean set operations (union, difference, intersect) applied to simple shapes



FvDFH Figure 12.27



H&B Figure 9.9

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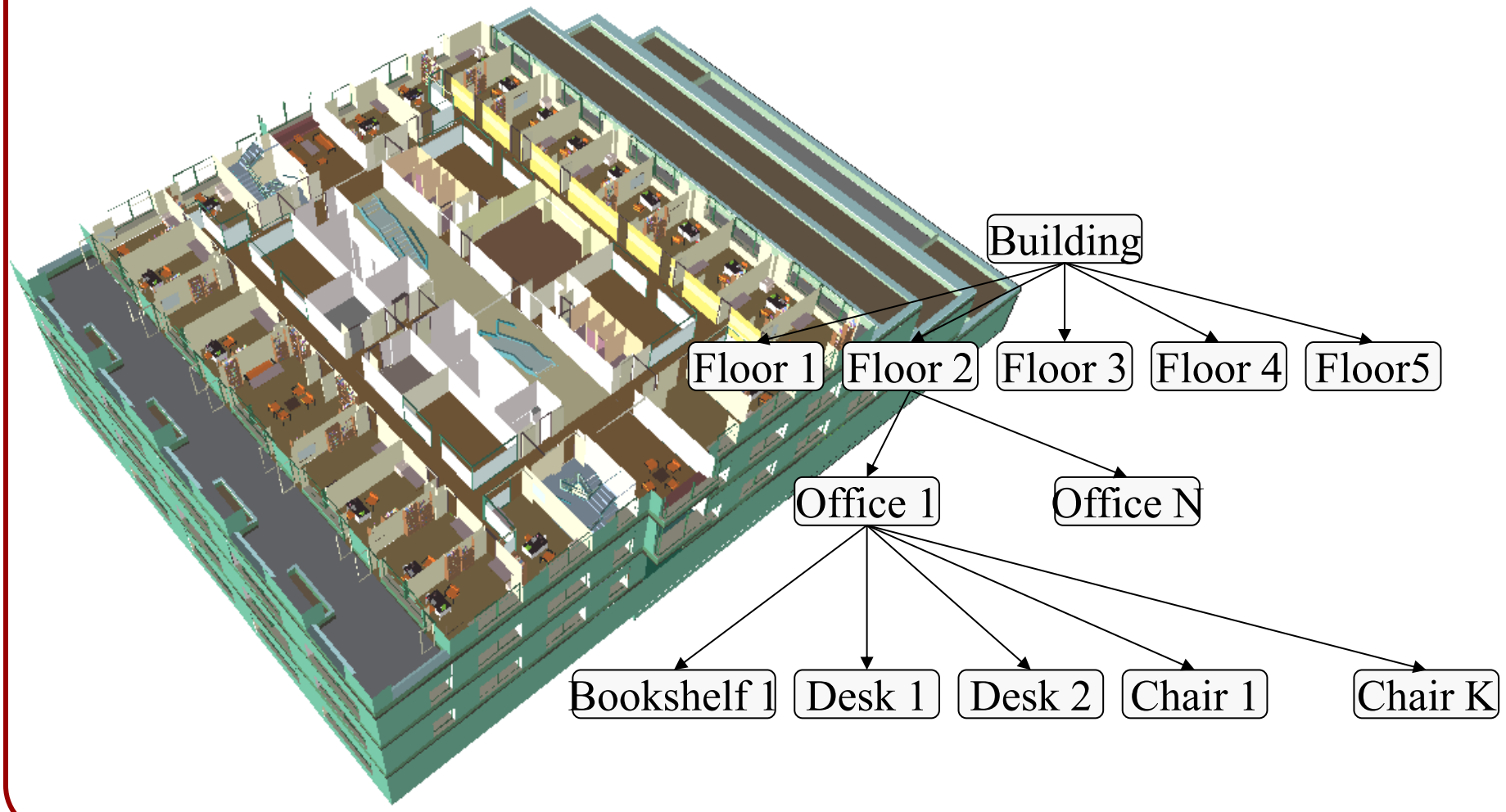
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High-level structures
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- Application specific

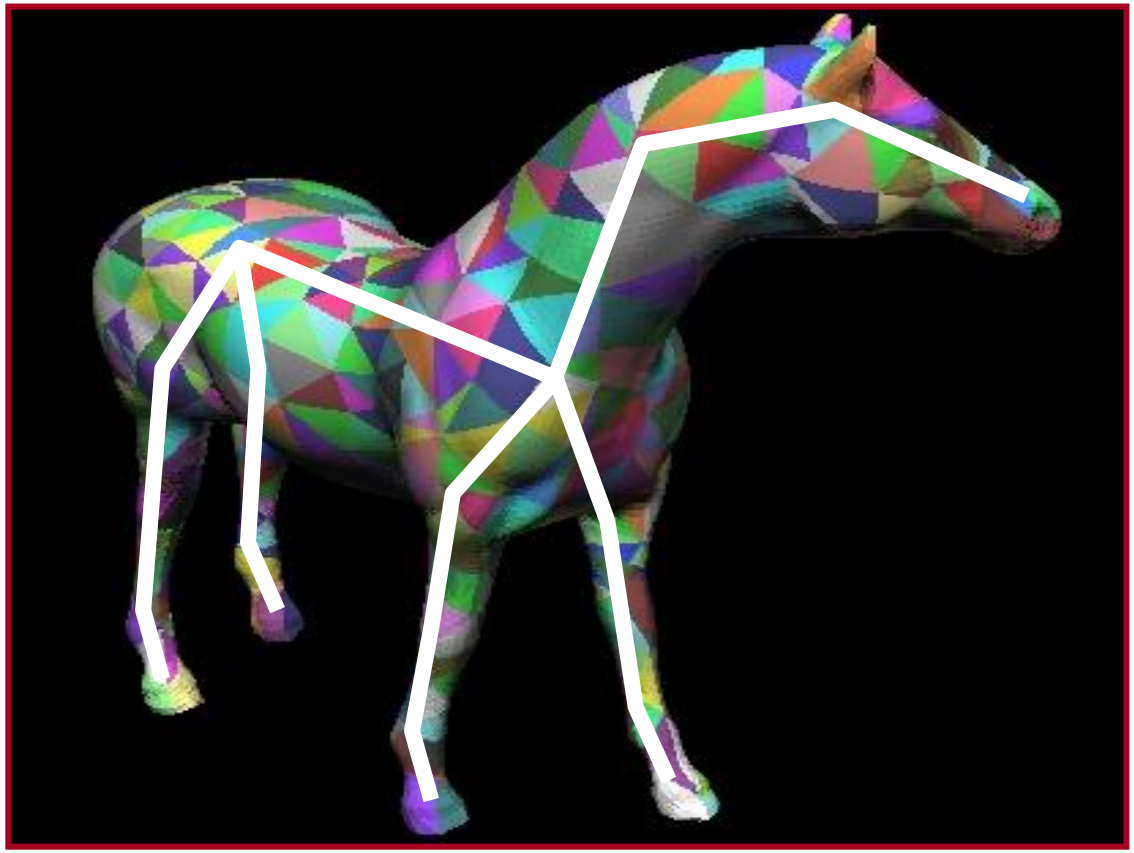
Scene Graphs

Union of objects at leaf nodes



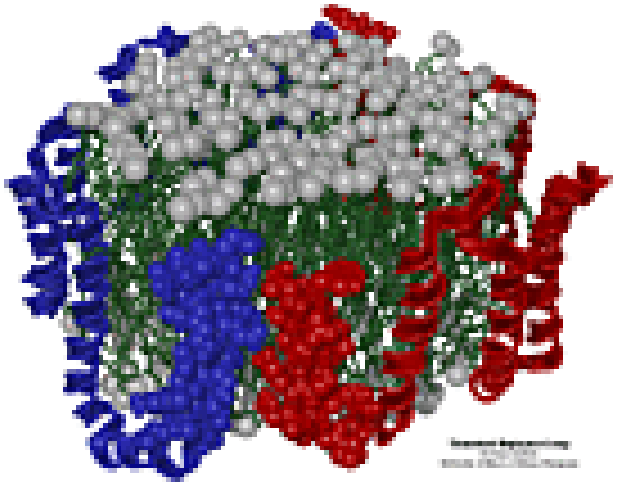
Skeletons

Graph of curves with geometry associated to curve vertices



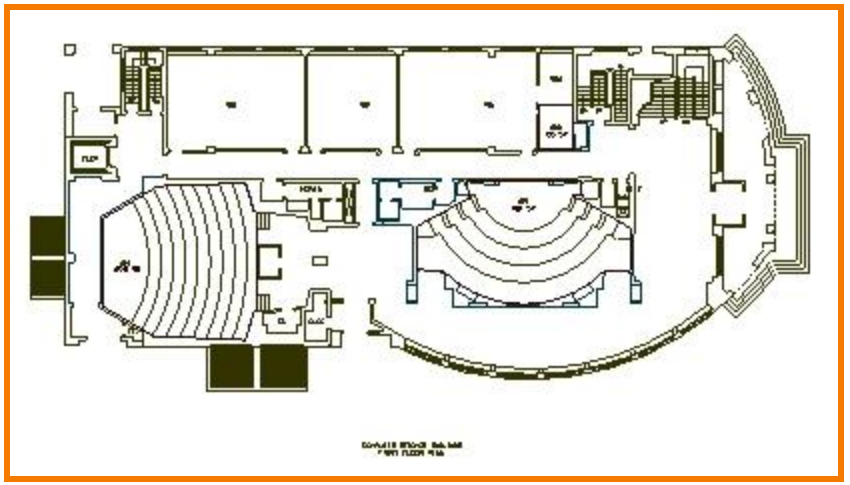
Stanford Graphics Laboratory

Application Specific



Apo A-1

(Theoretical Biophysics Group,
University of Illinois at Urbana-Champaign)



Architectural Floorplan

Surfaces



What makes a good surface representation?

- Concise
- Local support
- Affine invariant
- Arbitrary topology
- Guaranteed smoothness
- Natural parameterization
- Efficient display
- Efficient intersections

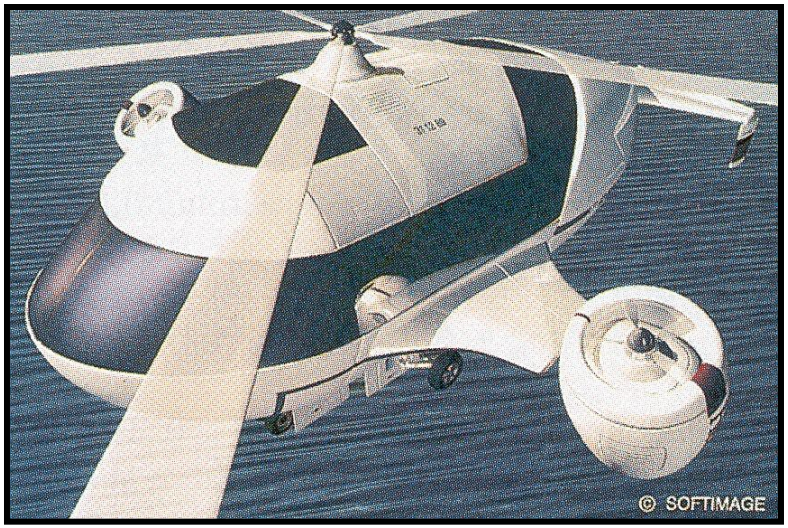
Surfaces



What makes a good surface representation?

