The Procrustes Method and 3D Scanning

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The Digital Michelangelo Project: 3D Scanning of Large Statues, Levoy *et al*. 2000

Recall

Any matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$ can be expressed in terms of its Singular Value Decomposition as:

$$\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}^{\top}$$

with:

- $\mathbf{U}, \mathbf{V} \in \mathbb{R}^{n \times n}$ orthogonal
- $\mathbf{D} \in \mathbb{R}^{n \times n}$ diagonal (i.e. off-diagonals are 0)
  - The diagonal entries are:
    - – Non-negative
    - – Decreasing

Recall

Given a square matrix:

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{11} & \cdots & \mathbf{M}_{n1} \\ \vdots & \ddots & \vdots \\ \mathbf{M}_{1n} & \cdots & \mathbf{M}_{nn} \end{pmatrix}$$

the *trace* is the sum of the diagonal entries:

$$\text{Trace}(\mathbf{M}) = \sum_i \mathbf{M}_{ii}$$

Recall

1. Given matrices $\mathbf{P}$ and $\mathbf{Q}$, we have:
$$(\mathbf{P}\cdot\mathbf{Q})^\top = \mathbf{Q}^\top\cdot\mathbf{P}^\top$$

2. Given a square matrix $\mathbf{P}$, we have:
$$\text{Trace}(\mathbf{P}) = \text{Trace}(\mathbf{P}^\top)$$

3. Given an $n\times m$ matrices $\mathbf{P}$ and $\mathbf{Q}$, we have:
$$\text{Trace}(\mathbf{P}^\top\cdot\mathbf{Q}) = \text{Trace}(\mathbf{Q}^\top\cdot\mathbf{P})$$

4. Given an $n\times n$ matrices $\mathbf{P}$ and $\mathbf{Q}$, we have:
$$\text{Trace}(\mathbf{P}) = \text{Trace}(\mathbf{Q}^{-1}\cdot\mathbf{P}\cdot\mathbf{Q})$$

5. Given vectors $\mathbf{v}$ and $\mathbf{w}$, we have:
$$\|\mathbf{v}\pm\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 \pm 2\langle\mathbf{v},\mathbf{w}\rangle$$

Recall

Given a point-set $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \subset \mathbb{R}^m$, we denote by $\mathbf{P} = (\mathbf{p}_1 | \cdots | \mathbf{p}_n) \in \mathbb{R}^{m \times n}$ the matrix whose columns are the points $\{\mathbf{p}_i\}$.

Given a transformation $\mathbf{M} \in \mathbb{R}^{m \times m}$, the matrix defined by the transformed points is:

$$(\mathbf{M}(\mathbf{p}_1) | \cdots | \mathbf{M}(\mathbf{p}_n)) = \mathbf{M} \cdot \mathbf{P}$$

Recall

For matrices $\mathbf{P} \in \mathbb{R}^{n\times m}, \mathbf{Q} \in \mathbb{R}^{m\times l}$, the (i, j)-th entry of $\mathbf{P} \cdot \mathbf{Q}$ is the dot-product of the i-th row of $\mathbf{P}$ and j-th column of $\mathbf{Q}$:

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \\ \mathbf{P}_{31} & \mathbf{P}_{32} \\ \mathbf{P}_{41} & \mathbf{P}_{42} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \end{pmatrix}$$

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Recall

For matrices $\mathbf{P} \in \mathbb{R}^{n\times m}, \mathbf{Q} \in \mathbb{R}^{m\times l}$, the (i,j)-th entry of $\mathbf{P}\cdot\mathbf{Q}$ is the dot-product of the i-th row of $\mathbf{P}$ and j-th column of $\mathbf{Q}$.

⇒ Given $\mathbf{P} = (\mathbf{p}_1|\cdots|\mathbf{p}_n), \mathbf{Q} = (\mathbf{q}_1|\cdots|\mathbf{q}_n) \in \mathbb{R}^{m\times n}$:

$$\mathbf{P}^\top \cdot \mathbf{Q} = \begin{pmatrix} \langle \mathbf{p}_1, \mathbf{q}_1\rangle & \cdots & \langle \mathbf{p}_1, \mathbf{q}_n\rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{p}_n, \mathbf{q}_1\rangle & \cdots & \langle \mathbf{p}_n, \mathbf{q}_n\rangle \end{pmatrix}$$

⇒ In particular, we have:

$$\mathrm{Trace}(\mathbf{P}^\top \cdot \mathbf{Q}) = \sum_{i=1}^{n} \langle \mathbf{p}_i, \mathbf{q}_i \rangle$$

Recall

We denote by $O(m)$ the group of *orthogonal* $m \times m$ matrices (i.e. rotations and reflections):

$$(\mathbf{O}^\top \cdot \mathbf{O}) = \mathbf{Id}. \Leftrightarrow \quad \mathbf{O}^\top = \mathbf{O}^{-1} \quad \forall \mathbf{O} \in O(m)$$

$\Rightarrow$ The determinant of any orthogonal matrix is ± 1:

$$\det(\mathbf{O}) = \pm 1 \qquad \forall \mathbf{O} \in O(m)$$

We denote by $SO(m) \subset O(m)$ the of orthonormal $m \times m$ matrices (i.e. just rotations):

$$SO(m) = \{\mathbf{O} \in O(m) | \det(\mathbf{O}) = 1\}$$

Recall

If $\mathbf{O} \in O(m)$ is an orthogonal transformation:

$$(\mathbf{O}^\top \cdot \mathbf{O}) = \mathbf{Id}.$$

$\Leftrightarrow$ The columns vectors of $\mathbf{O}$ are unit-length:

$$\sum_{j=1}^{m} \mathbf{O}_{ij}^2 = 1 \quad \forall\, 1 \leq i \leq m$$

$\Rightarrow |\mathbf{O}_{ij}| \leq 1$

Recall

1. Given a function $F(\mathbf{p})$, the point $\mathbf{p}$ is an extremum of F if the gradient of F vanishes at $\mathbf{p}$.

2. If $F(\mathbf{p}) = \|\mathbf{p}\|^2$ then:

$$\begin{aligned} \nabla F &= \nabla\left(p_x^2 + p_y^2 + p_z^2\right) \\ &= \left(2p_x, 2p_y, 2p_z\right) \\ &= 2\mathbf{p} \end{aligned}$$

3. If $F(\mathbf{p}) = \langle \mathbf{p}, \mathbf{q} \rangle$ then:

$$\begin{aligned} \nabla_{\mathbf{p}} F &= \nabla_{\mathbf{p}}\left(p_x q_x + p_y q_y + p_z q_z\right) \\ &= \left(q_x, q_y, q_z\right) \\ &= \mathbf{q} \end{aligned}$$

Recall

1. Given two real values $a, b \in \mathbb{R}$, we have:
$$|a \cdot b| = |a| \cdot |b|$$

2. Given two real values $a, b \in \mathbb{R}$, we have:
$$|a + b| \leq |a| + |b|$$

Claim

Given a diagonal matrix $\mathbf{D} \in \mathbb{R}^m$, the orthogonal transformation $\mathbf{O} \in O(m)$ maximizing the trace:

$$\text{Trace}(\mathbf{O} \cdot \mathbf{D})$$

is the diagonal matrix:

$$\mathbf{O} = \text{sign}(\mathbf{D}) = \begin{pmatrix} \text{sign}(\mathbf{D}_{11}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \text{sign}(\mathbf{D}_{nn}) \end{pmatrix}$$

This gives:

$$\begin{aligned} \text{Trace}(\mathbf{O} \cdot \mathbf{D}) &= \text{sign}(\mathbf{D}_{11})\mathbf{D}_{11} + \cdots + \text{sign}(\mathbf{D}_{nn})\mathbf{D}_{nn} \\ &= |\mathbf{D}_{11}| + \cdots + |\mathbf{D}_{nn}| \end{aligned}$$

