Modeling Transformations

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Announcements

Midterm is October 14th

Overview

- Ray-Tracing so far
- Modeling transformations

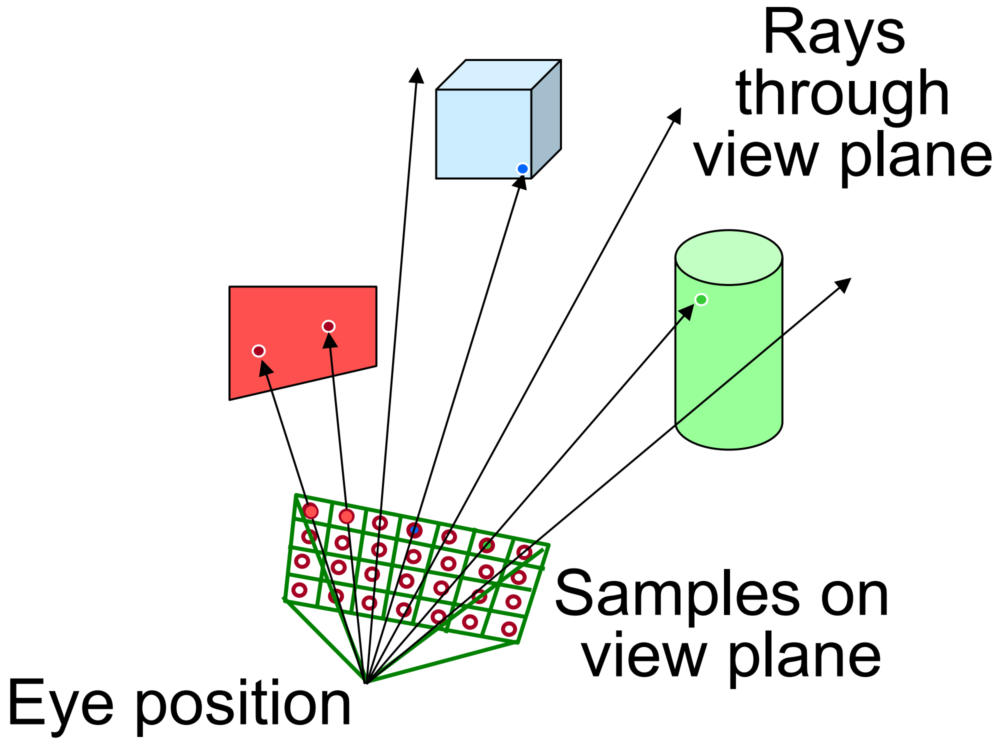
Ray Tracing

```
Image RayTrace( Camera camera , Scene scene , int width , int height , int depth , float cutoff )
{
	Image image( width , height );
	for( int i=0 ; i<width ; i++ ) for( int j=0 ; j<height ; j++ )
	{
		Ray< 3 > ray = ConstructRayThroughPixel( camera , i , j );
		image[i][j] = GetColor( scene , ray , 1. , depth , Color( cutOff , cutOff , cutOff ) );
	}
	return image;
}
```

Ray Tracing

For each sample …

- Construct ray from eye position through view plane
- Compute color contribution of the ray



Ray Tracing

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	}
	return image;
}
```

Constructing Ray Through a Pixel

2D Example: Side view of camera

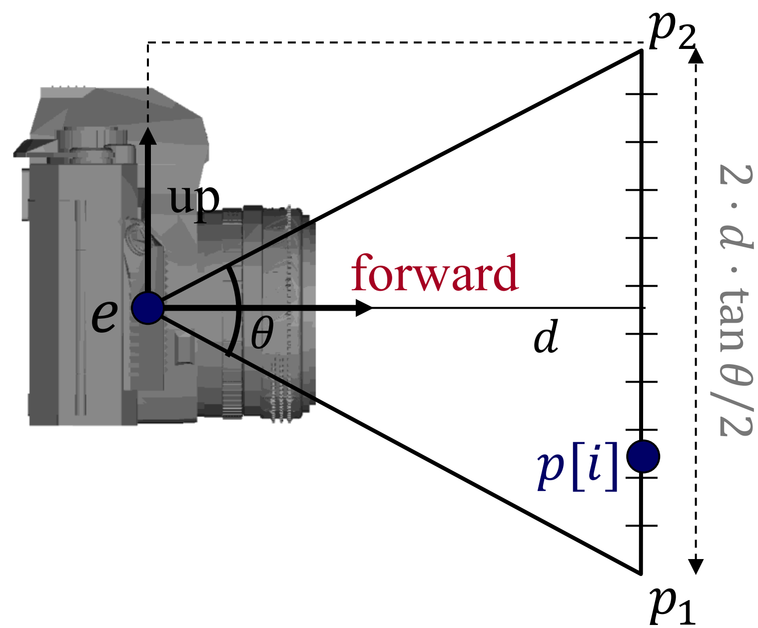
Where is the i-th pixel, $p[i]$, with $i \in [0, \text{height})$?

θ = field of view angle (given)
d = distance to view plane (arbitrary)

$$p_1 = e + d \cdot \text{forward} - d \cdot \tan\frac{\theta}{2} \cdot \text{up}$$

$$p_2 = e + d \cdot \text{forward} + d \cdot \tan\frac{\theta}{2} \cdot \text{up}$$

$$p[i] = p_1 + \left(\frac{i + 0.5}{\text{height}}\right) \cdot (p_2 - p_1)$$



Ray Tracing

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		image[i][j] = GetColor( scene , ray , 1. , depth , Color( cutOff , cutOff , cutOff ) );
	}
	return image;
}
```

Ray Tracing

```
Color GetColor( Scene scene , Ray< 3 > ray , float ir , int rDepth , Color cutOff )
{
	Color c(0,0,0);
	Ray< 3 > reflect , refract;
	if( !rDepth || ( cutOff[0]>1 && cutOff[1]>1 && cutOff[2]>1 ) ) return c;
	Intersection hit;

	if( FindIntersection( ray , scene , hit ) )
	{
		c += GetSurfaceColor( hit.position );

		if( Dot( ray.direction , hit.normal )<0 ) // Ray coming in from the outside
		{
			reflect.direction = Reflect( ray.direction , hit.normal );
			reflect.position = hit.position + reflect.direction*ε;
			c += GetColor( scene , reflect , ir , rDepth-1 , cutOff/hit.kSpec )*hit.kSpec;
		}

		if(  Refract( ray.direction , hit.normal , ir , hit.ir , refract.direction )
		{
			refract.position = hit.position + refract.direction*ε;
			c += GetColor( scene , refract , hit.ir , rDepth-1 , cutOff/hit.kTran )*hit.kTran;
		}
	}
	return c;
}
```

Ray-Scene Intersection

- Intersections with geometric primitives
  - Sphere
  - Triangle
  - Groups of primitives (scene)
- Acceleration techniques
  - Bounding volume hierarchies
  - Spatial partitions
    - Uniform grids
    - Octrees
    - BSP trees