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smooth ≠ complex



H&B Figure 10.46

Surfaces



What makes a good surface representation?

- Concise
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edits are localized



Not Locally Supported

Surfaces



What makes a good surface representation?

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applying an affine transformation (linear+translation) to the surface does not fundamentally change its representation.

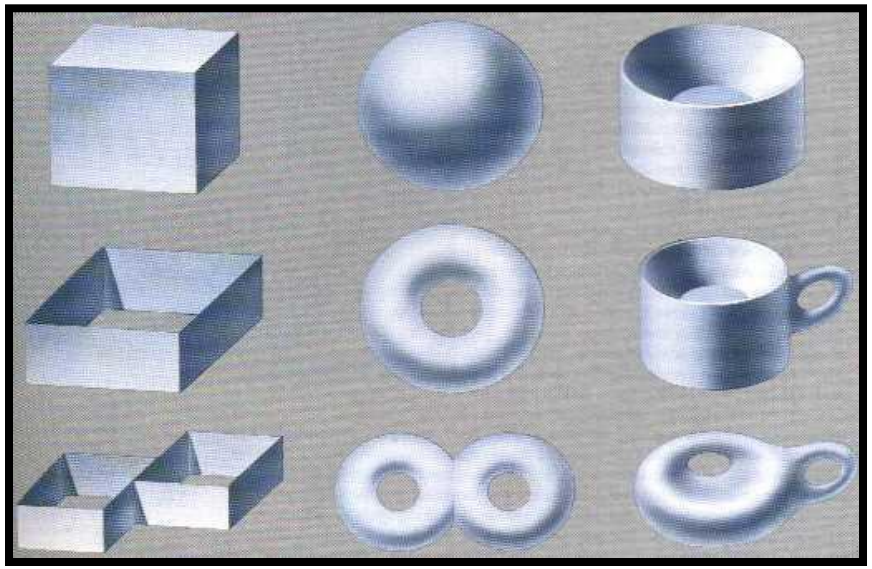
Surfaces



What makes a good surface representation?

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can represent surfaces with arbitrary on topology



Surfaces



What makes a good surface representation?

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positions/normal vary continuously/smoothly over the surface

Surfaces

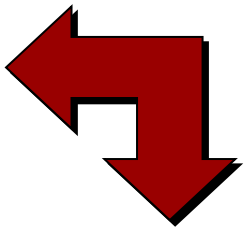


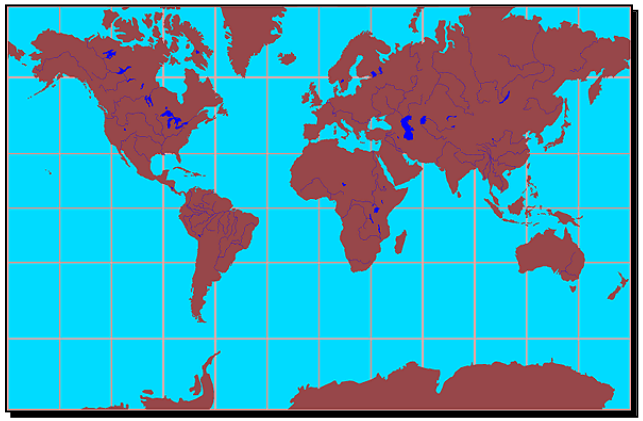
What makes a good surface representation?

- Concise
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supports texture mapping







Surfaces



What makes a good surface representation?

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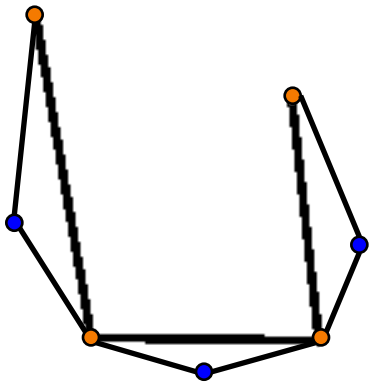
supports efficient ray-tracing and/or real-time rendering

Subdivision



Q: How can we interpret a coarse set of samples as a smooth curve?

A: Introduce new in-between vertices that smooth out the severe angles



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Course Notes

Subdivision



Q: How can we interpret a coarse set of samples as a smooth curve?

A: Introduce new in-between vertices that smooth out the severe angles



User: Specifies coarse geometry
Algorithm: Defines refined geometry

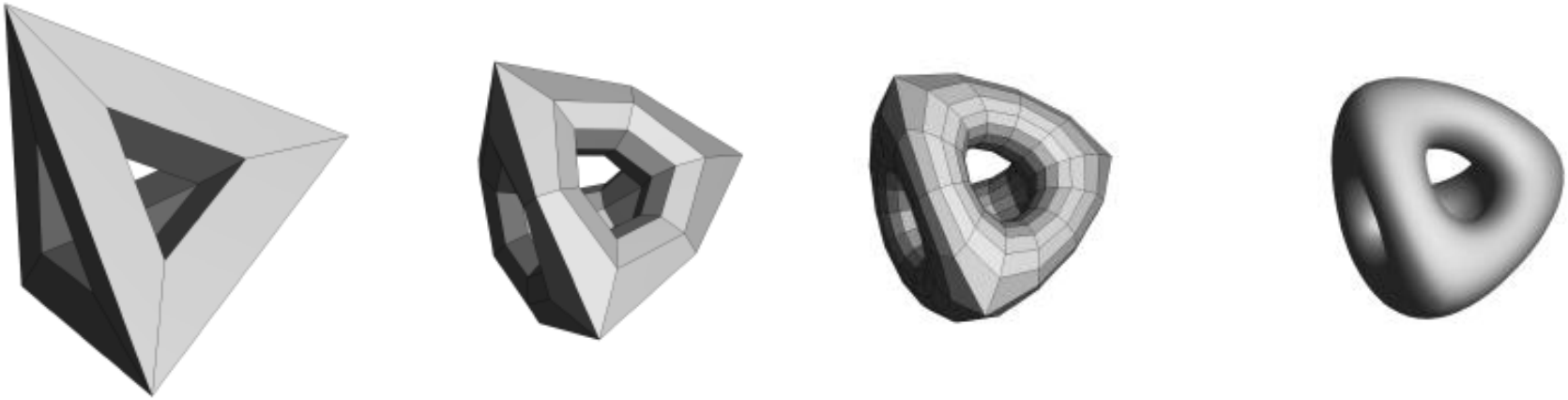
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Subdivision Surfaces



Coarse mesh & subdivision rule

- Define smooth surface as limit of a sequence of refinements



(a) (b) (c) (d)

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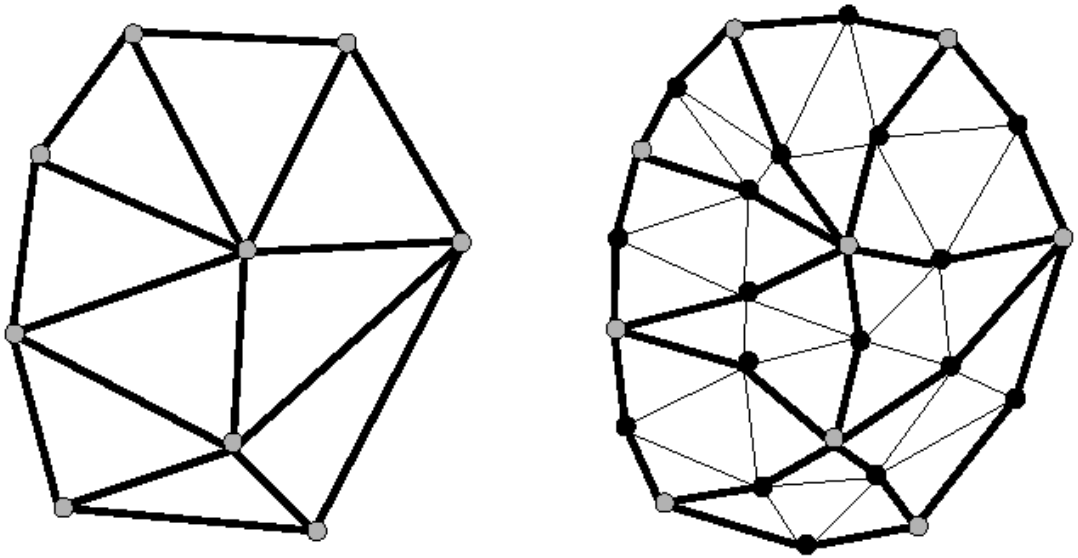
Key Questions

How to subdivide the mesh?

Aim for properties like smoothness

How to represent the mesh? (Next time)

Aim for efficiency of implementing subdivision rules



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General Subdivision Scheme



How to subdivide the mesh?

1. Refinement (topology)
2. Smoothing (geometry)

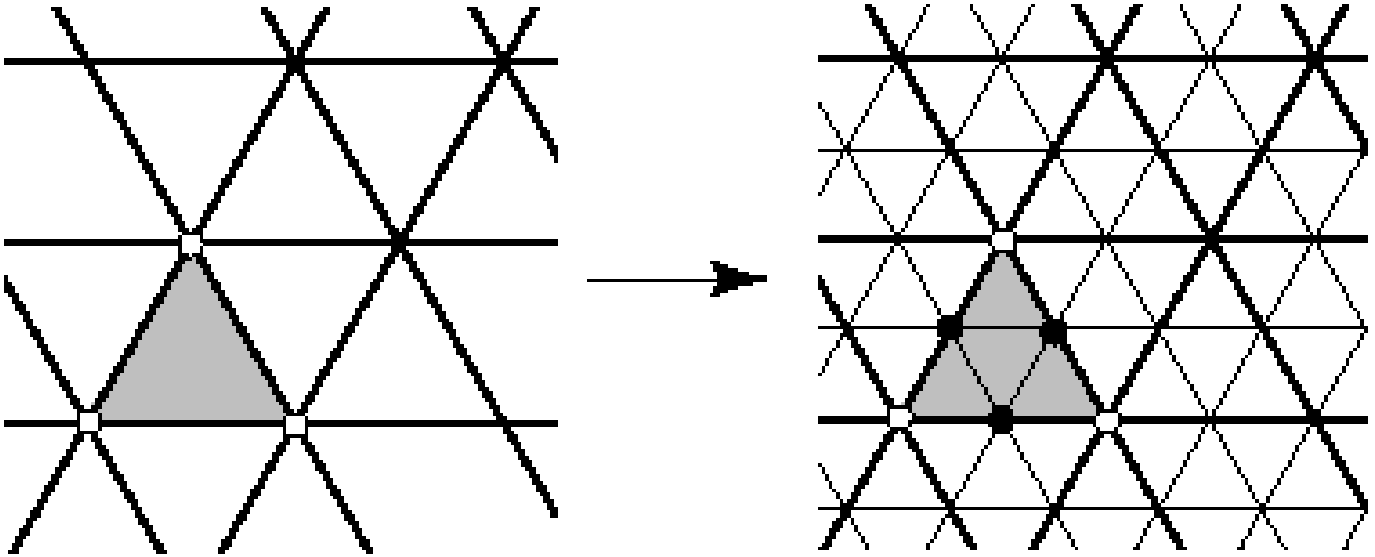
Loop Subdivision Scheme



How to subdivide the mesh?