Claim

Given a diagonal matrix $\mathbf{D} \in \mathbb{R}^m$, the orthogonal transformation $\mathbf{O} \in O(m)$ maximizing the trace:

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Will show that for any orthogonal $\mathbf{O}$:

$$\begin{aligned} \text{Trace}(\mathbf{O} \cdot \mathbf{D}) &\leq \text{Trace}(\text{sign}(\mathbf{D}) \cdot \mathbf{D}) \\ &= |\mathbf{D}_{11}| + \cdots + |\mathbf{D}_{nn}| \end{aligned}$$

Proof

$$\text{Trace}(\mathbf{O}\cdot\mathbf{D}) \leq |\mathbf{D}_{11}| + \cdots + |\mathbf{D}_{nn}|$$

Setting:

$$\mathbf{O} = \begin{pmatrix} \mathbf{O}_{11} & \cdots & \mathbf{O}_{n1} \\ \vdots & \ddots & \vdots \\ \mathbf{O}_{1n} & \cdots & \mathbf{O}_{nn} \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} \mathbf{D}_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{D}_{nn} \end{pmatrix}$$

we get:

$$\mathbf{O}\cdot\mathbf{D} = \begin{pmatrix} \mathbf{O}_{11}\cdot\mathbf{D}_{11} & \cdots & \mathbf{O}_{n1}\cdot\mathbf{D}_{nn} \\ \vdots & \ddots & \vdots \\ \mathbf{O}_{1n}\cdot\mathbf{D}_{11} & \cdots & \mathbf{O}_{nn}\cdot\mathbf{D}_{nn} \end{pmatrix}$$

Proof

$$\text{Trace}(\mathbf{O}\cdot\mathbf{D}) \leq |\mathbf{D}_{11}| + \cdots + |\mathbf{D}_{nn}|$$

Since:

$$\mathbf{O}\cdot\mathbf{D} = \begin{pmatrix} \mathbf{O}_{11}\cdot\mathbf{D}_{11} & \cdots & \mathbf{O}_{n1}\cdot\mathbf{D}_{nn} \\ \vdots & \ddots & \vdots \\ \mathbf{O}_{1n}\cdot\mathbf{D}_{11} & \cdots & \mathbf{O}_{nn}\cdot\mathbf{D}_{nn} \end{pmatrix}$$

we get:

$$\begin{aligned} \text{Trace}(\mathbf{O}\cdot\mathbf{D}) &= \mathbf{O}_{11}\mathbf{D}_{11} + \cdots + \mathbf{O}_{nn}\mathbf{D}_{nn} \\ &\leq |\mathbf{O}_{11}\mathbf{D}_{11} + \cdots + \mathbf{O}_{nn}\mathbf{D}_{nn}| \\ &\leq |\mathbf{O}_{11}\mathbf{D}_{11}| + \cdots + |\mathbf{O}_{nn}\mathbf{D}_{nn}| \\ &= |\mathbf{O}_{11}||\mathbf{D}_{11}| + \cdots + |\mathbf{O}_{nn}||\mathbf{D}_{nn}| \end{aligned}$$

Proof

$$\text{Trace}(\mathbf{O}\cdot\mathbf{D}) \leq |\mathbf{D}_{11}| + \cdots + |\mathbf{D}_{nn}|$$

Since:

$$\mathbf{O}\cdot\mathbf{D} = \begin{pmatrix} \mathbf{O}_{11}\cdot\mathbf{D}_{11} & \cdots & \mathbf{O}_{n1}\cdot\mathbf{D}_{nn} \\ \vdots & \ddots & \vdots \\ \mathbf{O}_{1n}\cdot\mathbf{D}_{11} & \cdots & \mathbf{O}_{nn}\cdot\mathbf{D}_{nn} \end{pmatrix}$$

we get:

$$\text{Trace}(\mathbf{O}\cdot\mathbf{D}) \leq |\mathbf{O}_{11}||\mathbf{D}_{11}| + \cdots + |\mathbf{O}_{nn}||\mathbf{D}_{nn}|$$

Since $\mathbf{O}$ is orthogonal, we have $|\mathbf{O}_{ii}| \leq 1$:

$$\text{Trace}(\mathbf{O}\cdot\mathbf{D}) \leq |\mathbf{D}_{11}| + \cdots + |\mathbf{D}_{nn}|$$

(Orthogonal) Procrustes Method

<u>Goal</u>:

Given points $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \subset \mathbb{R}^m$ and $\{\mathbf{q}_1, \ldots, \mathbf{q}_n\} \subset \mathbb{R}^m$, find the **translation** $\boldsymbol{\delta} \in \mathbb{R}^m$ and **orthogonal transform** $\mathbf{O} \in O(m)$ that best aligns $\{\mathbf{p}_i\}$ to $\{\mathbf{q}_i\}$.

That is, find $\boldsymbol{\delta}$ and $\mathbf{O}$ minimizing the alignment energy:

$$E(\boldsymbol{\delta}, \mathbf{O}) = \sum_{i=1}^{n} \|\mathbf{O}(\mathbf{p}_i + \boldsymbol{\delta}) - \mathbf{q}_i\|^2$$

(Orthogonal) Procrustes Method

$$\|\mathbf{v} \pm \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 \pm 2\langle \mathbf{v}, \mathbf{w} \rangle$$

Goal:

1. Find the **translation** $\boldsymbol{\delta} \in \mathbb{R}^m$ minimizing:*

$$\begin{aligned} E(\boldsymbol{\delta}) &= \sum_{i=1}^{n} \|(\mathbf{p}_i + \boldsymbol{\delta}) - \mathbf{q}_i\|^2 \\ &= \sum_{i=1}^{n} \|(\mathbf{p}_i - \mathbf{q}_i) + \boldsymbol{\delta}\|^2 \\ &= \sum_{i=1}^{n} \left(\|\mathbf{p}_i - \mathbf{q}_i\|^2 + \|\boldsymbol{\delta}\|^2 + 2\langle \mathbf{p}_i - \mathbf{q}_i, \boldsymbol{\delta} \rangle\right) \end{aligned}$$

*We'll see why we can ignore orthogonal transformations shortly.

(Orthogonal) Procrustes Method

$$\nabla_{\mathbf{p}}\|\mathbf{p}\|^2 = 2\mathbf{p} \quad \text{and} \quad \nabla_{\mathbf{p}}\langle\mathbf{p},\mathbf{q}\rangle = \mathbf{q}$$

Goal:

$$E(\boldsymbol{\delta}) = \sum_{i=1}^{n}\left(\|\mathbf{p}_i - \mathbf{q}_i\|^2 + \|\boldsymbol{\delta}\|^2 + 2\langle\mathbf{p}_i - \mathbf{q}_i, \boldsymbol{\delta}\rangle\right)$$

1. Find the **translation** $\boldsymbol{\delta} \in \mathbb{R}^m$ minimizing $E(\boldsymbol{\delta})$.*

   Taking the gradient gives:

$$\nabla E(\boldsymbol{\delta}) = \sum_{i=1}^{n} 2\boldsymbol{\delta} + 2(\mathbf{p}_i - \mathbf{q}_i)$$

*We'll see why we can ignore orthogonal transformations shortly.

(Orthogonal) Procrustes Method

Goal:

$$\nabla E(\boldsymbol{\delta}) = \sum_{i=1}^{n} 2\boldsymbol{\delta} + 2(\mathbf{p}_i - \mathbf{q}_i)$$

1. Find the **translation** $\boldsymbol{\delta} \in \mathbb{R}^m$ minimizing $E(\boldsymbol{\delta})$.*

   The minimizing translation must satisfy:

$$\nabla E(\boldsymbol{\delta}) = 0$$
$$\Downarrow$$
$$\boldsymbol{\delta} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{q}_i - \mathbf{p}_i)$$

*We'll see why we can ignore orthogonal transformations shortly.