Ray Tracing

```
Color GetColor( Scene scene , Ray< 3 > ray , float ir , int rDepth , Color cutOff )
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			c += GetColor( scene , refract , hit.ir , rDepth-1 , cutOff/hit.kTran )*hit.kTran;
		}
	}
	return c;
}
```
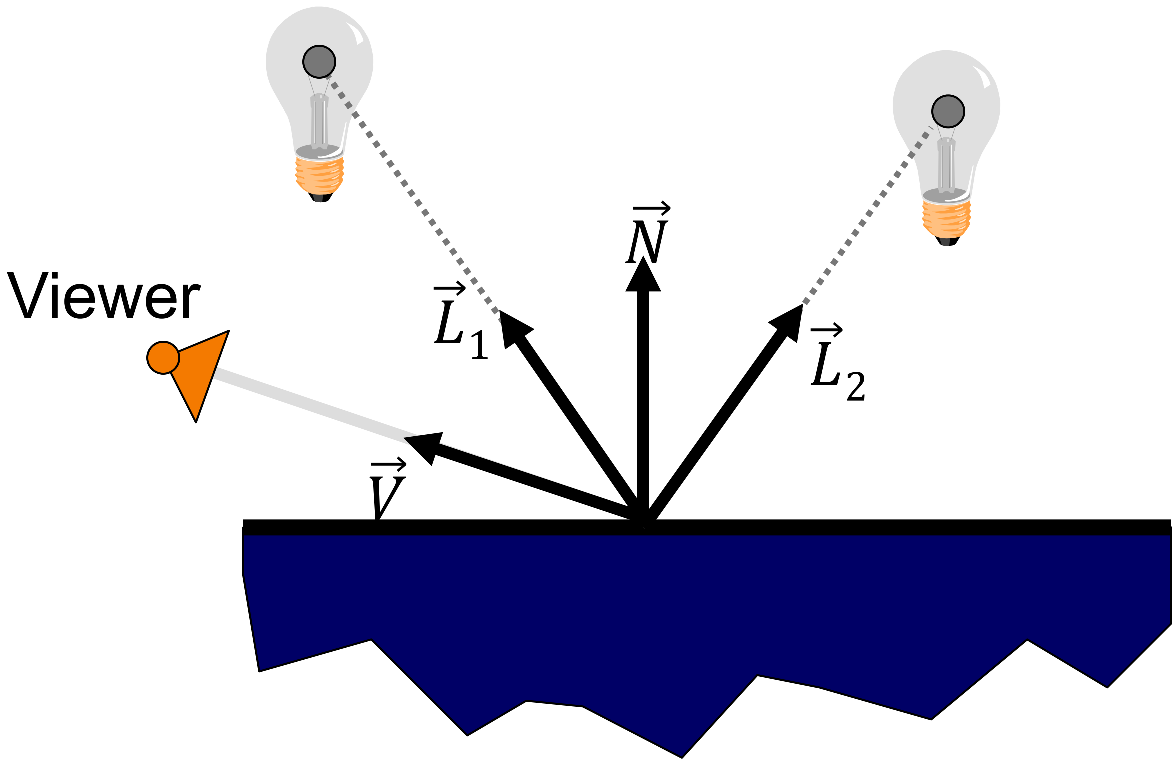

Surface Illumination Calculation



$$I = \boxed{K_E + \sum_L \left[K_A \cdot I_L^A + \left(K_D \cdot \langle \vec{N}, \vec{L} \rangle + K_S \cdot \langle \vec{V}, \vec{R}(\vec{L}) \rangle^{K_n}\right) \cdot I_L \cdot S_L\right]} + K_s \cdot I_R + K_T \cdot I_T$$

Ray Tracing

```
Color GetColor( Scene scene , Ray< 3 > ray , float ir , int rDepth , Color cutOff )
{
	Color c(0,0,0);
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			reflect.position = hit.position + reflect.direction*ε;
			c += GetColor( scene , reflect , ir , rDepth-1 , cutOff/hit.kSpec )*hit.kSpec;
		}

		if(  Refract( ray.direction , hit.normal , ir , hit.ir , refract.direction )
		{
			refract.position = hit.position + refract.direction*ε;
			c += GetColor( scene , refract , hit.ir , rDepth-1 , cutOff/hit.kTran )*hit.kTran;
		}
	}
	return c;
}
```
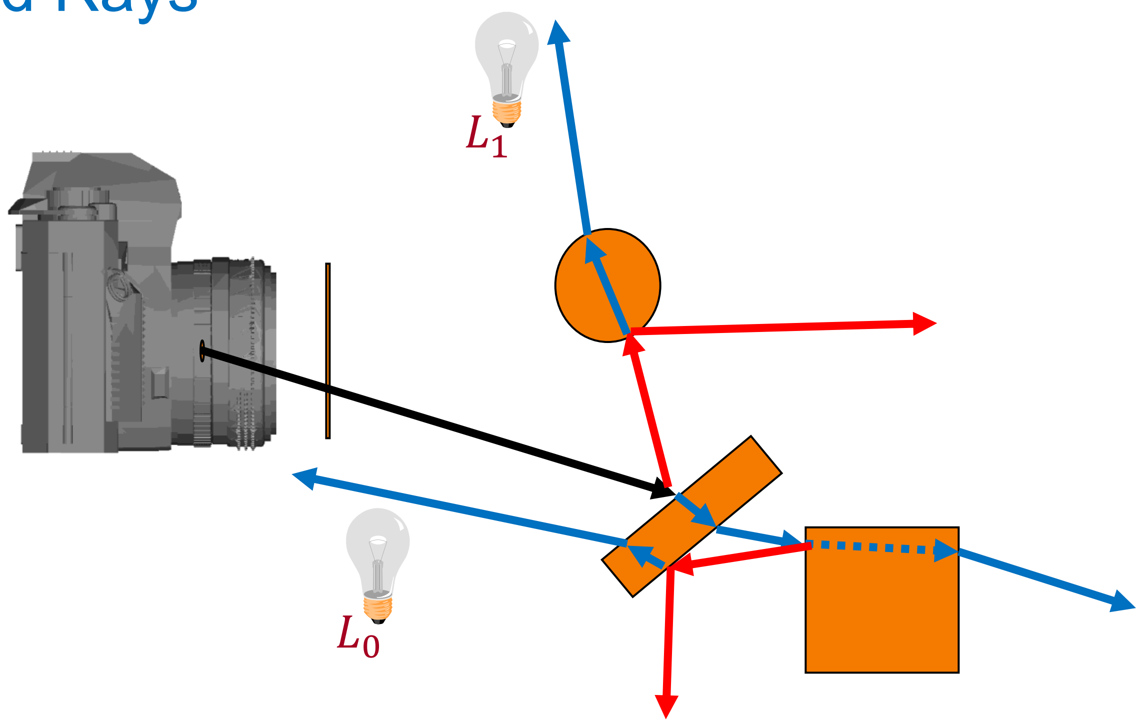

Ray Tracing (Recursive)

Consider the contribution of:

Reflected Rays

Refracted Rays



$$I = K_E + \sum_L \left[K_A \cdot I_L^A + \left(K_D \cdot \langle \vec{N}, \vec{L} \rangle + K_S \cdot \langle \vec{V}, \vec{R}(\vec{L}) \rangle^{K_n} \right) \cdot I_L \cdot S_L \right] + \boxed{K_s \cdot I_R + K_T \cdot I_T}$$

Overview

- Raytracing so far
- Modeling transformations

Modeling Transformations

Specify transformations for objects allows:

- Defining objects in their own coordinate systems
- Using one object definition multiple times in a scene

Overview

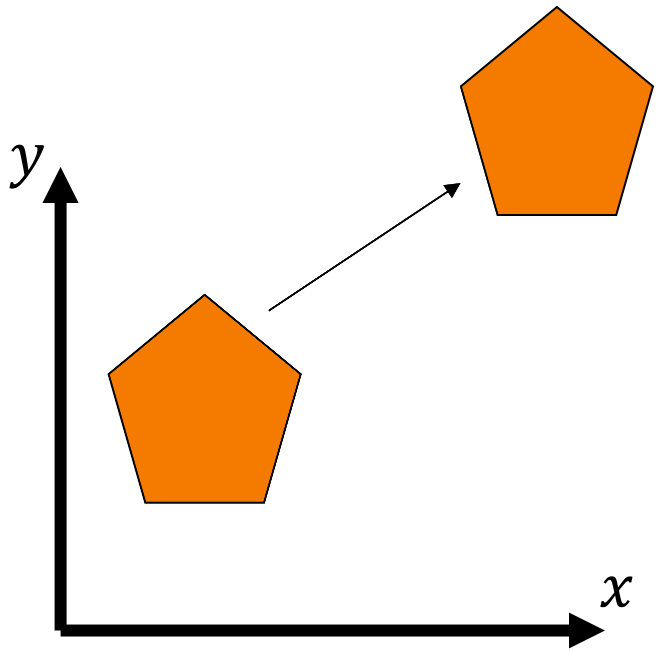
- 2D Transformations
  - Basic 2D transformations
  - Matrix representation
  - Matrix composition
- 3D Transformations
  - Basic 3D transformations
  - Matrix representation
  - Matrix composition

Simple 2D Transformations

Translation

$$p' = p + t$$

$$\begin{bmatrix} p'_x \\ p'_y \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