Refinement:

Subdivide each triangle into 4 by introducing edge mid-points and connecting vertices



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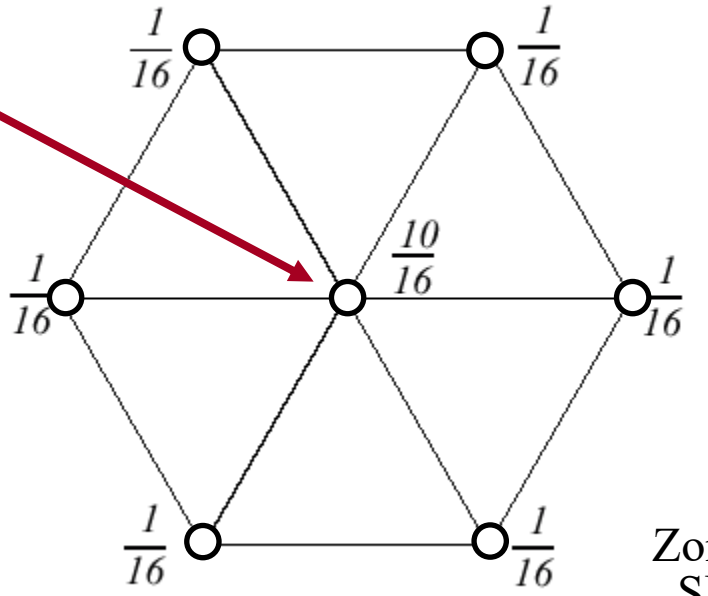
Loop Subdivision Scheme



How to subdivide the mesh?

- Refinement
- Smoothing (existing vertices):
  - Choose *new* location as weighted average of *original* vertex and its neighbors

Existing vertex being moved from one level to the next



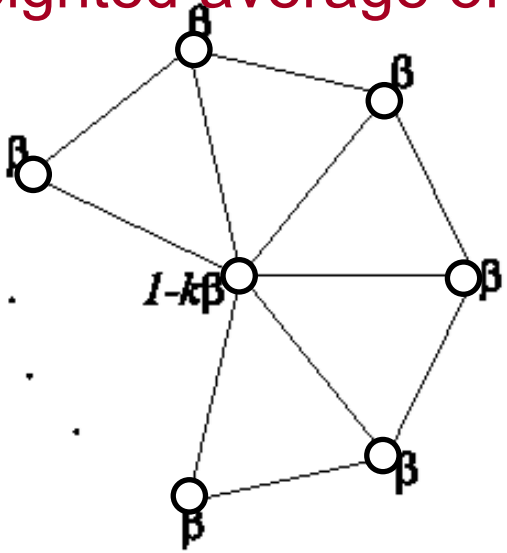
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Loop Subdivision Scheme



How to subdivide the mesh?

- Refinement
- Smoothing (existing vertices):
  - Choose *new* location as weighted average of *original* vertex and its neighbors



What about *extraordinary* vertices with $k \neq 6$ neighboring faces?

new_position = $(1 - k\beta)$original_position + sum(β*each_original_vertex)

Loop Subdivision Scheme



How to subdivide the mesh?

- Refinement
- Smoothing (existing vertices):
  - Choose *new* location as weighted average of *original* vertex and its neighbors

$0 \leq \beta \leq 1/k$:

As β increases, the contribution from adjacent vertices plays a more important role.

If $\beta = 0$, the subdivision is interpolatory.

new_position = $(1 - k\beta)$original_position + sum(β*each_original_vertex*)

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Loop Subdivision Scheme



Challenge:

Choose β so that the limit surface has guaranteed smoothness properties

Original Loop

$$\beta = \frac{1}{k}\left(\frac{5}{8} - \left(\frac{3}{8} + \frac{1}{4}\cos\frac{2\pi}{k}\right)^2\right)$$

Warren

$$\beta = \begin{cases} \frac{3}{8k} & k > 3 \\ \frac{3}{16} & k = 3 \end{cases}$$

Why are valence-6 vertices “regular”?



Definition:

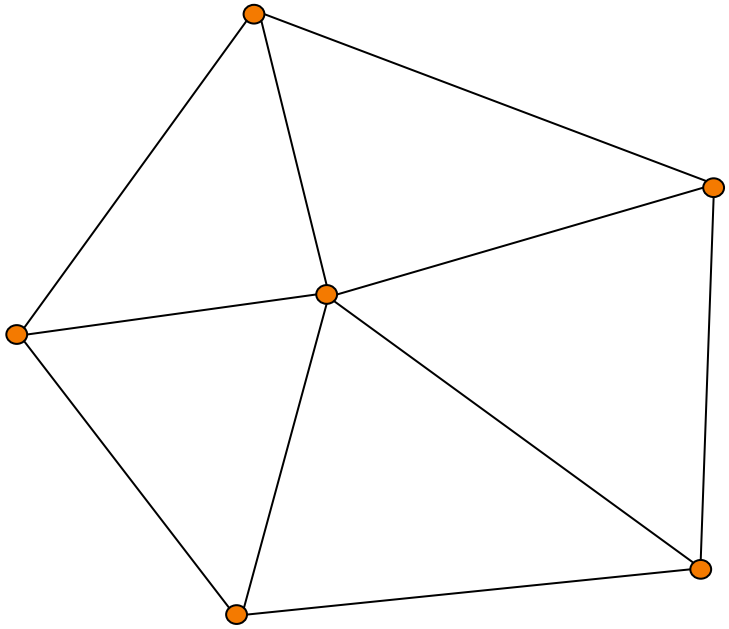
Given an undirected graph, the *valence* of a vertex/node in the graph is the number of edges emanating from it.

Why are valence-6 vertices “regular”?

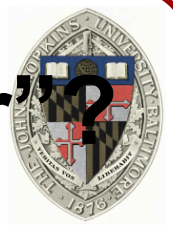


Subdivision:

Q: What happens after we refine?



Why are valence-6 vertices “regular”?

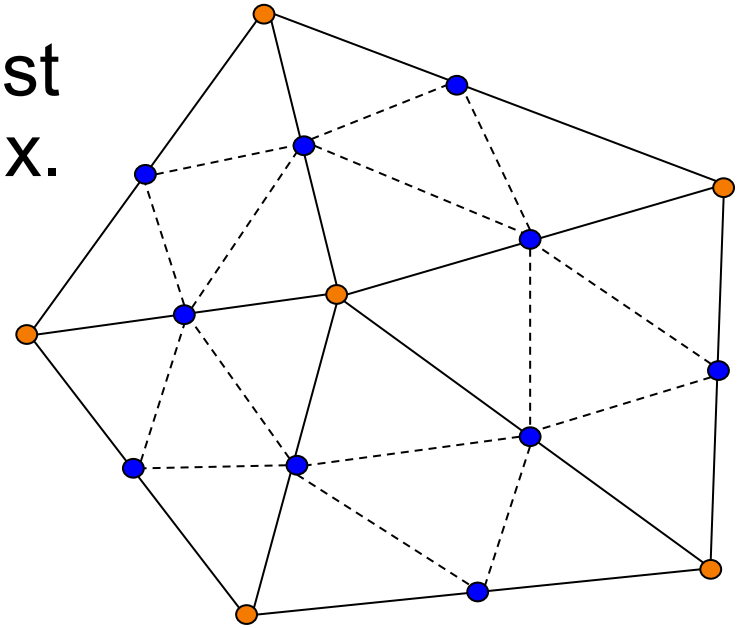


Subdivision:

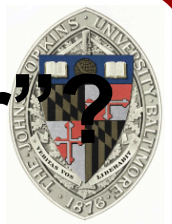
Q: What happens after we refine?

A: Valence of old vertices unchanged.
Valence of **all** new vertices is six.

⇒ Iterating the refinement, most vertices will have valence six.



Why are valence-6 vertices “regular”?



Euler Characteristic:

For connected, water-tight meshes, the number of vertices, edges, and faces satisfy:

$$|V| - |E| + |F| = 2 - 2g$$

where g is the genus of the surface (how many topological holes it has).

For water-tight triangle meshes, each face has three edges and each edge is shared by two faces, so the number of edges is

$$|E| = \frac{3}{2}|F|$$

Why are valence-6 vertices “regular”?



Euler Characteristic:

$$|V| - |E| + |F| = 2 - 2g$$

For water-tight triangle meshes:

$$|E| = \frac{3}{2}|F|$$

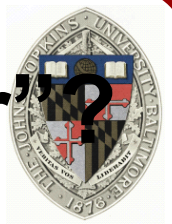
Putting this together we get:

$$|V| - |E| + \frac{2}{3}|E| = 2 - 2g$$

$$|V| - \frac{1}{3}|E| = 2 - 2g$$

$$3|V| \approx |E|$$

Why are valence-6 vertices "regular"?



$$3|V| \approx |E|$$

$$\Downarrow$$

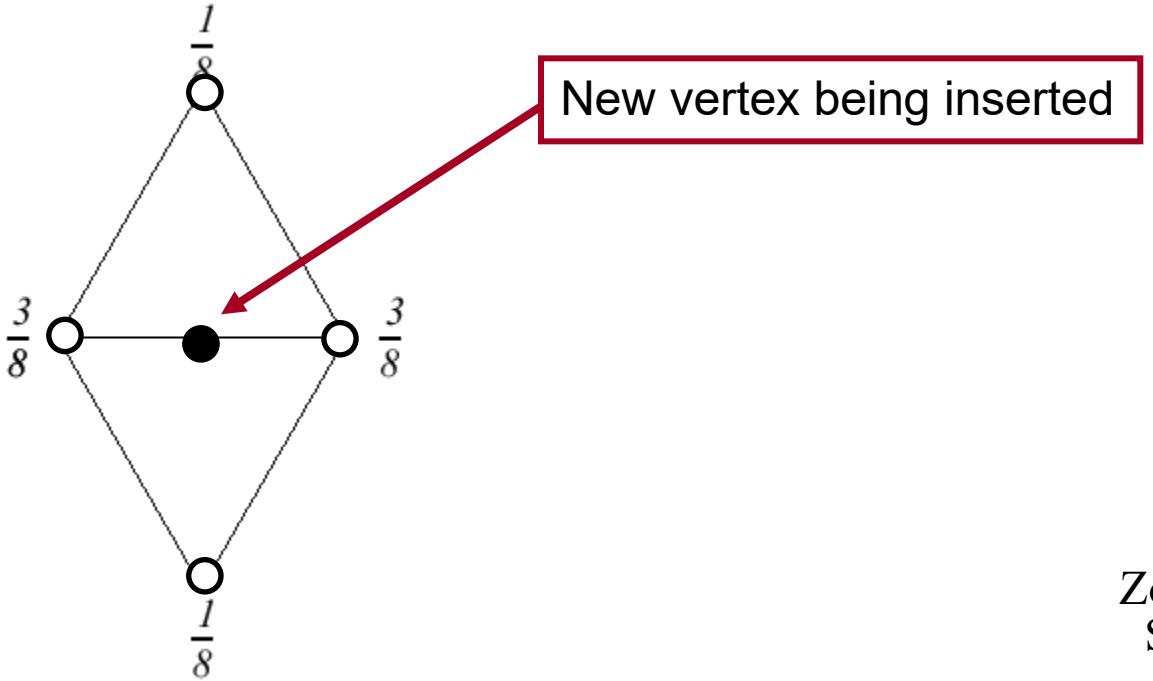
$$\begin{aligned}\text{Average Valence} &= \frac{1}{|V|}\sum_{v \in V} valence(v) \\ &= \frac{1}{|V|}(2|E|) \\ &\approx \frac{1}{|V|}(6|V|) \\ &= 6\end{aligned}$$

Loop Subdivision Scheme



How to subdivide the mesh?