(Orthogonal) Procrustes Method

Goal:

$$\nabla E(\boldsymbol{\delta}) = \sum_{i=1}^{n} 2\boldsymbol{\delta} + 2(\mathbf{p}_i - \mathbf{q}_i)$$

1. Find the **translation** $\boldsymbol{\delta} \in \mathbb{R}^m$ minimizing $E(\boldsymbol{\delta})$.*

   The minimizing translation must satisfy:

$$\nabla E(\boldsymbol{\delta}) = 0$$
$$\Downarrow$$
$$\boldsymbol{\delta} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{q}_i - \mathbf{p}_i) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{q}_i - \frac{1}{n}\sum_{i=1}^{n}\mathbf{p}_i$$

*We'll see why we can ignore orthogonal transformations shortly.

The minimizing translation takes the center of mass of $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ to the center of mass of $\{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$.

Goa

⇓

The point-sets are translationally aligned when their centers of mass coincide.

⇓

1. Fin If the centers are both at the origin, the point-sets are translationally aligned $(\boldsymbol{\delta})$.*

Th

Note:
If a point-set is translated so its center of mass is at the origin, any linear transformation (e.g. rotation) of the point-set will still have its center of mass at the origin.

⇓

If the point-sets are centered at the origin, they are optimally translationally aligned regardless of the rotation.

(Orthogonal) Procrustes Method

$\|\mathbf{v} \pm \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 \pm 2\langle \mathbf{v}, \mathbf{w} \rangle$

Goal:

2. Find the **transform** $\mathbf{O} \in O(m)$ minimizing:

$$E(\mathbf{O}) = \sum_{i=1}^{n} \|\mathbf{O}(\mathbf{p}_i) - \mathbf{q}_i\|^2$$

$$= \sum_{i=1}^{n} \|\mathbf{O}(\mathbf{p}_i)\|^2 + \|\mathbf{q}_i\|^2 - 2\langle \mathbf{O}(\mathbf{p}_i), \mathbf{q}_i \rangle$$

$$= \sum_{i=1}^{n} \|\mathbf{p}_i\|^2 + \|\mathbf{q}_i\|^2 - 2\langle \mathbf{O}(\mathbf{p}_i), \mathbf{q}_i \rangle$$

Minimizing $E(\mathbf{O})$ is the same as maximizing:

$$\tilde{E}(\mathbf{O}) = \sum_{i=1}^{n} \langle \mathbf{O}(\mathbf{p}_i), \mathbf{q}_i \rangle$$

(Orthogonal) Procrustes Method

Goal:

2. Find the **transform** $\mathbf{O} \in O(m)$ maximizing:

$$\tilde{E}(\mathbf{O}) = \sum_{i=1}^{n} \langle \mathbf{O}(\mathbf{p}_i), \mathbf{q}_i \rangle$$

- Set $\mathbf{P} = (\mathbf{p}_1 | \cdots | \mathbf{p}_n)$ and $\mathbf{Q} = (\mathbf{q}_1 | \cdots | \mathbf{q}_n)$.
- Use the facts that:

$$\mathbf{O} \cdot \mathbf{P} = \left(\mathbf{O}(\mathbf{p}_1) | \cdots | \mathbf{O}(\mathbf{p}_n)\right)$$

$$\text{Trace}(\mathbf{P}^\top \cdot \mathbf{Q}) = \sum_{i=1}^{n} \langle \mathbf{p}_i, \mathbf{q}_i \rangle$$

$$\Downarrow$$

$$\tilde{E}(\mathbf{O}) = \text{Trace}\left((\mathbf{O} \cdot \mathbf{P})^\top \cdot \mathbf{Q}\right)$$

(Orthogonal) Procrustes Method

Goal:

2. Find the **transform** $\mathbf{O} \in O(m)$ maximizing:

$$\begin{aligned}\tilde{E}(\mathbf{O}) &= \mathrm{Trace}\big((\mathbf{O}\cdot\mathbf{P})^\top \cdot \mathbf{Q}\big)\\ &= \mathrm{Trace}\big((\mathbf{P}^\top\cdot\mathbf{O}^\top)\cdot\mathbf{Q}\big)\\ &= \mathrm{Trace}\big(\mathbf{Q}\cdot(\mathbf{P}^\top\cdot\mathbf{O}^\top)\big)\\ &= \mathrm{Trace}\big((\mathbf{Q}\cdot\mathbf{P}^\top)\cdot\mathbf{O}^\top\big)\\ &= \mathrm{Trace}\big(\mathbf{O}\cdot(\mathbf{Q}\cdot\mathbf{P}^\top)^\top\big)\\ &= \mathrm{Trace}\big(\mathbf{O}\cdot(\mathbf{P}\cdot\mathbf{Q}^\top)\big)\end{aligned}$$

$$(\mathbf{A}\cdot\mathbf{B})^\top = \mathbf{B}^\top\cdot\mathbf{A}^\top$$
$$\mathbf{O}^{-1} = \mathbf{O}^\top \quad \forall \mathbf{O}\in O(m)$$
$$\mathrm{Trace}(\mathbf{B}) = \mathrm{Trace}(\mathbf{A}^{-1}\cdot\mathbf{B}\cdot\mathbf{A})$$
$$\mathrm{Trace}(\mathbf{A}^\top) = \mathrm{Trace}(\mathbf{A})$$

Massage so we are trying to maximize $\mathrm{Trace}(\mathbf{O}\cdot\mathbf{D})$, for some diagonal $\mathbf{D}$.

(Orthogonal) Procrustes Method

Goal:

$$\text{Trace}(\mathbf{B}) = \text{Trace}(\mathbf{A}^{-1} \cdot \mathbf{B} \cdot \mathbf{A})$$

2. Find the **transform** $\mathbf{O} \in O(m)$ maximizing:

$$\tilde{E}(\mathbf{O}) = \text{Trace}\left(\mathbf{O} \cdot (\mathbf{P} \cdot \mathbf{Q}^\top)\right)$$

Compute the singular value decomposition:

$$\mathbf{P} \cdot \mathbf{Q}^\top = \mathbf{U} \cdot \mathbf{D} \cdot \mathbf{V}^\top$$

with $\mathbf{U}$ and $\mathbf{V}$ orthogonal and $\mathbf{D}$ diagonal.

$$\Downarrow$$

$$\begin{aligned}\tilde{E}(\mathbf{O}) &= \text{Trace}(\mathbf{O} \cdot \mathbf{U} \cdot \mathbf{D} \cdot \mathbf{V}^\top) \\ &= \text{Trace}(\mathbf{V}^\top \cdot \mathbf{O} \cdot \mathbf{U} \cdot \mathbf{D}) \\ &= \text{Trace}\left((\mathbf{V}^\top \cdot \mathbf{O} \cdot \mathbf{U}) \cdot \mathbf{D}\right)\end{aligned}$$

Massage so we are trying to maximize $\text{Trace}(\mathbf{O} \cdot \mathbf{D})$, for some diagonal $\mathbf{D}$.