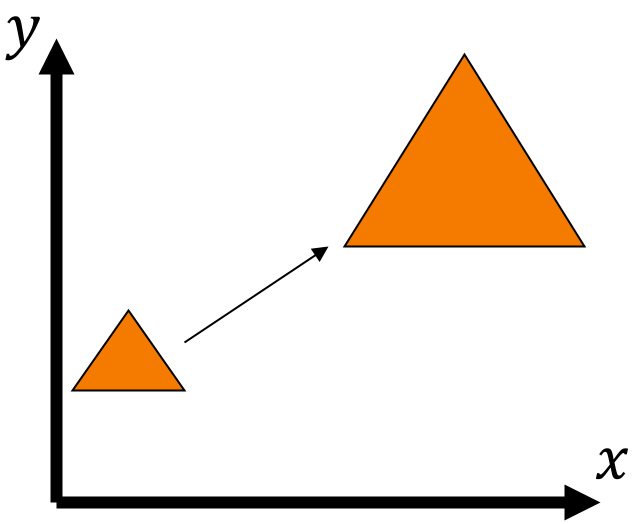


Simple 2D Transformations

Scale

$$p' = S \cdot p$$

$$\begin{bmatrix} p'_x \\ p'_y \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

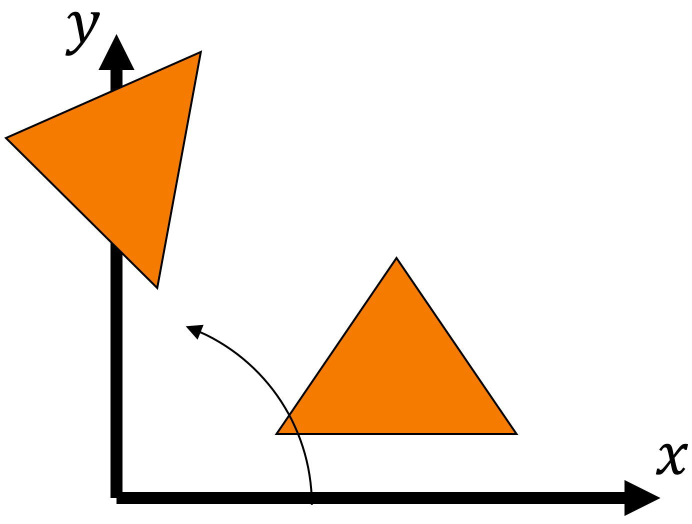


Simple 2D Transformation

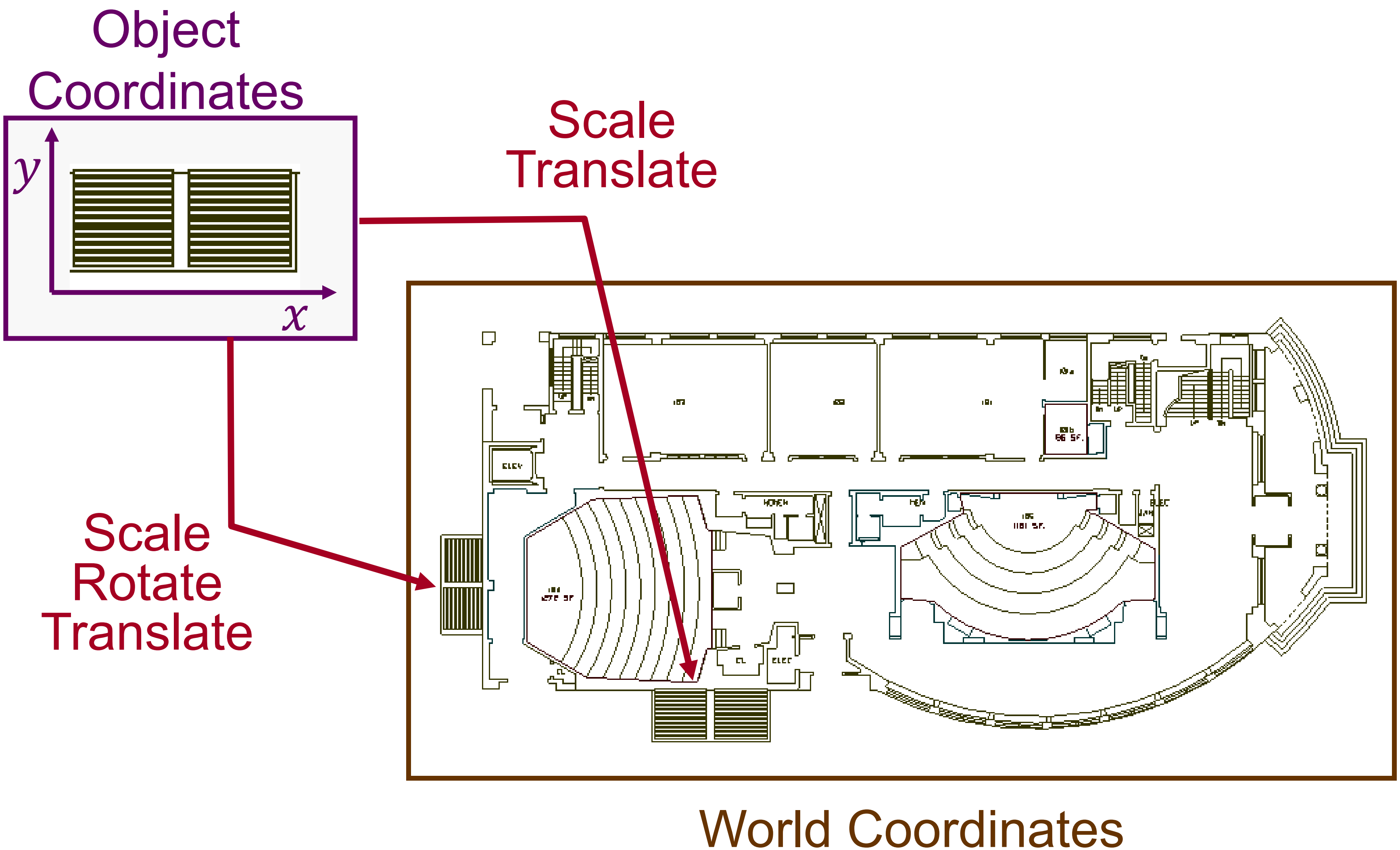
Rotation

$$p' = R \cdot p$$

$$\begin{bmatrix} p'_x \\ p'_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