- Refinement
- Smoothing (inserted vertices):
  - Choose location as weighted average of *original* vertices in local neighborhood



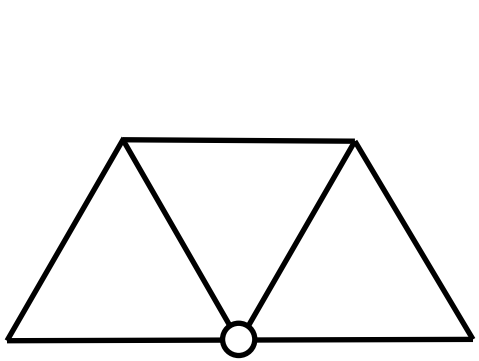
Zorin & Schroeder
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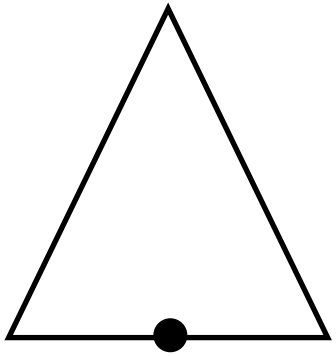
Boundary Cases?



What about *boundary vertices / edges?*

- Existing vertex adjacent to an incomplete “triangle fan”
- New vertex bordered by only one triangle



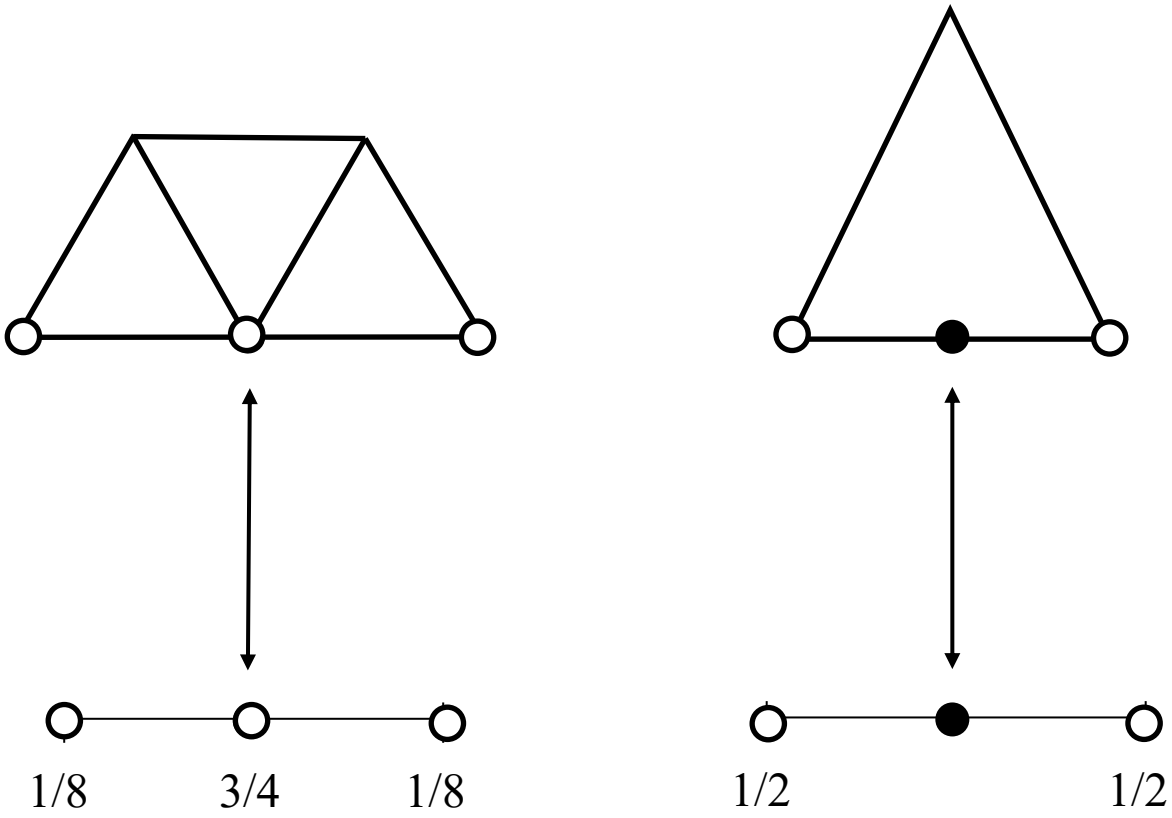


Boundary Cases?



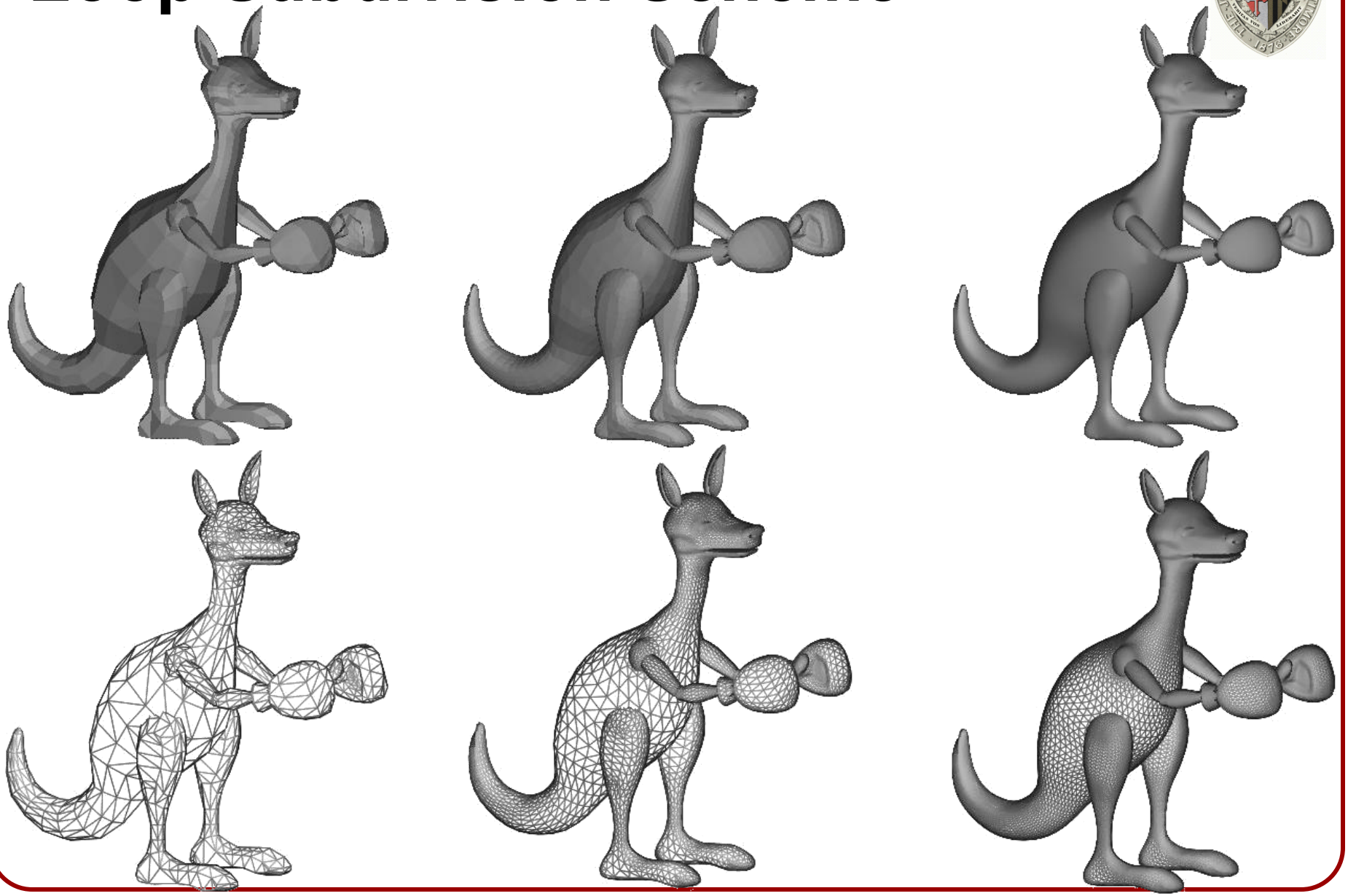
What about *boundary vertices / edges?*

Refine using (boundary) curve refinement rules.