(Orthogonal) Procrustes Method

Goal:

2. Find the **transform** $\mathbf{O} \in O(m)$ maximizing:

$$\tilde{E}(\mathbf{O}) = \text{Trace}\left((\mathbf{V}^\top \cdot \mathbf{O} \cdot \mathbf{U}) \cdot \mathbf{D}\right)$$

Since $\mathbf{V}^\top \cdot \mathbf{O} \cdot \mathbf{U}$ is orthogonal, this is maximized if:

$$\mathbf{V}^\top \cdot \mathbf{O} \cdot \mathbf{U} = \text{sign}(\mathbf{D})$$

$$\Downarrow$$

$$\mathbf{O} = \mathbf{V} \cdot \text{sign}(\mathbf{D}) \cdot \mathbf{U}^\top$$

Since the diagonal entries of $\mathbf{D}$ are non-negative:

$$\mathbf{O} = \mathbf{V} \cdot \mathbf{U}^\top$$

(Orthonormal) Procrustes Method

$$\mathbf{O} = \mathbf{V} \cdot \text{sign}(\mathbf{D}) \cdot \mathbf{U}^\top$$

In practice, we often want the best *orthonormal* transformation, $\mathbf{O} \in SO(m)$, not just orthogonal transformation. This requires:

$$\begin{aligned} 1 &= \det(\mathbf{O}) \\ &= \det(\mathbf{V}) \cdot \text{sign}(\mathbf{D}) \cdot \det(\mathbf{U}^\top) \\ &= \det(\mathbf{V}) \cdot \det(\mathbf{U}^\top) \\ &= \det(\mathbf{V} \cdot \mathbf{U}^\top) \end{aligned}$$

As with SVD normalization, we achieve this by setting one of the entries of $\text{sign}(\mathbf{D})$ to $\det(\mathbf{V} \cdot \mathbf{U}^\top)$.

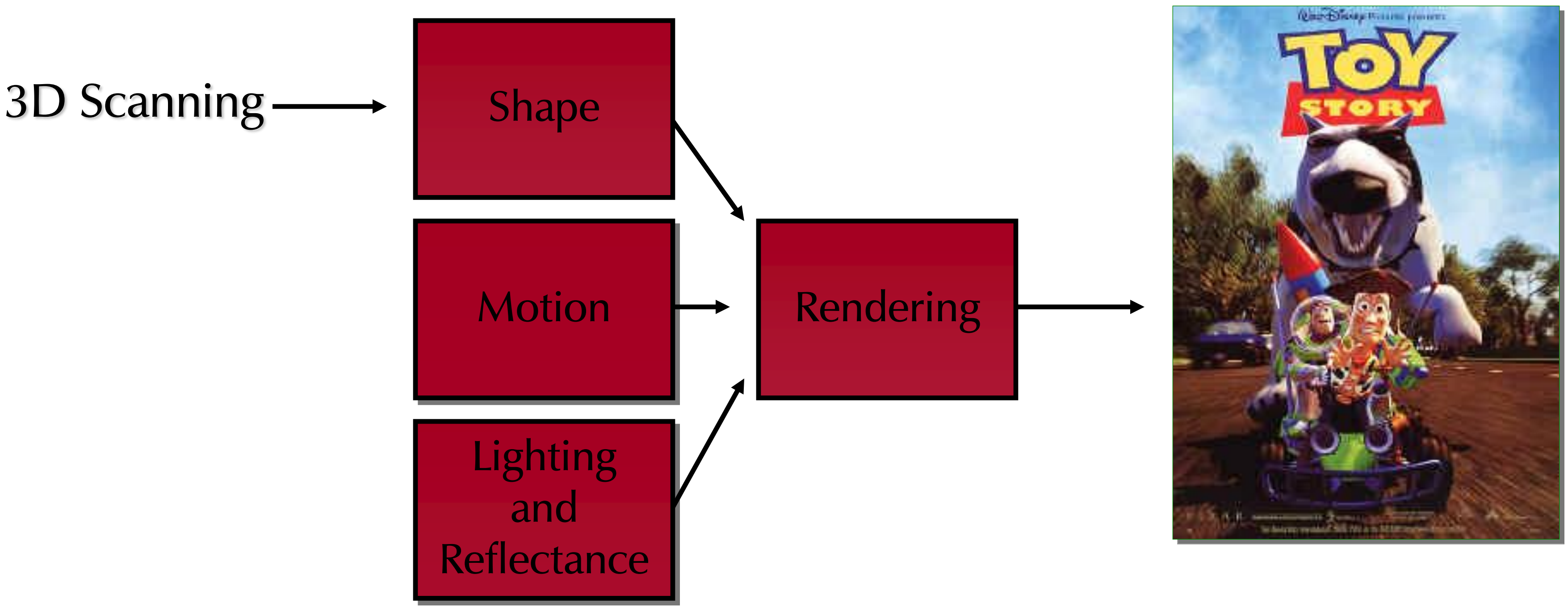
Since the diagonals are non-negative and decreasing, we flip the last one.

3D Scanning

Lecture courtesy of

Szymon Rusinkiewicz
Princeton University

Computer Graphics Pipeline



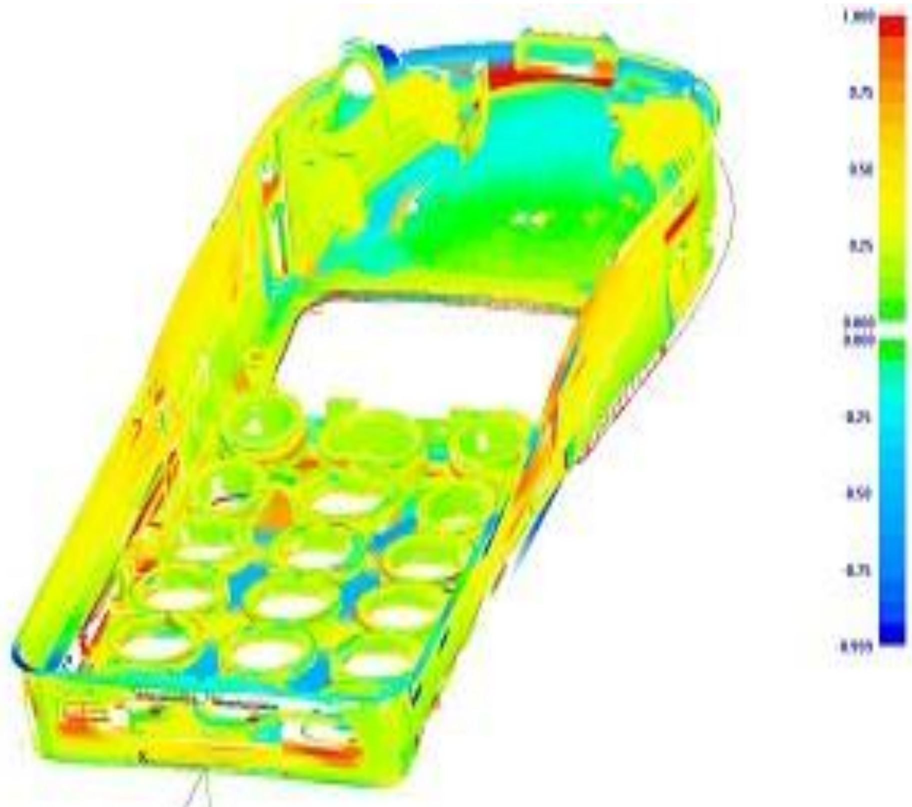
- Human time = expensive
- Sensors = cheap
  - Computer graphics increasingly relies on measurements of the real world

3D Scanning Applications

- Computer graphics
- Product inspection
- Robot navigation
- Product design
- Archaeology
- Clothes fitting

Industrial Inspection

Are manufactured parts within a tolerance?