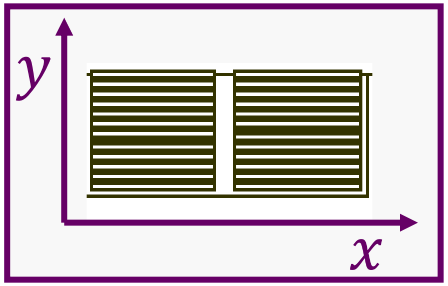


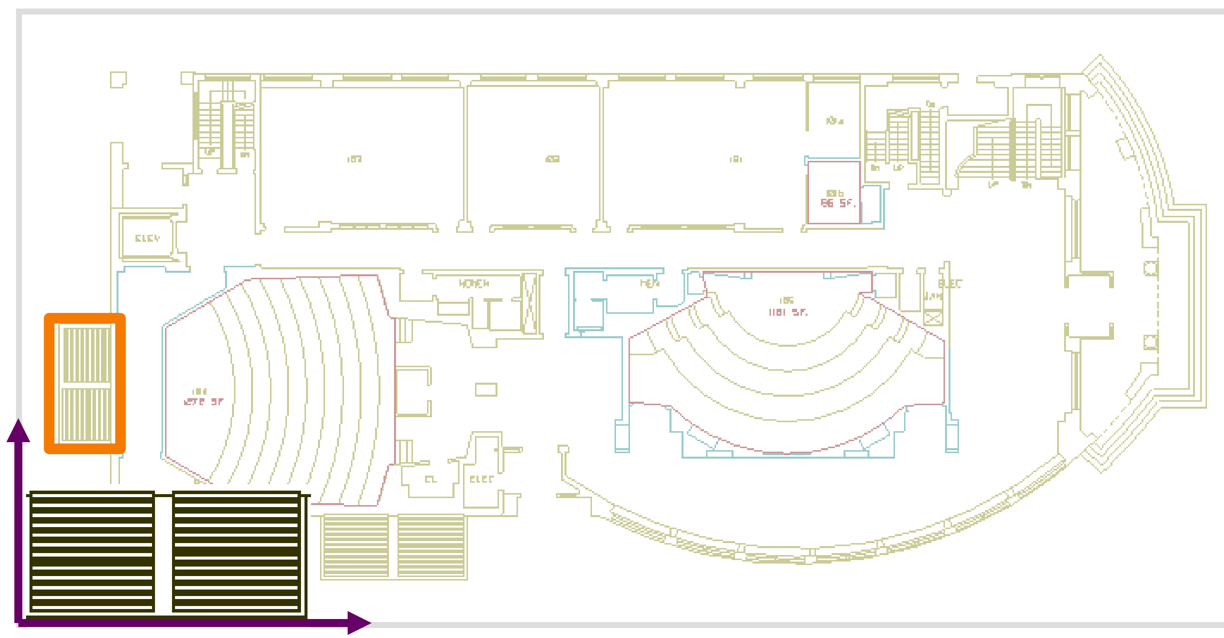
2D Modeling Transformations



2D Modeling Transformations

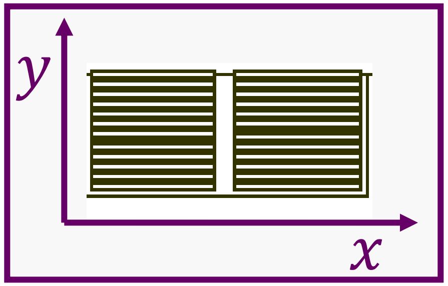
Object Coordinates



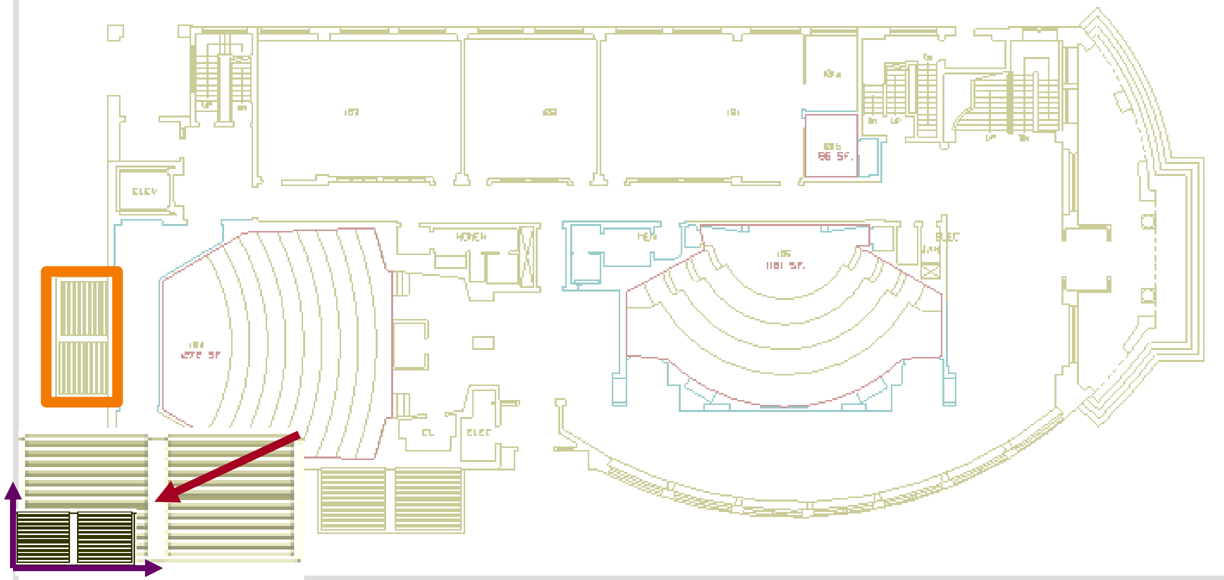


2D Modeling Transformations

Object Coordinates

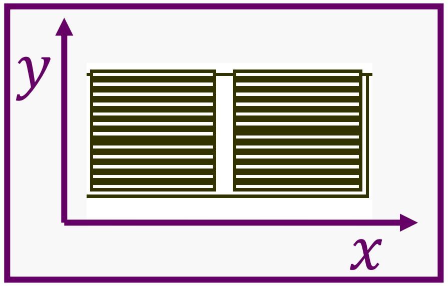


Scale .3, .3

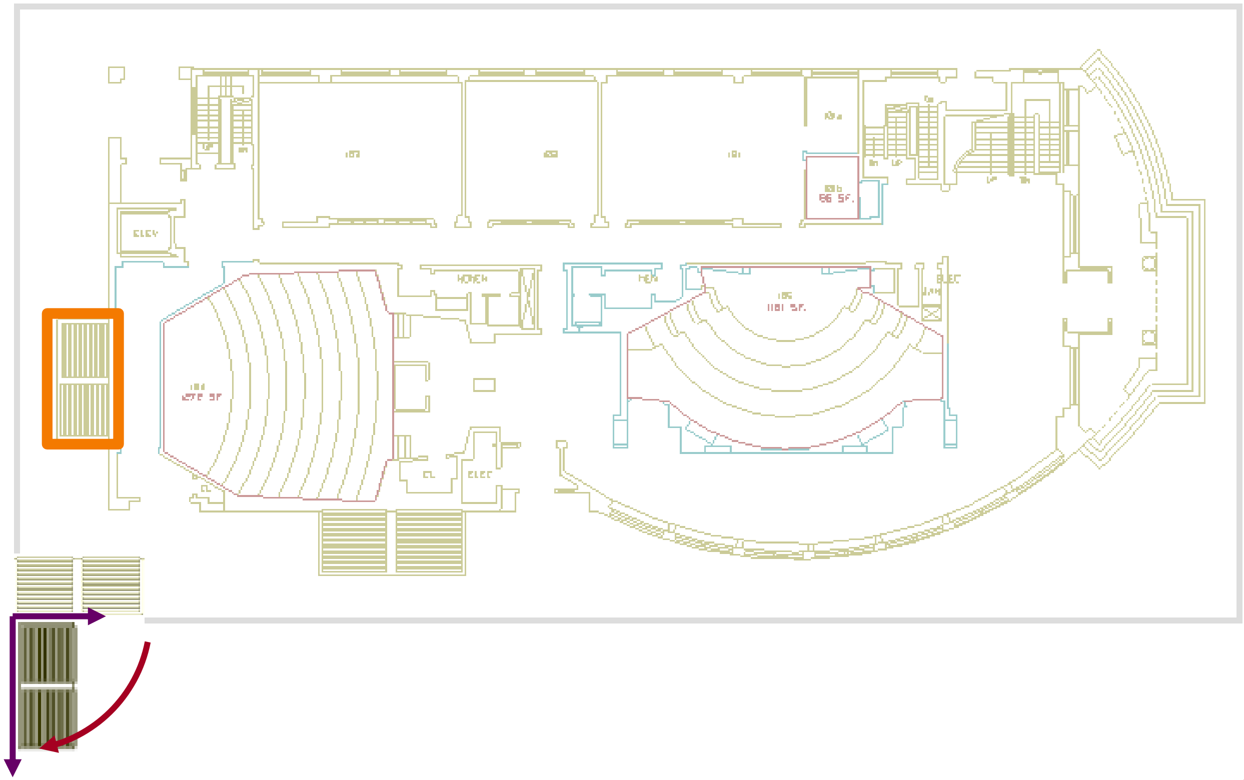


2D Modeling Transformations

Object Coordinates

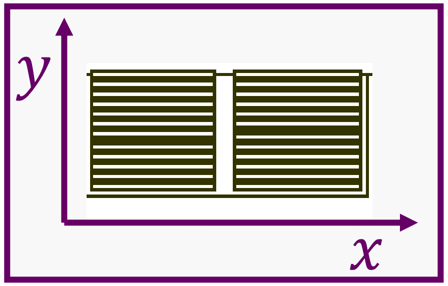


Scale .3, .3
Rotate -90

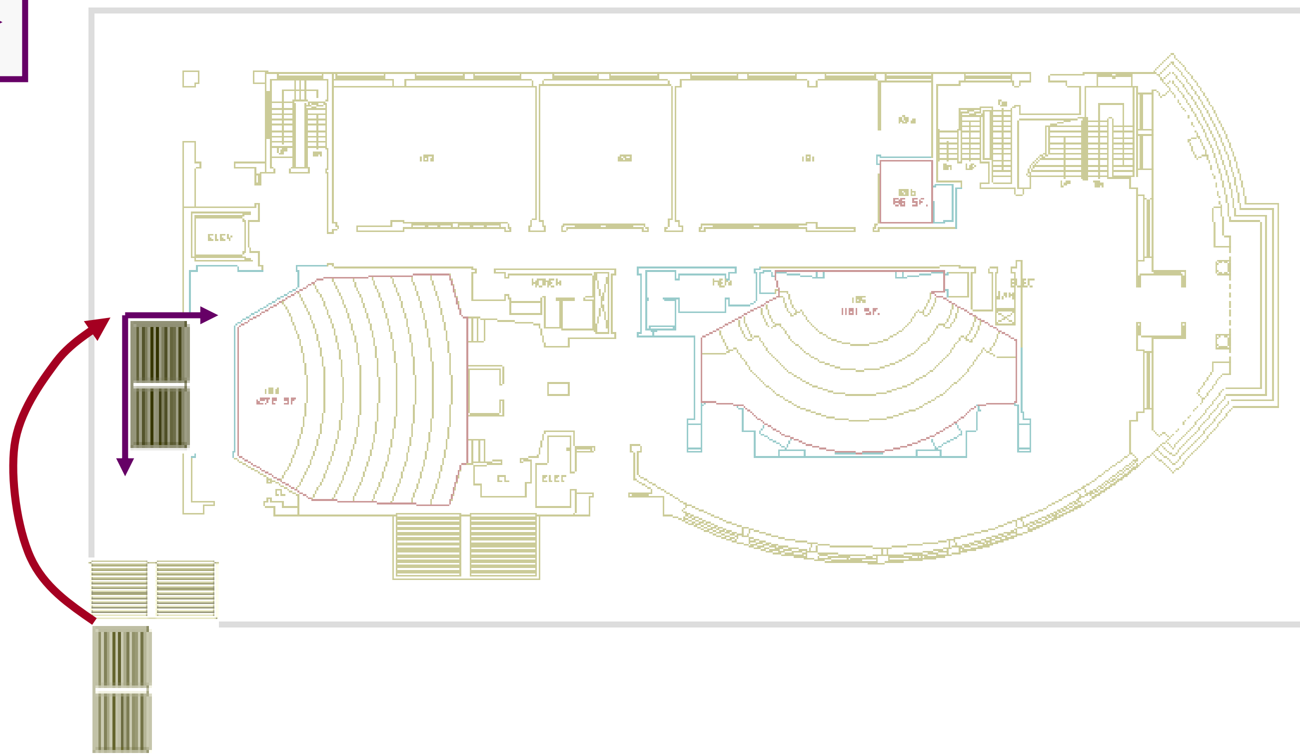


2D Modeling Transformations

Object
Coordinates

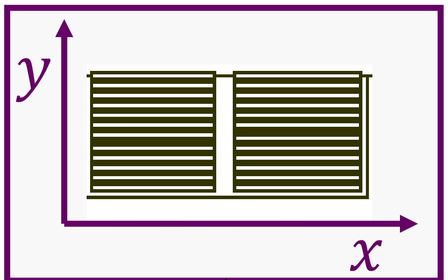