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Course Notes

Loop Subdivision Scheme

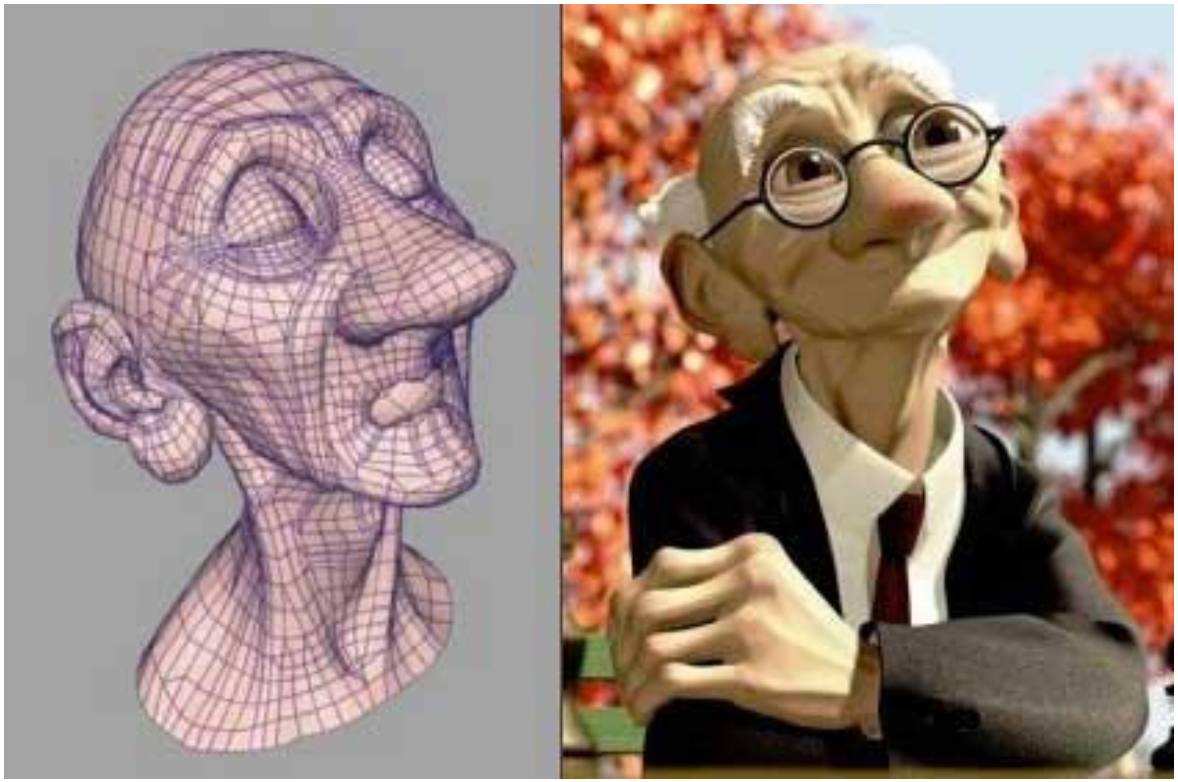


Loop Subdivision Scheme



Geri's Game, *Pixar*

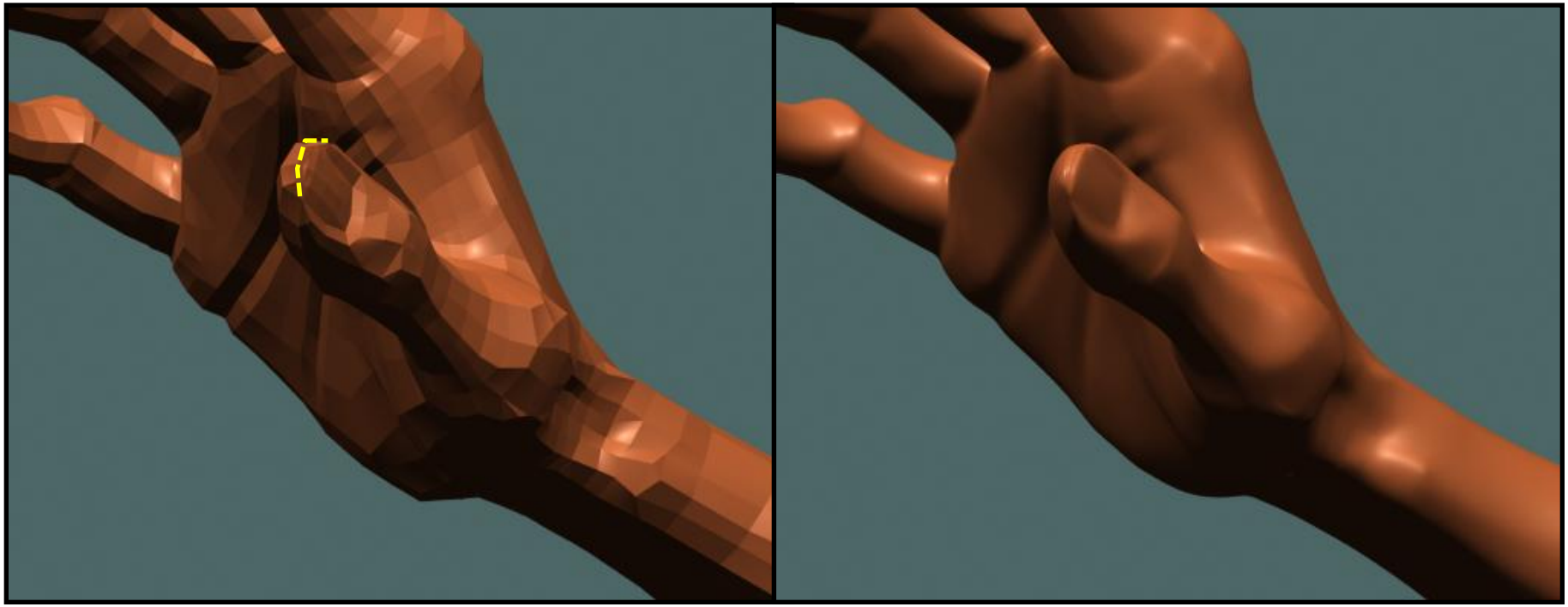
Loop Subdivision Scheme



Pixar

Smooth surfaces can be constructed from coarse meshes!

Loop Subdivision Scheme



Pixar

Sharp creases can be specified by specifying that certain curves should subdivide as boundary curves.

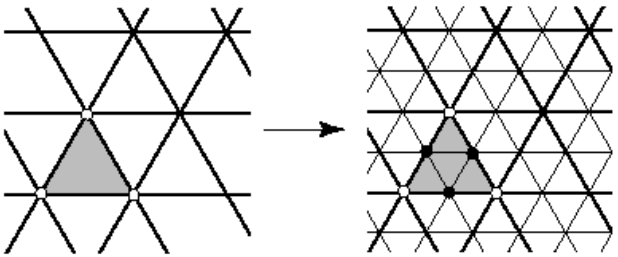
Zorin & Schroeder
SIGGRAPH 99
Course Notes

Subdivision Schemes

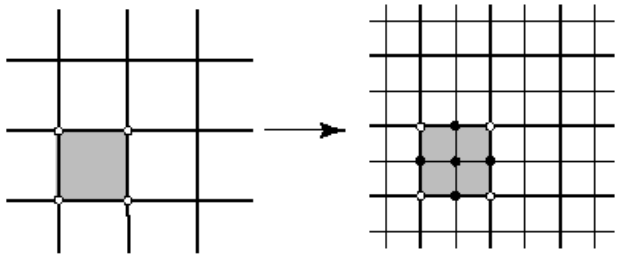


There are different subdivision schemes

- Different methods for refining topology
  - Triangles versus quads
- Different rules for positioning vertices
  - Interpolating versus approximating



Face split for triangles



Face split for quads

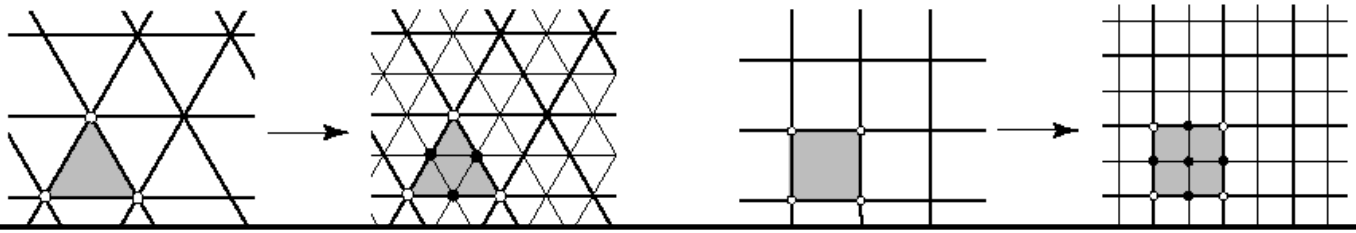
Face split		
	Triangular meshes	*Quad. meshes*
Approximating	Loop (C^2)	Catmull-Clark (C^2)
Interpolating	Mod. Butterfly (C^1)	Kobbelt (C^1)

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Course Notes

Subdivision Schemes

There are different subdivision schemes

- Different methods for refining topology
  - Triangles versus quads
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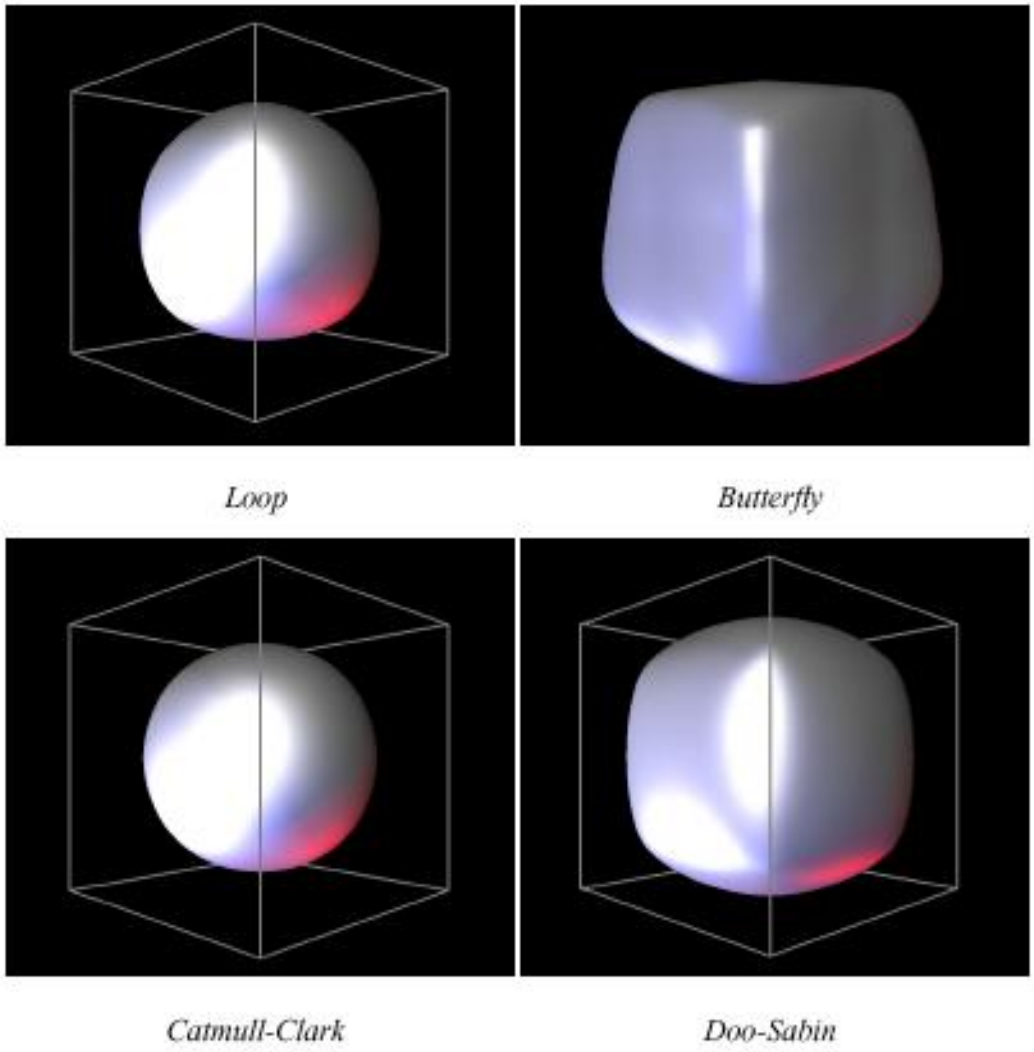


In general, forcing the subdivision to be interpolating removes degrees of freedom, making the solution less smooth.