Clothing

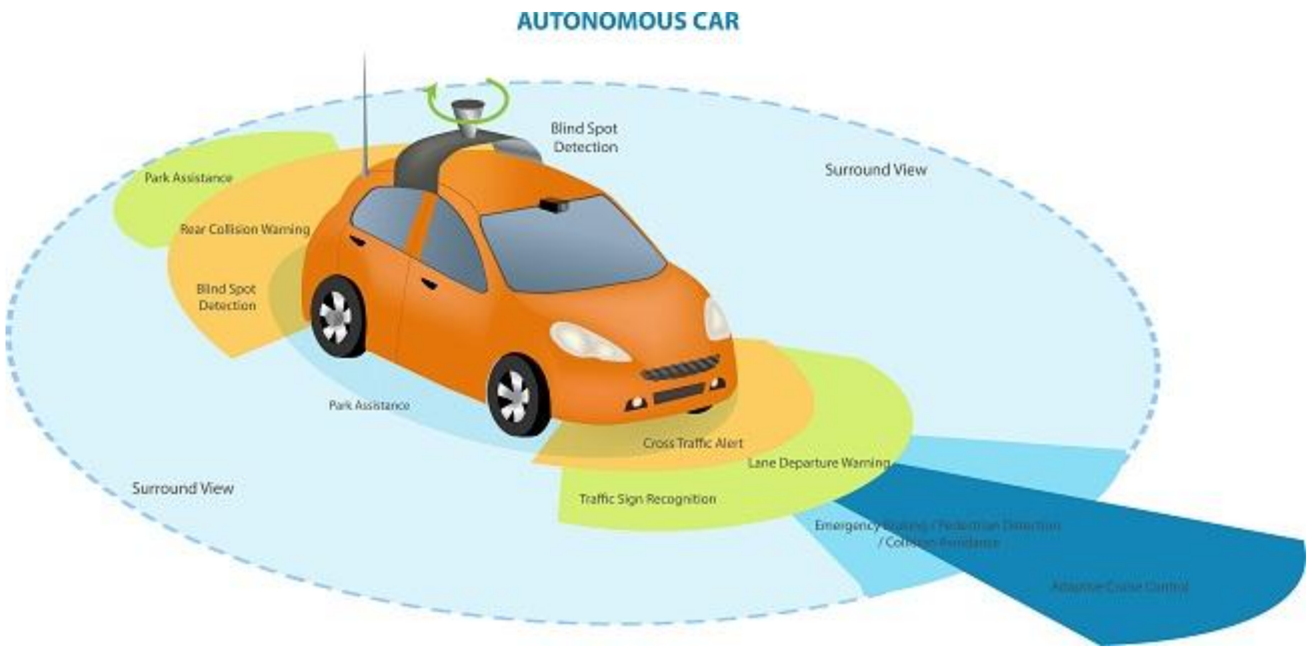
Scan a person, custom-fit clothing

- U.S. Army; booths in malls

Driving

Autonomous navigation

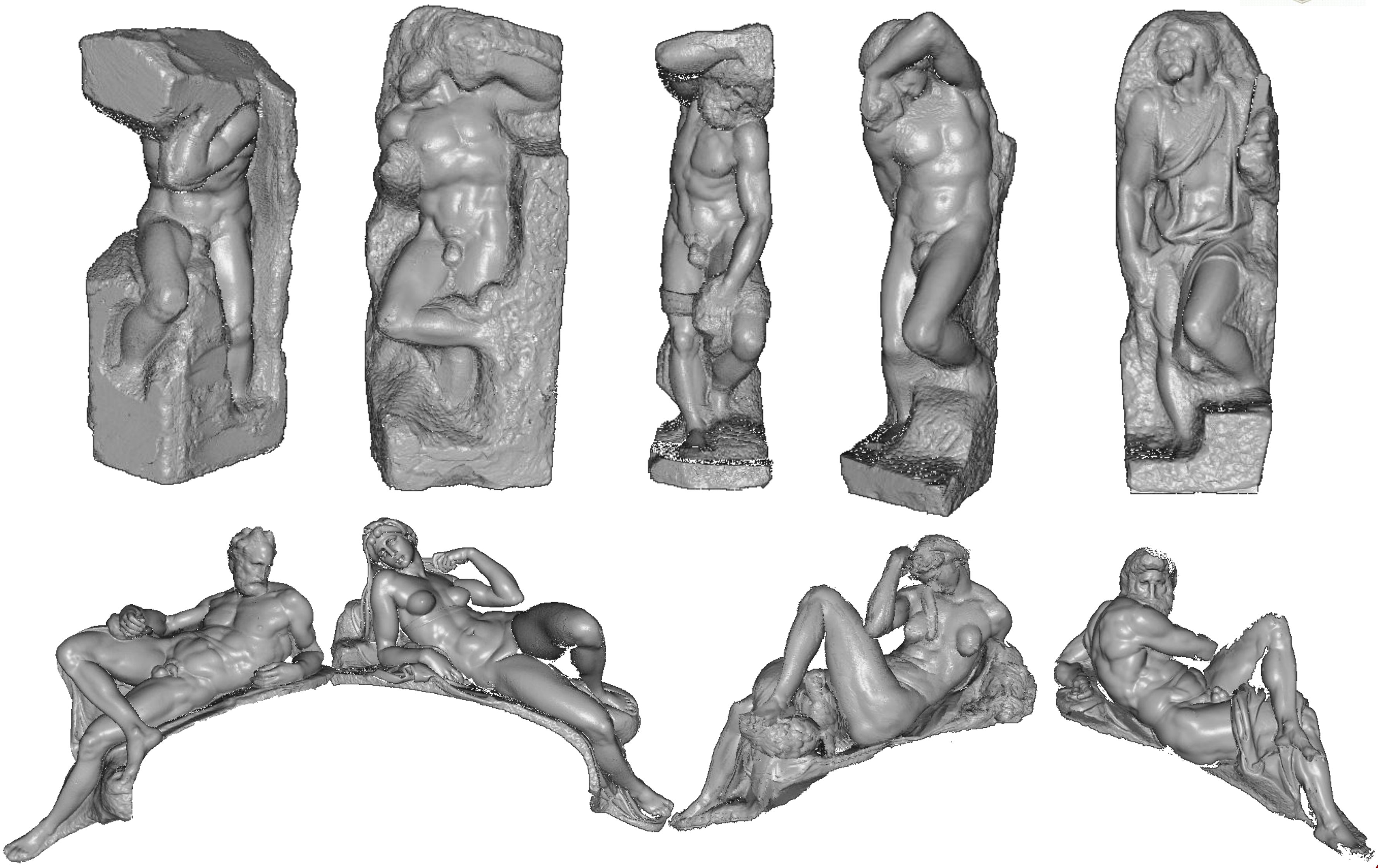
Collision avoidance



https://www.geotab.com/blog/crash-avoidance/

The Digital Michelangelo Project

The Digital Michelangelo Project



Why Scan Sculptures?

Virtual museums

Controlled interaction (lighting, proximity, etc.)