Scale .3, .3
Rotate -90
Translate 3, 5



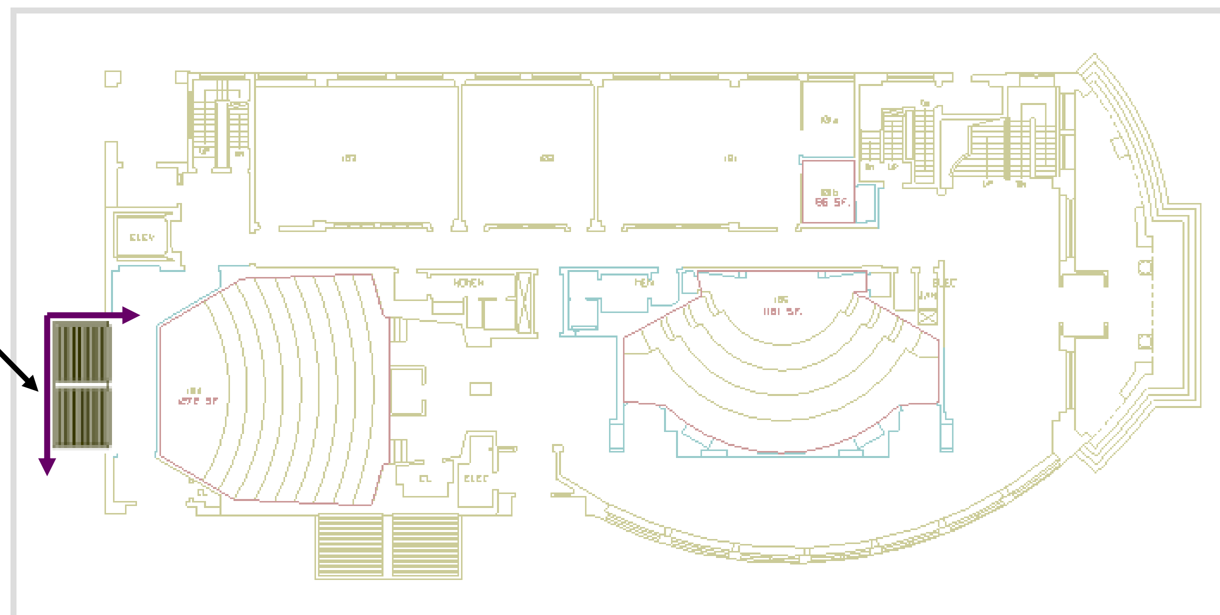
2D Modeling Transformations

Object Coordinates



The composition take us from object to world coordinates

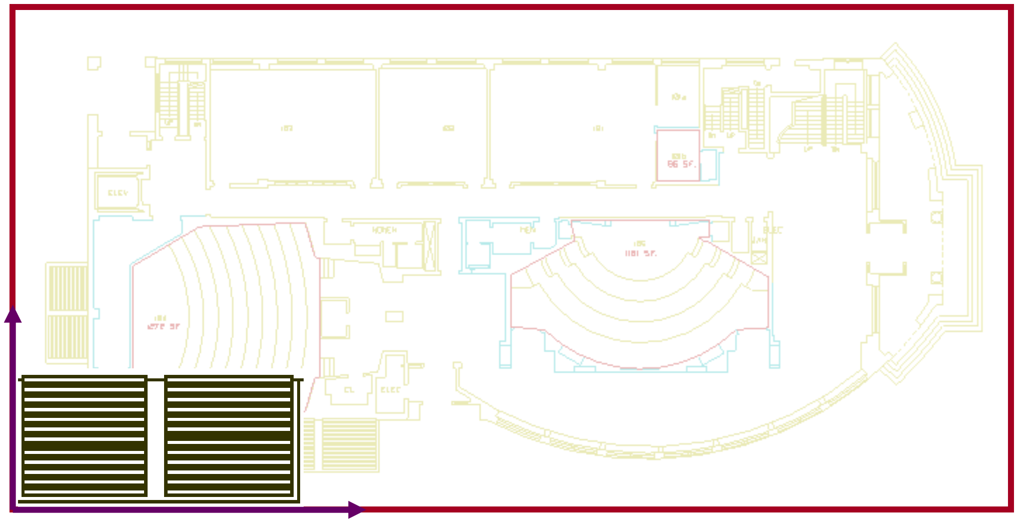
Scale .3, .3
Rotate -90
Translate 3, 5



Composing 2D Transformations

Translation:

$$x' = x + t_x$$
$$y' = y + t_y$$



Scale:

$$x' = x \cdot s_x$$
$$y' = y \cdot s_y$$

Rotation:

$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$
$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$

Composing 2D Transformations

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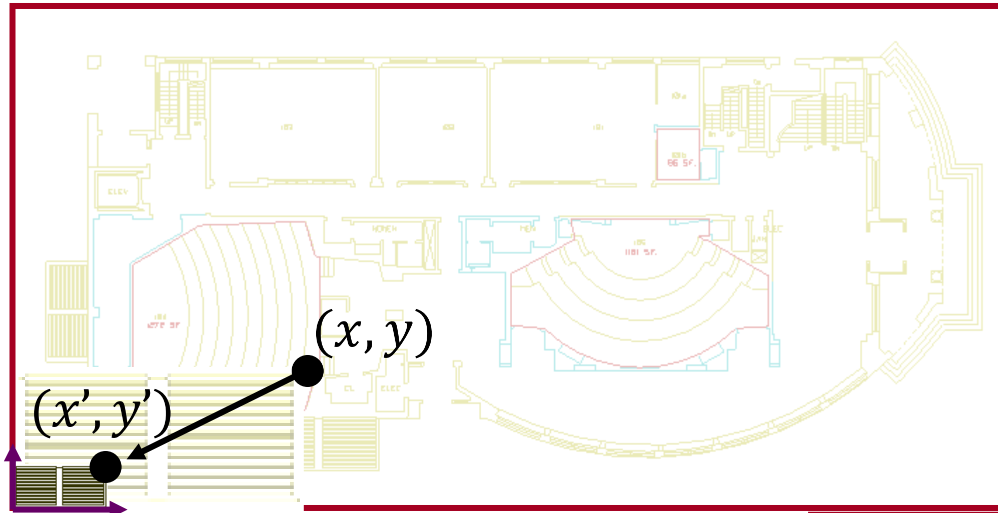