	Triangular meshes	*Quad. meshes*
Approximating	Loop (C^2)	Catmull-Clark (C^2)
Interpolating	Mod. Butterfly (C^1)	Kobbelt (C^1)

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Course Notes

Subdivision Schemes



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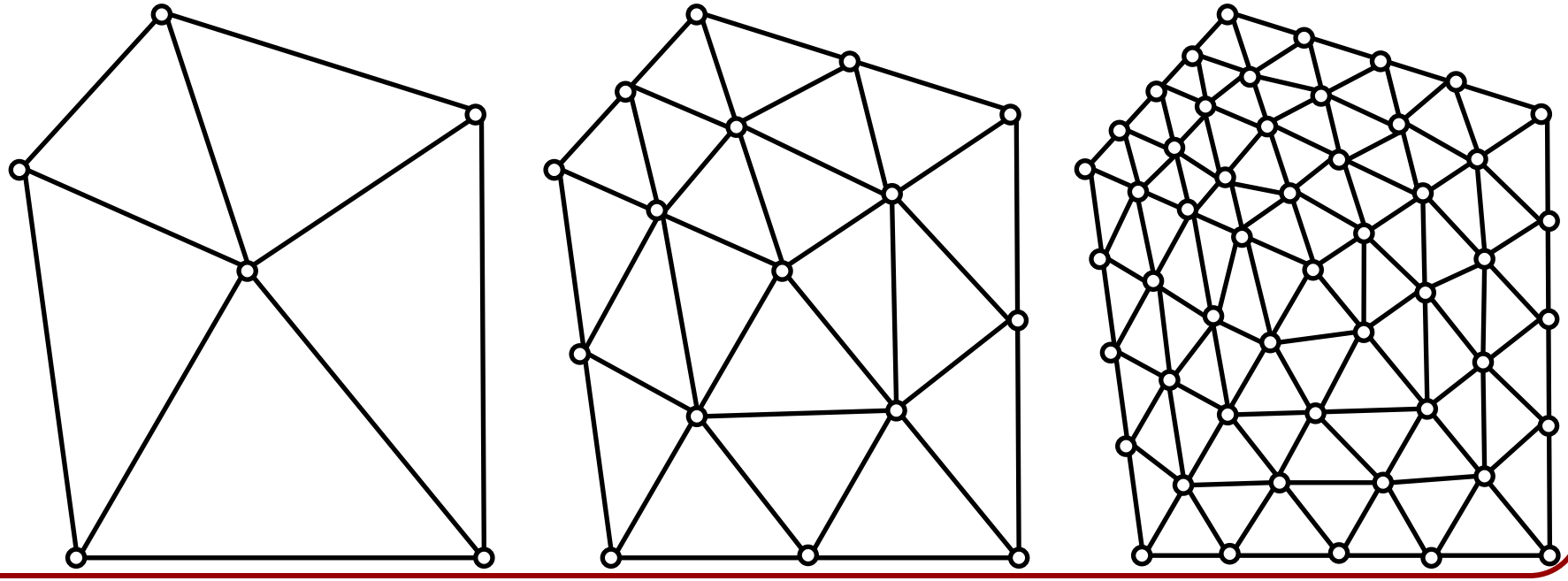
Subdivision Surfaces

- Properties:
  - ✓ Concise
  - Local support
  - Affine invariant
  - Arbitrary topology
  - Guaranteed smoothness
  - Natural parameterization
  - Efficient display
  - Efficient intersections



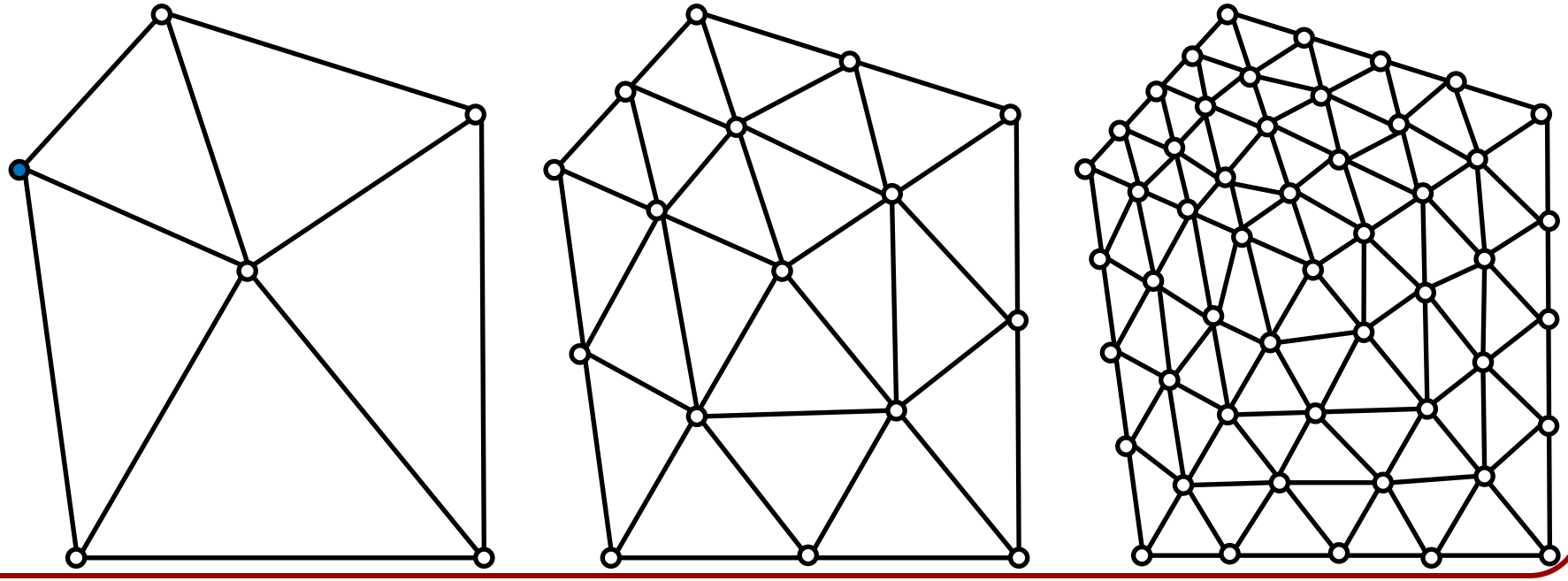
Pixar

Local support?



Local support?

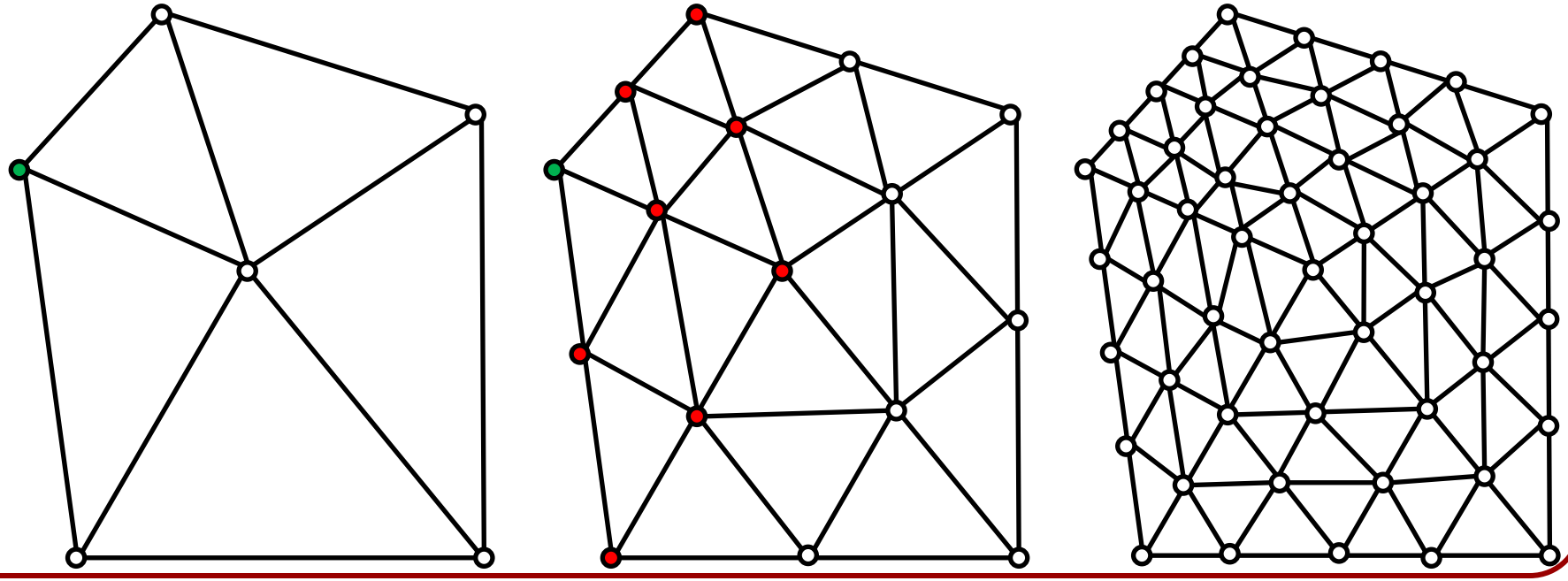
Modifying a vertex position at the coarser level



Local support?