Study working techniques

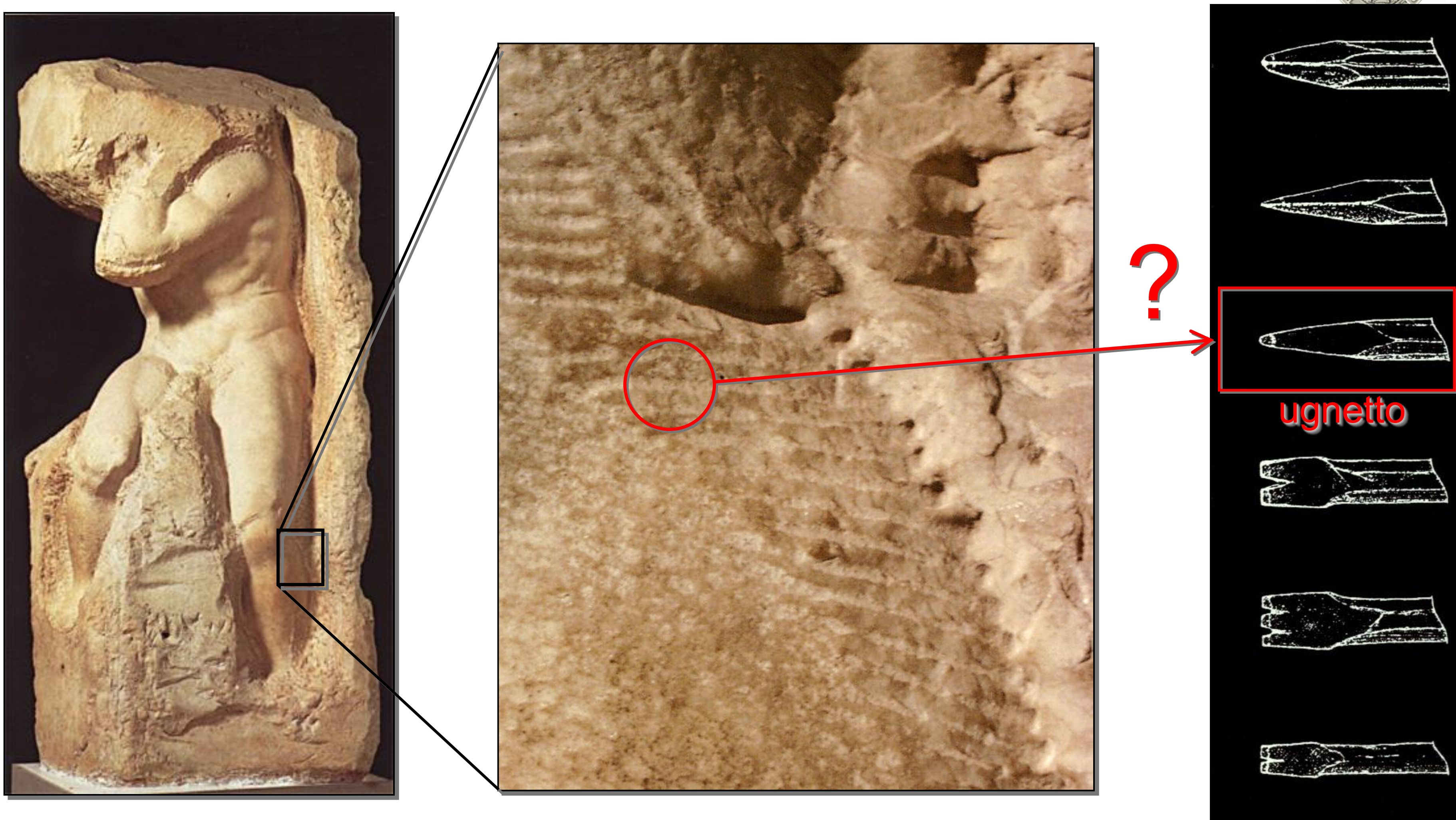
Cultural heritage preservation

Goals

Scan 10 sculptures by Michelangelo

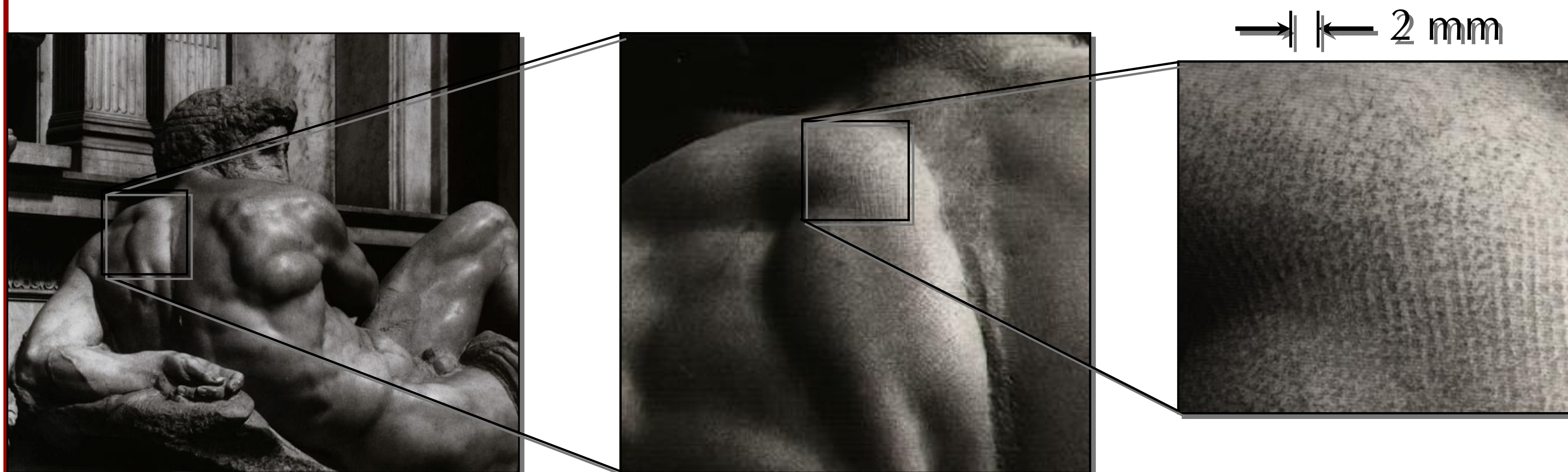
High-resolution (i.e. quarter-millimeter) resolution

Why Capture Chisel Marks?



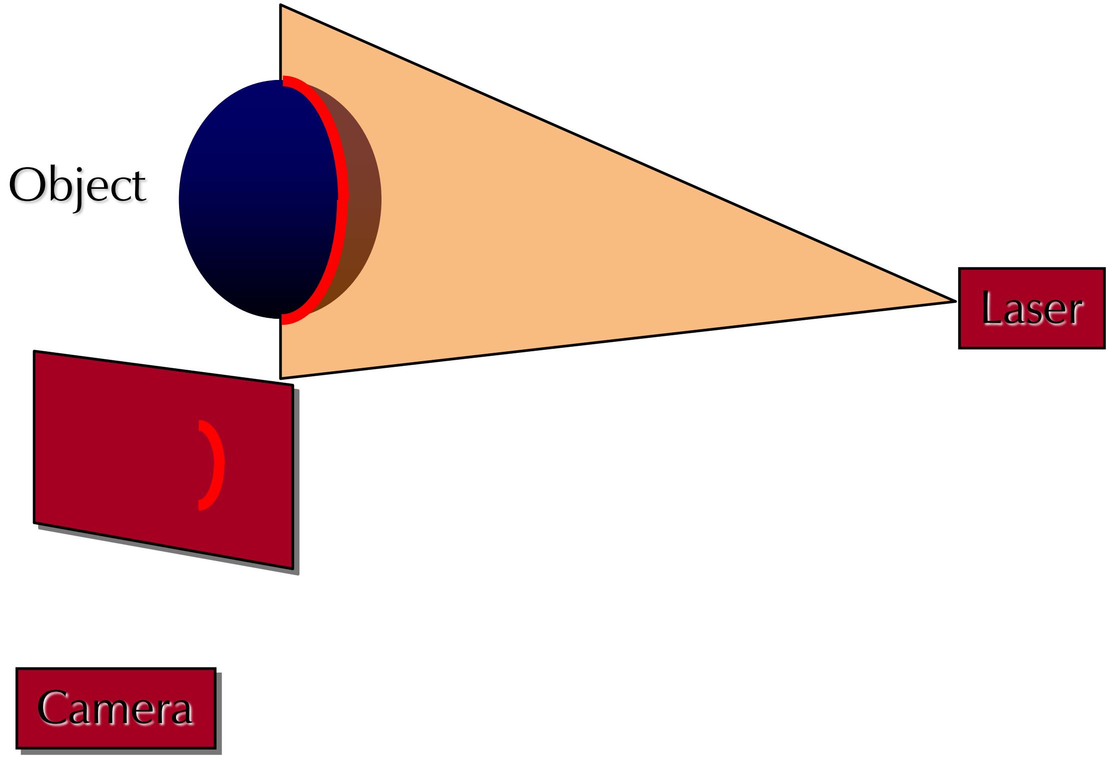
Atlas (Accademia)

Why Capture Chisel Mark **Geometry**?