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Composing 2D Transformations

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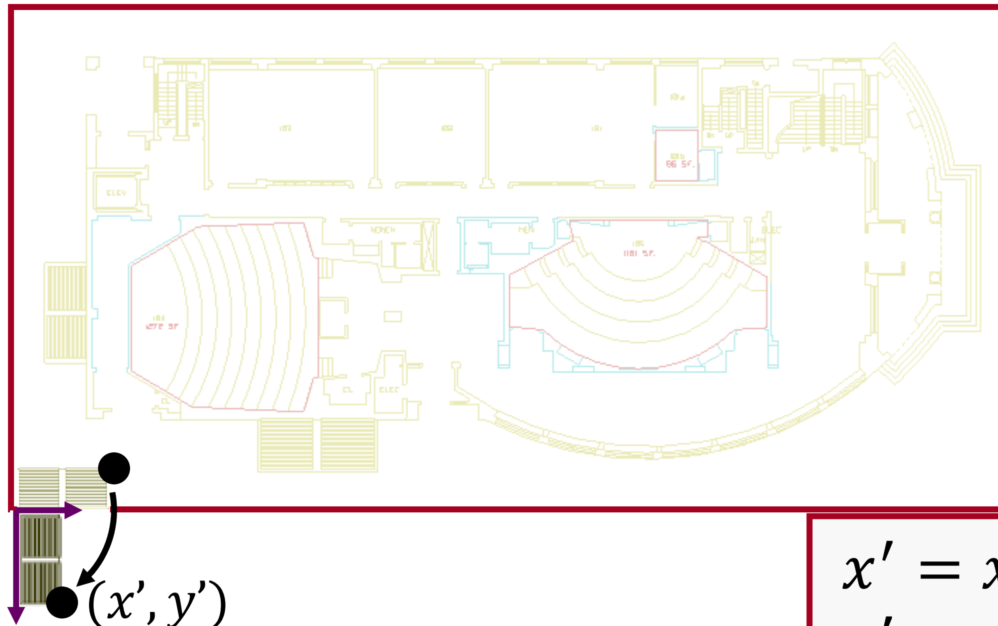

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Composing 2D Transformations

Translation:

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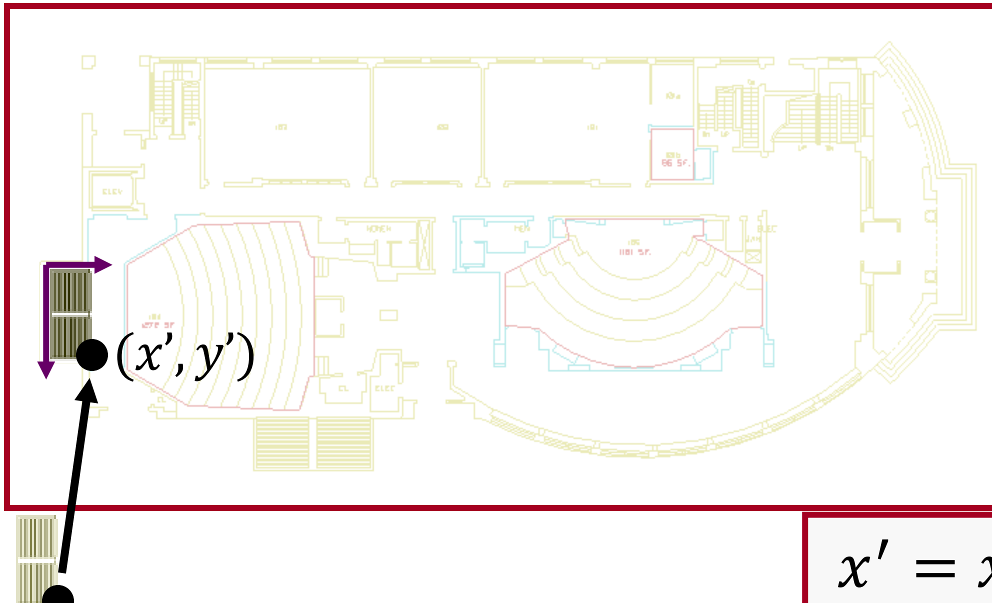

$$x' = x \cdot s_x$$
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$$x' = (x \cdot s_x) \cdot \cos\theta - (y \cdot s_y) \cdot \sin\theta$$
$$y' = (x \cdot s_x) \cdot \sin\theta + (y \cdot s_y) \cdot \cos\theta$$

$$x' = (x \cdot s_x) \cdot \cos\theta - (y \cdot s_y) \cdot \sin\theta + t_x$$
$$y' = (x \cdot s_x) \cdot \sin\theta + (y \cdot s_y) \cdot \cos\theta + t_y$$

Composing 2D Transformations

Naïve composition makes the expression more complicated as more transformations are applied!

Trans

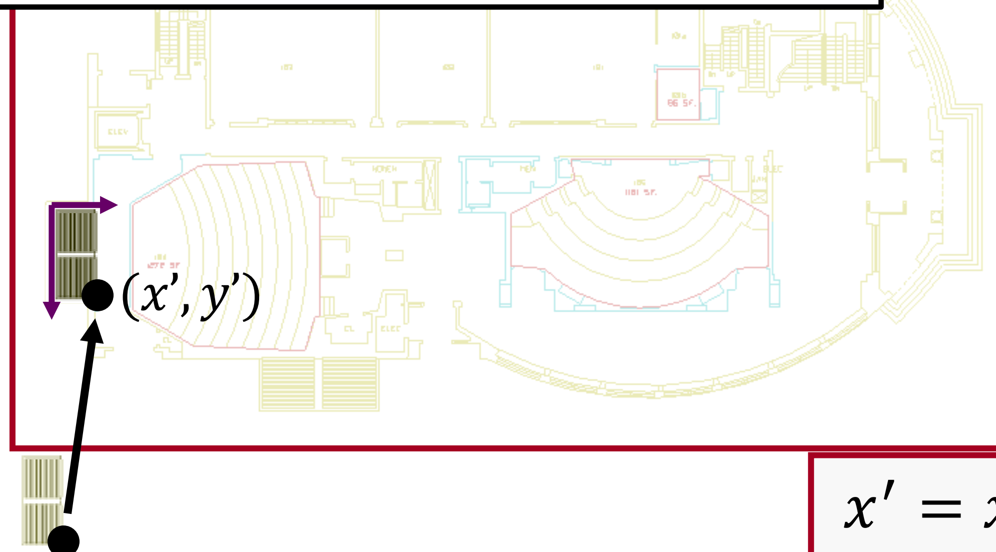
$$x' = x + t_x$$
$$y' = y + t_y$$

Scale:

$$x' = x \cdot s_x$$
$$y' = y \cdot s_y$$

Rotation:

$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$
$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$



$$x' = x \cdot s_x$$
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$$x' = (x \cdot s_x) \cdot \cos\theta - (y \cdot s_y) \cdot \sin\theta$$
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Overview

- 2D Transformations
  - Basic 2D transformations
  - Matrix representation
  - Matrix composition
- 3D Transformations
  - Basic 3D transformations
  - Matrix representation
  - Matrix composition

Matrix Representation

Represent 2D transformation by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Multiply matrix by column vector

$\Updownarrow$

apply transformation to point

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \Leftrightarrow \quad \begin{aligned} x' &= a \cdot x + b \cdot y \\ y' &= c \cdot x + d \cdot y \end{aligned}$$

Matrix Representation

Transformation composition is matrix multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix}$$

$\Rightarrow$ The composition is still represented by matrix

2x2 Matrices

What transforms can we represent with a matrix?