Modifying a vertex position at the coarser level

We modify positions in the one-ring at the next level



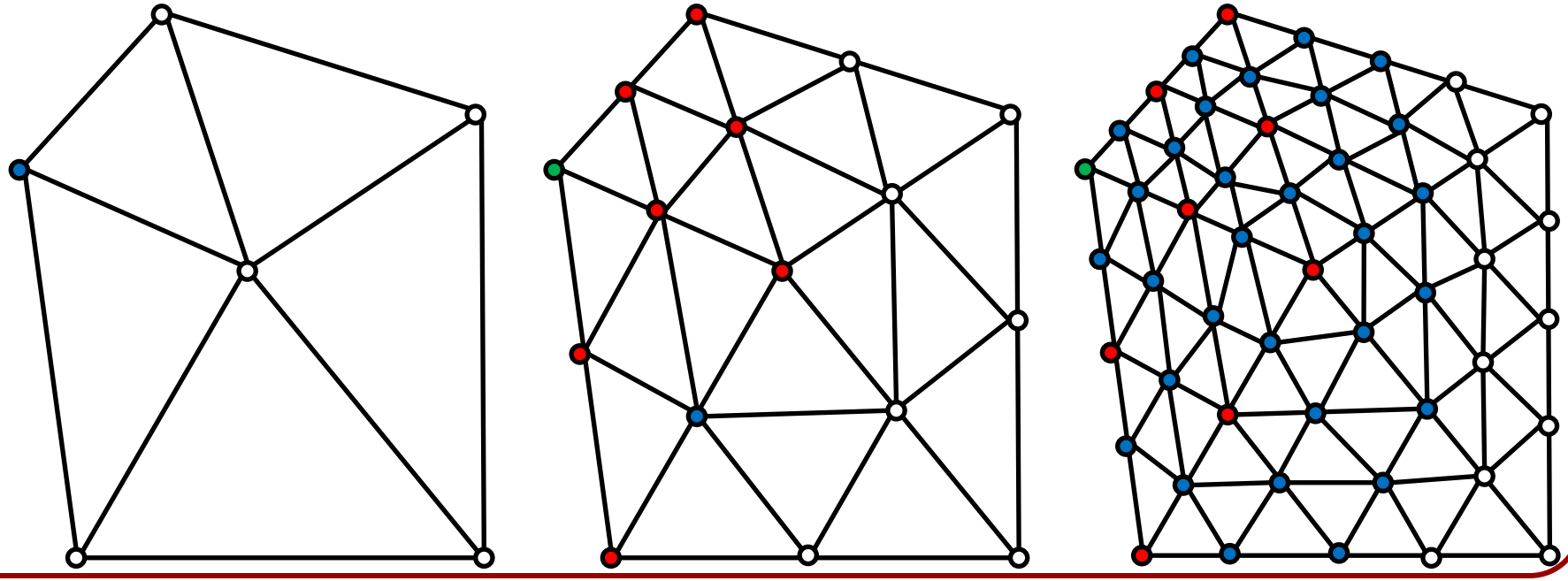
Local support?

Modifying a vertex position at the coarser level

We modify positions in the one-ring at the next level

Which modifies positions in the one-ring at the next level

Because we refine by a factor of two at each level, the effects are limited within the two-ring at the original level.



Subdivision Surfaces

- Properties:
  - ✓ Concise
  - ✓ Local support
  - ✓ Affine invariant
  - ✓ Arbitrary topology
  - ◦ Guaranteed smoothness
  - ◦ Natural parameterization
  - ◦ Efficient display
  - ◦ Efficient intersections

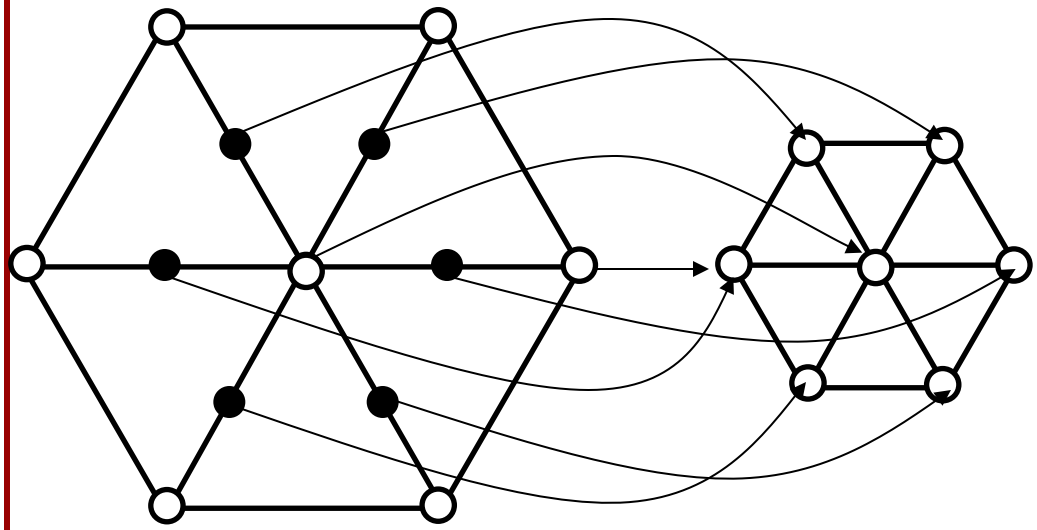


Pixar

Guaranteed Smoothness?

To determine the smoothness of the subdivision:

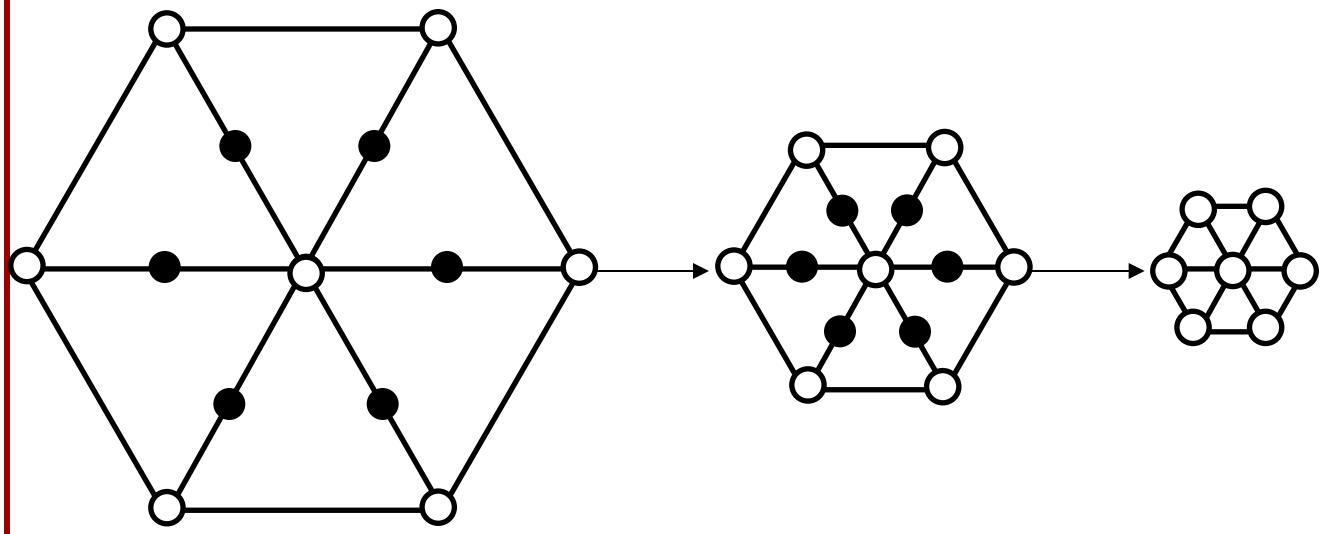
- Repeatedly apply the subdivision scheme
- Look at the neighborhood in the limit



Guaranteed Smoothness?

To determine the smoothness of the subdivision:

- Repeatedly apply the subdivision scheme
- Look at the neighborhood in the limit.



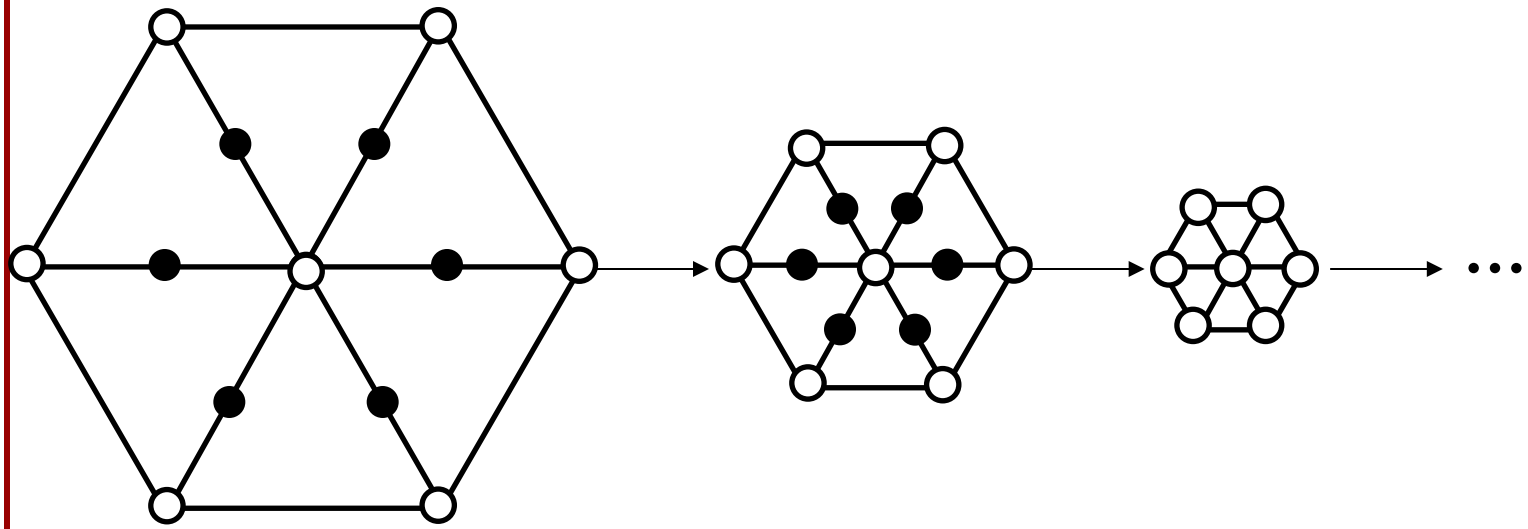
Guaranteed Smoothness?



To determine the smoothness of the subdivision:

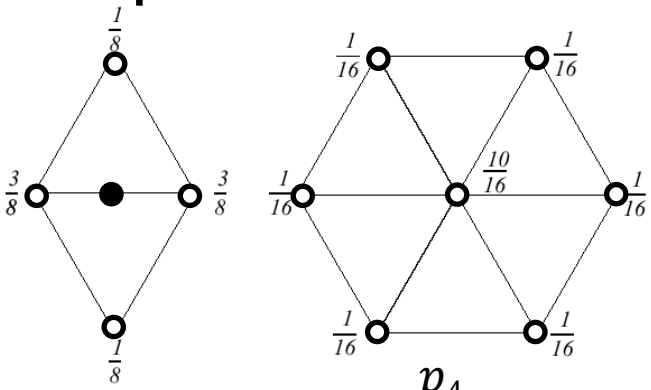
- Repeatedly apply the subdivision scheme
- Look at the neighborhood in the limit.