Day (Medici Chapel)

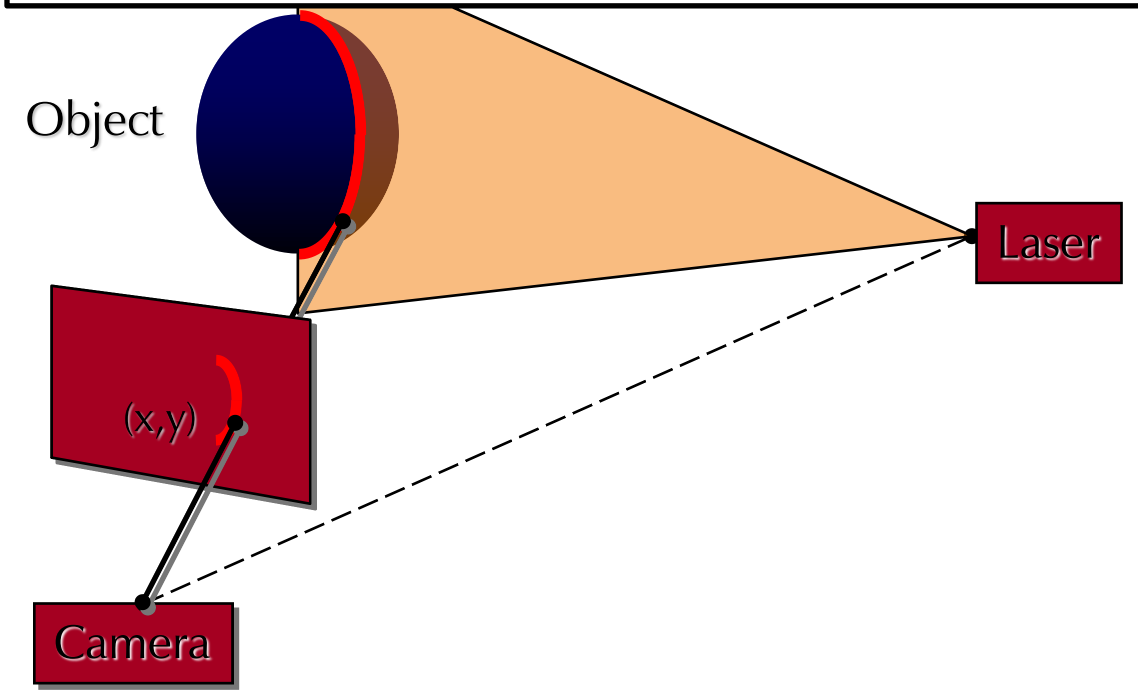
Triangulation



- Project laser stripe onto object
- Detect laser stripe in image

Triangulation

Gives the depth of the point (x, y) with respect to the camera.

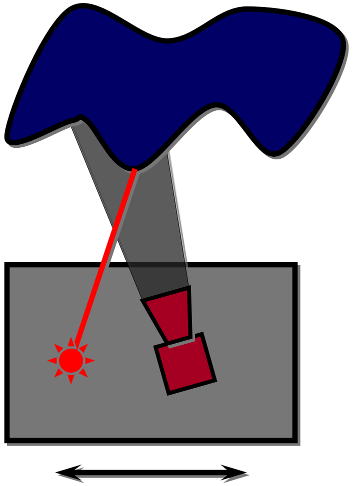


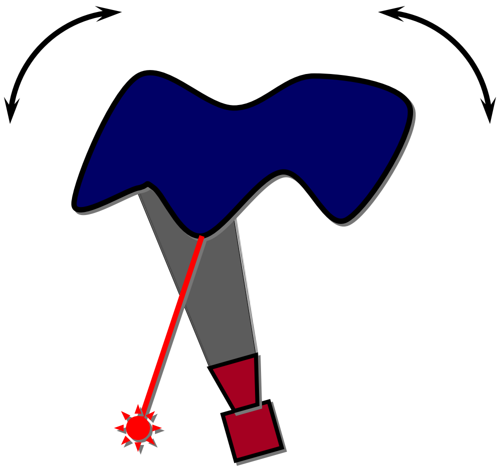
- Project laser stripe onto object
- Detect laser stripe in image
- Get depth from ray-plane triangulation

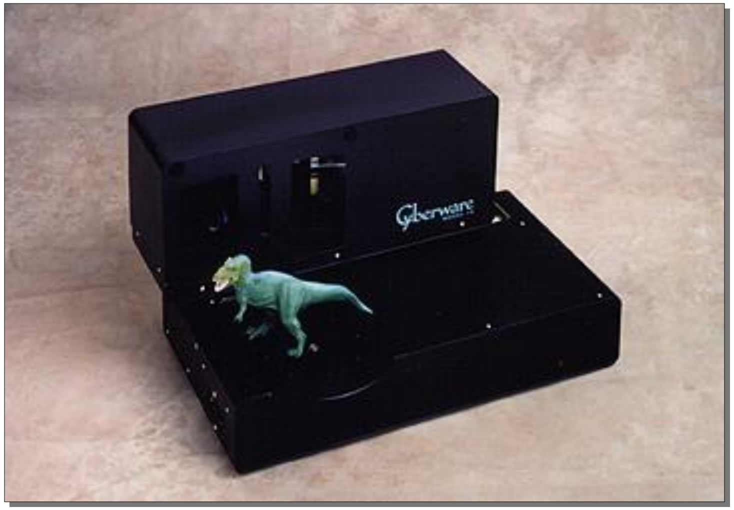
Triangulation: Moving the Camera and Illumination

- ✗ Moving independently leads to problems with calibration
- ✓ Most scanners mount camera and light source rigidly, move them as a unit

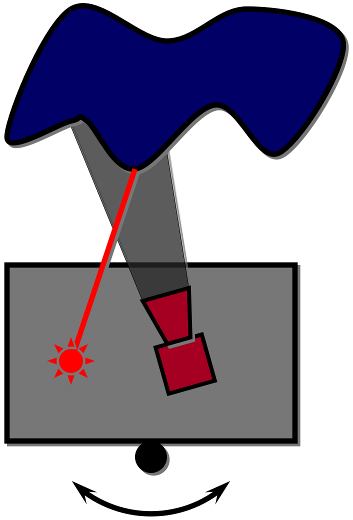
Triangulation: Moving the Camera and Illumination