2D Scale around (0,0)?

$$x' = s_x \cdot x \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$y' = s_y \cdot y$$

2D Rotate around (0,0)?

$$x' = \cos\theta \cdot x - \sin\theta \cdot y \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$y' = \sin\theta \cdot x + \cos\theta \cdot y$$

2D Mirror over Y axis?

$$x' = -x \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$y' = y$$

2x2 Matrices

What transforms can we represent with a matrix?

2D Scale around (0,0)?

$$x' = s_x \cdot x$$
$$y' = s_y \cdot y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Like scale with negative scale values

2D Mirror over Y axis?

$$x' = -x$$
$$y' = \quad y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

What transforms can we represent with a matrix?

2D Translation?

$$x' = x + t_x$$
$$y' = y + t_y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

What transforms can we represent with a matrix?

2D Translation?

$$x' = x + t_x$$
$$y' = y + t_y$$

NO!

Only <u>linear</u> 2D transformations can be represented with a 2×2 matrix

Linear Transformations

Linear transformations are combinations of …

- Scale, and rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Properties of linear transformations:

- Satisfies: $T(s_1 \cdot p_1 + s_2 \cdot p_2) = s_1 \cdot T(p_1) + s_2 \cdot T(p_2)$
- ⇒ Origin maps to origin
- ⇒ Lines map to lines
- ⇒ Preserves (weighted) average
- ⇒ Parallel lines remain parallel
- ⇒ Closed under composition

Linear Transformations

Linear transformations are combinations of …

Scale, and rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Properties of linear transformations:

Satisfies: $T(s_1 \cdot p_1 + s_2 \cdot p_2) = s_1 \cdot T(p_1) + s_2 \cdot T(p_2)$

$\Rightarrow$ Origin maps to origin

$\Rightarrow$ Lines map to lines

$\Rightarrow$ Preserves (weighted) averages

$\Rightarrow$ Parallel lines remain parallel

$\Rightarrow$ Closed under composition

Translations do not map the origin to the origin

2D Translation

Treat 2D positions as 3D positions by adding a third, *homogenous*, coordinate with fixed value "1":

$$(x, y) \to (x, y, 1)$$

Represent translations using a 3x3 matrix:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow \begin{aligned} x' &= x + t_x \\ y' &= y + t_y \end{aligned}$$

Basic 2D Transformations

Basic 2D transformations as 3×3 matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

Homog. Coordinates: $(x, y, 1) \leftrightarrow (x, y)$

More generally:

For $w \neq 0$ we associate $(x, y, w) \leftrightarrow \left(\frac{x}{w}, \frac{y}{w}\right)$

What about when $w = 0$?

Consider the limit:

$$\lim_{w \to 0} (x, y, w) \leftrightarrow \lim_{w \to 0} \left(\frac{x}{w}, \frac{y}{w}\right)$$

$\Rightarrow$ In the limit this is the *ideal point* at infinity in direction (x, y)…
… also the ideal point in direction $(-x, -y)$

Homog. Coordinates: $(x, y, 1) \leftrightarrow (x, y)$

More generally:

For $w \neq 0$ we associate $(x, y, w) \leftrightarrow \left(\frac{x}{w}, \frac{y}{w}\right)$

We associate $(x, y, 0) \leftrightarrow \lim_{w \to 0} \left(\frac{x}{w}, \frac{y}{w}\right)$

$(0, 0, 0)$ is not allowed!

$\Rightarrow$ In addition to supporting translation, homogenous coordinates describe geometry at infinity.

Homog. Coordinates: $(x, y, 1) \leftrightarrow (x, y)$

More generally:

For $w \neq 0$ we associate $(x, y, w) \leftrightarrow \left(\frac{x}{w}, \frac{y}{w}\right)$

We associate $(x, y, 0) \leftrightarrow \lim_{w \to 0} \left(\frac{x}{w}, \frac{y}{w}\right)$

Note:

As defined, the points $(a, b, 0)$ and $(-a, -b, 0)$ represent the *same* point at infinity.

Warning:

OpenGL distinguishes these *antipodal* points at infinity.

Affine Transformations

Affine transformations are combinations of …

- Linear transformations, and
- Translations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Properties of affine transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Preserves (weighted) average
- Parallel lines remain parallel
- Closed under composition

Affine Transformations

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Properti

- Origin
- Lines map to lines
- Preserves (weighted) average
- Parallel lines remain parallel
- Closed under composition

Note that with affine transformations $(x, y, 1)$ has to map to $(x', y', 1)$

Projective Transformations

Projective transformations …

- Affine transformations, and
- Projective warps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of projective transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- (Weighted) average is not necessarily preserved
- Parallel lines do not necessarily remain parallel
- Closed under composition

Projective Transformations

Projective transformations …

- Affine transformations, and
- Projective warps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properti

- Origin

Note that under projective transformations $(x, y, 1)$ does **not** have to map to $(x', y', 1)$

- Lines map to lines
- (Weighted) average is not necessarily preserved
- Parallel lines do not necessarily remain parallel
- Closed under composition

Overview

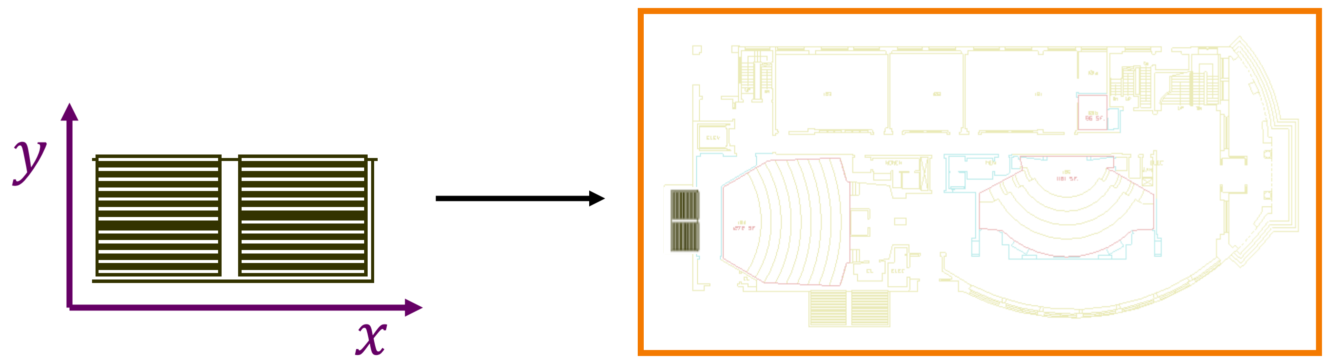
- 2D Transformations
  - Basic 2D transformations
  - Matrix representation
  - Matrix composition
- 3D Transformations
  - Basic 3D transformations
  - Matrix representation
  - Matrix composition

Matrix Composition

Transformations combine with matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$p' = T(t_x, t_y) \circ R(\theta) \circ S(s_x, s_y) \; p$$



Matrix Composition

Transformations combine with matrix multiplication

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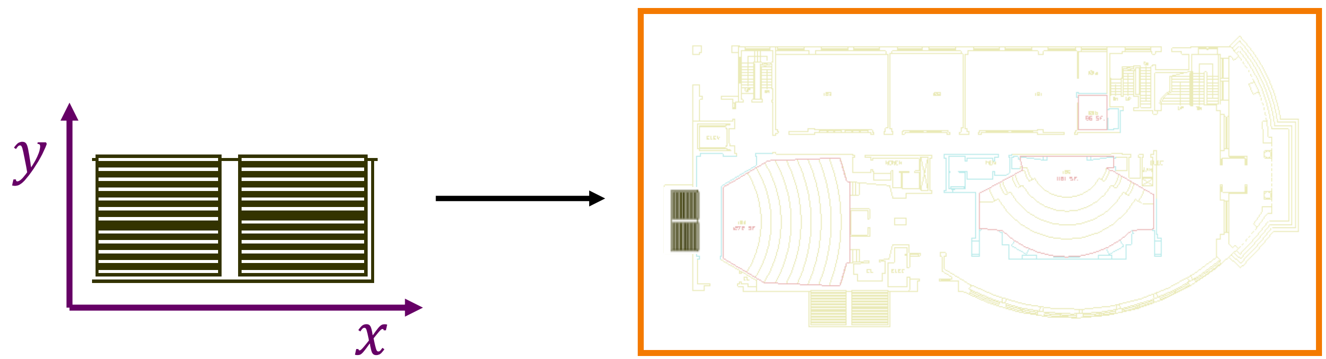
$$= \begin{bmatrix} s_x \cdot \cos\theta & -s_y \cdot \sin\theta & t_x \\ s_x \cdot \sin\theta & s_y \cdot \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Matrix Composition