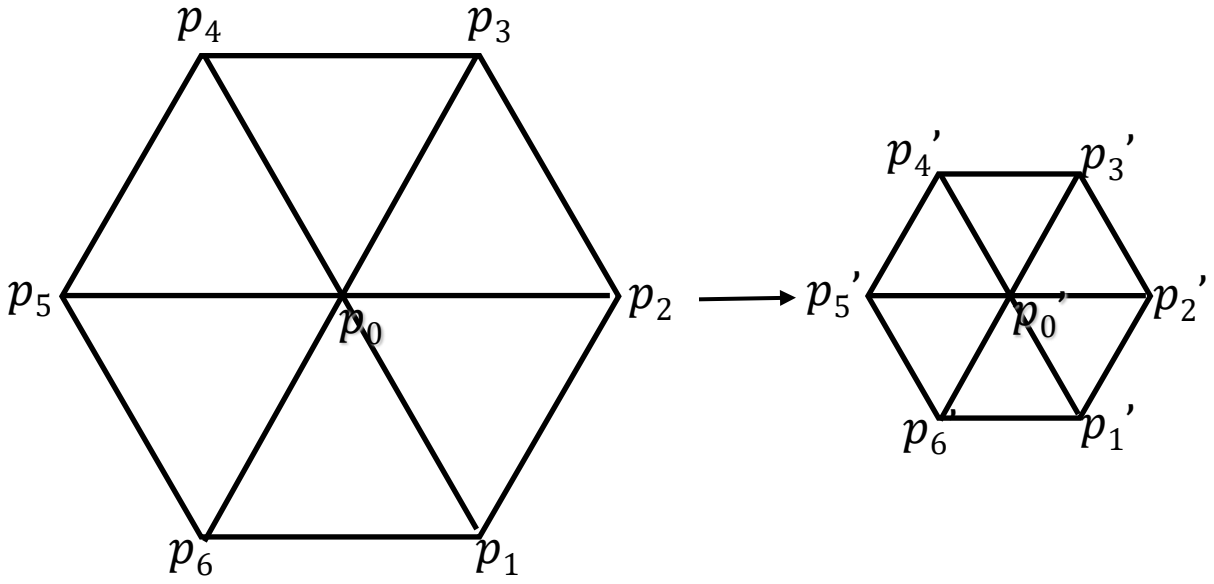
What happens after infinitely many iterations?



Guaranteed Smoothness?

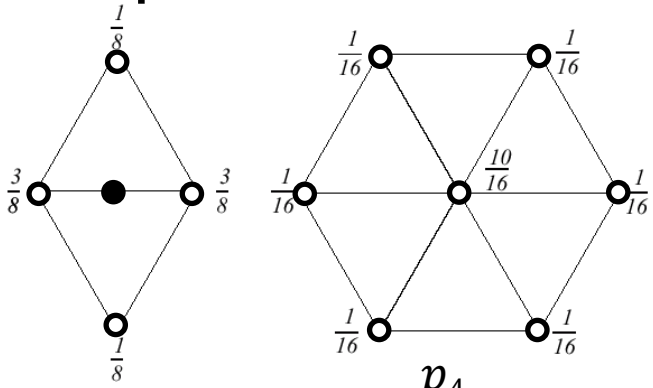
Compute the new positions as a linear combination of previous ones.





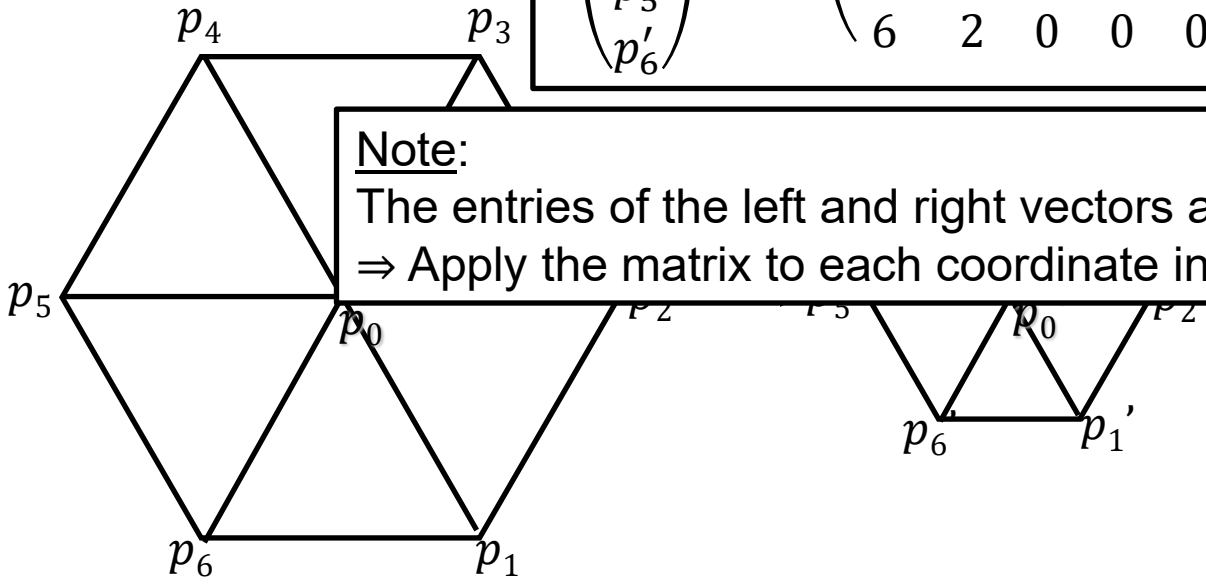
Guaranteed Smoothness?

Compute the new positions as a linear combination of previous ones.



Subdivision Matrix

$$\begin{pmatrix} p_0' \\ p_1' \\ p_2' \\ p_3' \\ p_4' \\ p_5' \\ p_6' \end{pmatrix} = \frac{1}{16} \begin{pmatrix} 10 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 6 & 2 & 0 & 0 & 0 & 2 \\ 6 & 2 & 6 & 2 & 0 & 0 & 0 \\ 6 & 0 & 2 & 6 & 2 & 0 & 0 \\ 6 & 0 & 0 & 2 & 6 & 2 & 0 \\ 6 & 0 & 0 & 0 & 2 & 6 & 2 \\ 6 & 2 & 0 & 0 & 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix}$$



Note:
The entries of the left and right vectors are 3D positions.
⇒ Apply the matrix to each coordinate independently

Guaranteed Smoothness?



Compute the new positions as a linear combination of previous ones.

To find the limit position of p_0, repeatedly apply the **subdivision matrix.**

Use eigenvalue decomposition to compute the n^{th} power of the matrix efficiently.

$$\begin{pmatrix} p_0^{(n)} \\ p_1^{(n)} \\ p_2^{(n)} \\ p_3^{(n)} \\ p_4^{(n)} \\ p_5^{(n)} \\ p_6^{(n)} \end{pmatrix} = \left[\frac{1}{16} \underbrace{\begin{pmatrix} 10 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 6 & 2 & 0 & 0 & 0 & 2 \\ 6 & 2 & 6 & 2 & 0 & 0 & 0 \\ 6 & 0 & 2 & 6 & 2 & 0 & 0 \\ 6 & 0 & 0 & 2 & 6 & 2 & 0 \\ 6 & 0 & 0 & 0 & 2 & 6 & 2 \\ 6 & 2 & 0 & 0 & 0 & 2 & 6 \end{pmatrix}}_{\mathbf{S}} \right]^n \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix}$$

Guaranteed Smoothness?



If, after a change of basis we have $\mathbf{S} = \mathbf{A}^{-1}\mathbf{D}\mathbf{A}$, where $\mathbf{D}$ is a diagonal matrix, then:

$$\mathbf{S}^n = (\mathbf{A}^{-1}\mathbf{D}\mathbf{A})(\mathbf{A}^{-1}\mathbf{D}\mathbf{A})\cdots(\mathbf{A}^{-1}\mathbf{D}\mathbf{A})(\mathbf{A}^{-1}\mathbf{D}\mathbf{A})$$
$$= \mathbf{A}^{-1}\mathbf{D}^n\mathbf{A}$$

Since $\mathbf{D}$ is diagonal, raising $\mathbf{D}$ to the n^{th} power is the same as raising each of the diagonal entries of $\mathbf{D}$ to the n^{th} power.

decomp

compute

power o

$$\mathbf{D}^n = \begin{pmatrix} \lambda_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_6 \end{pmatrix}^n = \begin{pmatrix} \lambda_0^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_6^n \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}^n \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

- If $|\lambda_i| > 1$ <u>for any</u> $0 \le i \le 6$, $\mathbf{D}^n$ blows up as $n \to \infty$.
- If $|\lambda_i| < 1$ <u>for all</u> $0 \le i \le 6$, $\mathbf{D}^n$ collapses as $n \to \infty$.
- If $\lambda_i = -1$ for any $0 \le i \le 6$, $\mathbf{D}^n$ does not converge as $n \to \infty$.

Guaranteed Smoothness?



Set $\mathbf{S}^\infty$ to be the matrix:

$$\mathbf{S}^\infty = \mathbf{A}^{-1}\begin{pmatrix} \lambda_0^\infty & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_6^\infty \end{pmatrix}\mathbf{A}$$

with $\lambda_i^\infty = 1$ if $\lambda_i = 1$, and $\lambda_i^\infty = 0$ otherwise.

The limit of the point p_0 and its 1-ring neighborhood under repeated subdivision is:

$$\begin{pmatrix} p_0^\infty \\ \hline \vdots \\ \hline p_6^\infty \end{pmatrix} = \mathbf{S}^\infty \begin{pmatrix} p_0 \\ \hline \vdots \\ \hline p_6 \end{pmatrix}$$

Note that if the subdivision scheme is continuous:

$$p_0^\infty = p_1^\infty = p_2^\infty = p_3^\infty = p_4^\infty = p_5^\infty = p_6^\infty$$

Guaranteed Smoothness?



Set $\mathbf{S}^{\infty}$ to be the matrix:

$$\mathbf{S}^{\infty} = \mathbf{A}^{-1} \begin{pmatrix} \lambda_0^{\infty} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_6^{\infty} \end{pmatrix} \mathbf{A}$$

with $\lambda_i^{\infty} = 1$ if $\lambda_i = 1$, and $\lambda_i^{\infty} = 0$ otherwise.

The limit of the point p_0 and its 1-ring neighborhood under repeated subdivision is:

Using a similar approach we can derive an expression for the normal at the limit point.
⇒ For the normal to be well-defined, we get additional constraints on diagonal values.

Subdivision Surfaces

- Properties:
  - ✓ Concise
  - ✓ Local support
  - ✓ Affine invariant
  - ✓ Arbitrary topology
  - ✓ Guaranteed smoothness
  - ✓ Natural parameterization
  - ◦ Efficient display
  - ◦ Efficient intersections



Pixar

Given data (e.g. texture coordinates) at the vertices of the base mesh, the subdivision weights can also be used to set the data.

Note:
Could be problematic if using a multi-chart atlas.

Subdivision Surfaces

- Properties:
  - ✓ Concise
  - ✓ Local support
  - ✓ Affine invariant
  - ✓ Arbitrary topology
  - ✓ Guaranteed smoothness
  - ✓ Natural parameterization
  - ✓ Efficient display
  - ◦ Efficient intersections

Can refine so that triangle projections are pixel-sized. (Can even use the limit positions as the vertex coordinates.)



Pixar

Subdivision Surfaces



- Properties:
  - ✓ Concise
  - ✓ Local support
  - ✓ Affine invariant
  - ✓ Arbitrary topology
  - ✓ Guaranteed smoothness
  - ✓ Natural parameterization
  - ✓ Efficient display
  - ✗ Efficient intersections



Given a ray, cannot tell where it would intersect the limit surface.
(But can ray-trace sufficiently refined mesh.)

Pixar