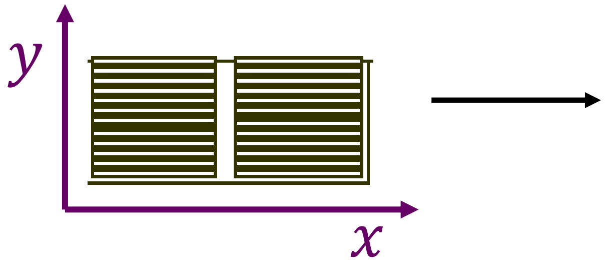
Transformations combine with matrix multiplication

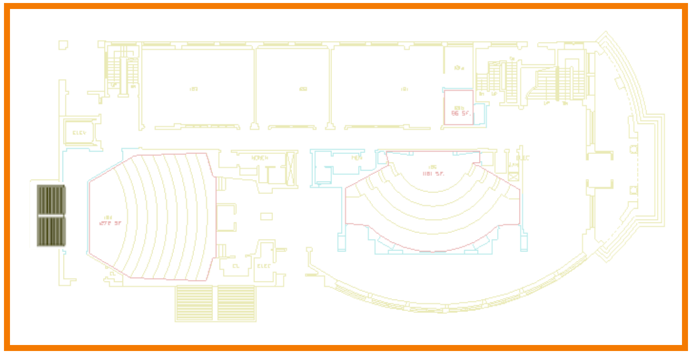
- General purpose representation
- Efficiently implemented with matrix (pre-)multiplication

$$p' = T\Big(R\big(S(p)\big)\Big)$$

$$\Updownarrow$$

$$p' = (T \circ R \circ S)(p)$$





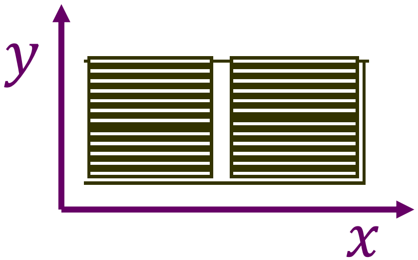
Matrix Composition

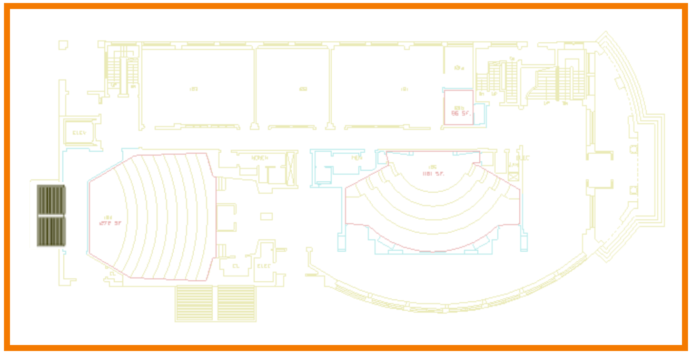
[NOTE] order of transformations matters

Matrix multiplication is not commutative

$$p' = T \cdot R \cdot S \cdot p$$

"Global" "Local"

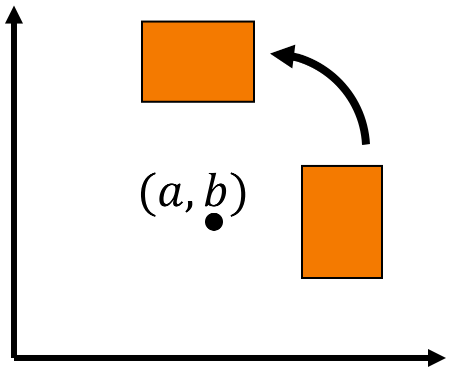




Matrix Composition

Rotate by θ around arbitrary point (a, b)

$$M = T(a,b) \circ R(\theta) \circ T(-a,-b)$$

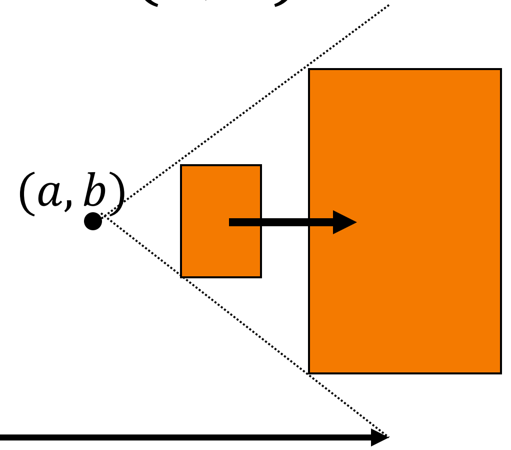


<u>Approach</u>:

1. Translate (a, b) to the origin.
2. Do the rotation about origin.
3. Translate back.

Scale by (s_x, s_y) around arbitrary point (a, b)

$$M = T(a,b) \circ S(s_x, s_y) \circ T(-a,-b)$$



(Use the same approach.)

Overview

2D Transformations

- Basic 2D transformations
- Matrix representation
- Matrix composition

3D Transformations

- Basic 3D transformations
- Matrix representation
- Matrix composition

3D Transformations

Same idea as 2D transformations

- Homogeneous coordinates: (x, y, z, w)
- 4×4 transformation matrices
  - Affine

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ \boxed{0 \quad 0 \quad 0 \quad 1} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

  - Projective

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ \boxed{m \quad n \quad o \quad p} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Basic 3D Transformations

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Identity

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Translation

Basic 3D Transformations

Pitch-Roll-Yaw Convention:

Any rotation can be expressed as the combination of a rotation about the x-, the y-, and the z-axis.

Rotate around z axis: $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotate around y axis: $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotate around x axis: $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Basic 3D Transformations

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Rotate aroun

How do you rotate around an arbitrary axis U by angle ψ?

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotate around x axis: $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

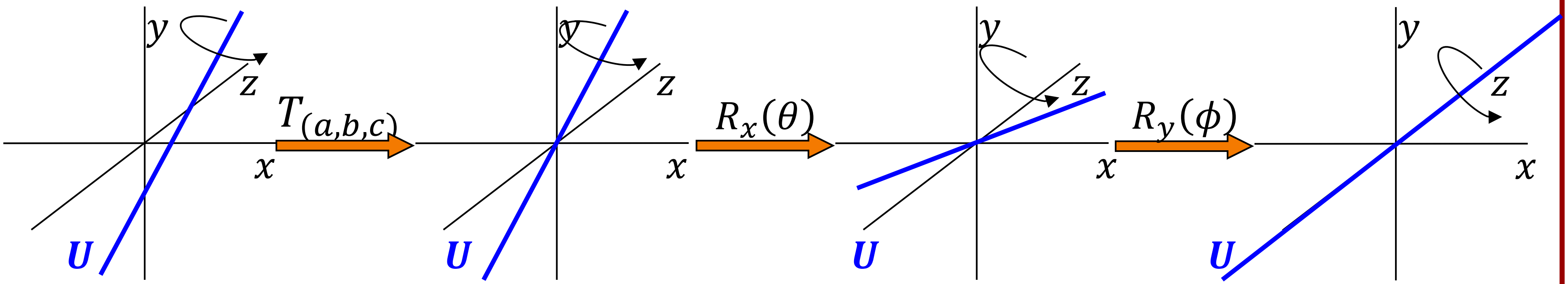
Rotation By ψ Around Arbitrary Axis U

Align U (w.l.o.g.) with the z-axis:

- $T_{(a,b,c)}$: Translate U by (a, b, c) to pass through origin
- $R_x(\theta)$: Rotate about the x-axis by θ to get U in the xz-plane
- $R_y(\phi)$: Rotate about the y-axis by ϕ to align U with the z-axis

$R_z(\psi)$: Perform rotation by ψ around the z-axis.

Do inverse of original transformation for alignment.



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$R_z(\psi)$: Perform rotation by ψ around the z-axis.

Do inverse of original transformation for alignment.

$$p' = \underbrace{\left(R_y(\phi)\cdot R_x(\theta)\cdot T_{(a,b,c)}\right)}_{\text{Aligning Transformation}}{}^{-1}\cdot R_z(\psi)\cdot \underbrace{\left(R_y(\phi)\cdot R_x(\theta)\cdot T_{(a,b,c)}\right)}_{\text{Aligning Transformation}} p$$