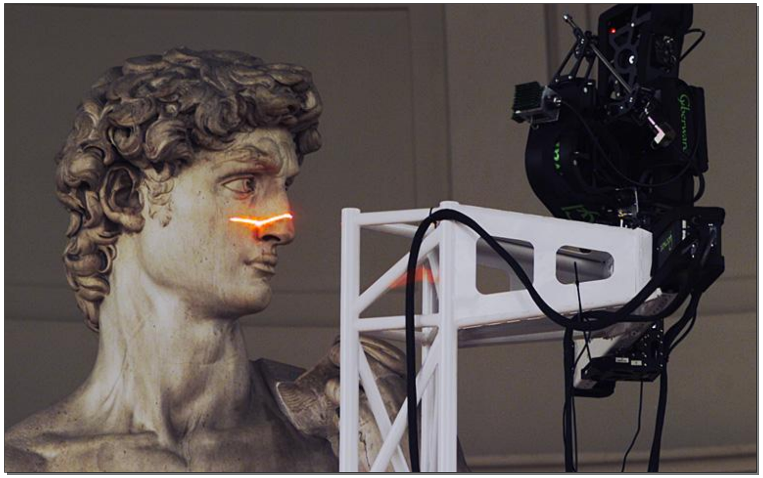




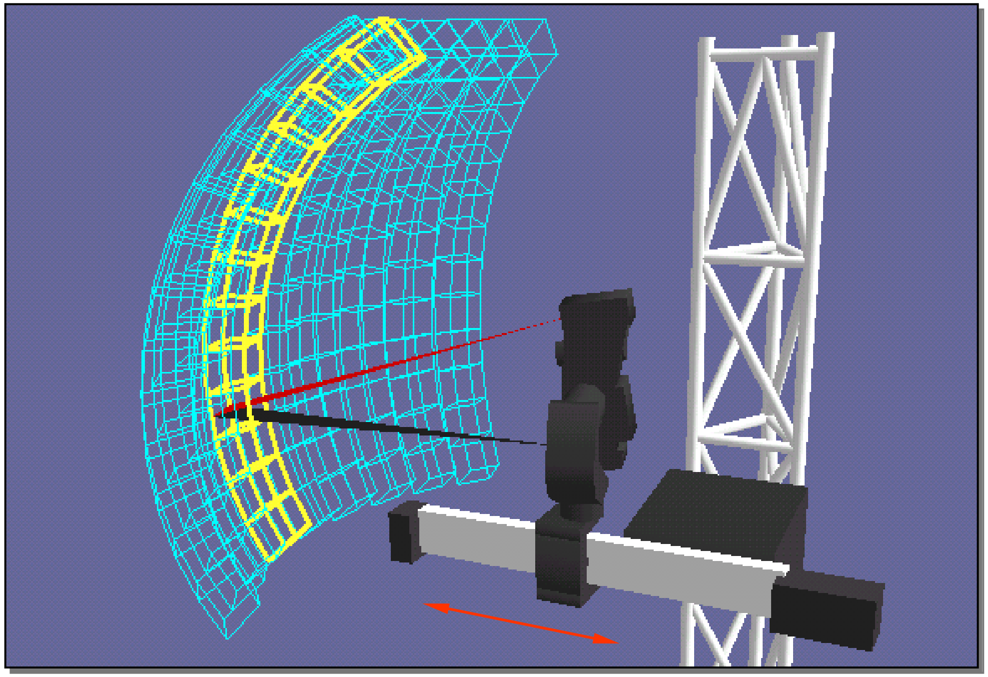


Triangulation: Moving the Camera and Illumination





Scanning a Large Object



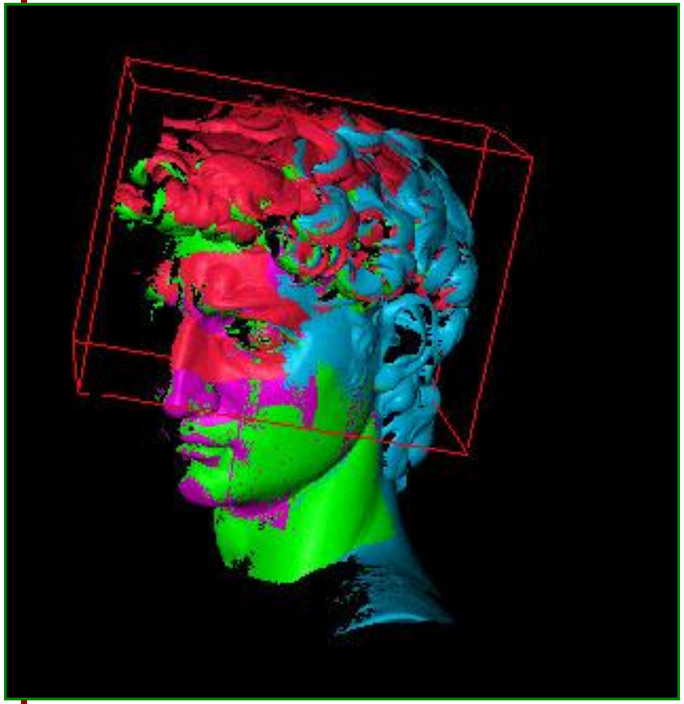
Calibrated motions

- pitch (yellow)
- pan (blue)
- horizontal translation (orange)

Uncalibrated motions

- vertical translation
- rolling the gantry
- remounting the scan head

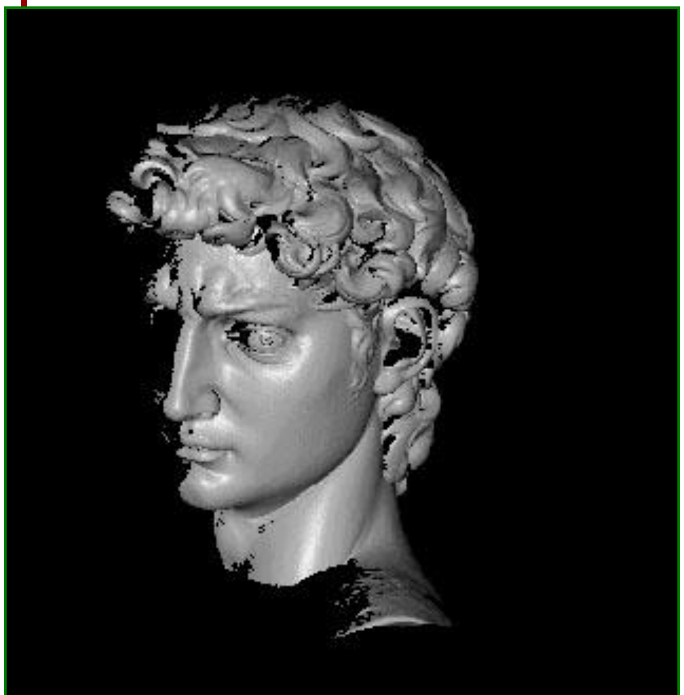
Range Processing Pipeline



Steps

1. Manual initial alignment
2. Automatic ICP to an existing scan
3. Global relaxation to diffuse error
4. Merging using volumetric method

Range Processing Pipeline



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Iterative Closest Point (ICP)

Goal:

Given two point-sets $\mathbf{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_n\} \subset \mathbb{R}^d$, and $\mathbf{Q} = \{\mathbf{q}_1, \dots, \mathbf{q}_m\} \subset \mathbb{R}^d$, find:

- The correspondence, $\Phi: \{1, \dots, n\} \to \{1, \dots, m\}$,
- The translation $\boldsymbol{\delta} \in \mathbb{R}^d$, and
- The rotation $\mathbf{O} \in SO(d)$

that minimize the sum of squared distances:

$$E(\Phi, \boldsymbol{\delta}, \mathbf{O}) = \sum_{i=1}^{n} \left\| \mathbf{O}(\mathbf{p}_i + \boldsymbol{\delta}) - \mathbf{q}_{\Phi(i)} \right\|^2$$

Iterative Closest Point (ICP)

$$E(\Phi, \boldsymbol{\delta}, \mathbf{O}) = \sum_{i=1}^{n} \left\| \mathbf{O}(\mathbf{p}_i + \boldsymbol{\delta}) - \mathbf{q}_{\Phi(i)} \right\|^2$$

Approach:

Create a sequence of correspondences, translations, and rotations:

$$\left\{ \{\Phi_0, \boldsymbol{\delta}_0, \mathbf{O}_0\}, \{\Phi_1, \boldsymbol{\delta}_1, \mathbf{O}_1\}, \cdots \right\}$$

that monotonically reduces the energy.

Implementation:

Alternately solve for the correspondence Φ_i and the transformation $\{\boldsymbol{\delta}_i, \mathbf{O}_i\}$.

Iterative Closest Point (ICP)

Algorithm:

0. Initialize $k = 0, \boldsymbol{\delta}_k = \vec{0}$, and $\mathbf{O}_k = \mathbf{Id}$.

1. Fix $\boldsymbol{\delta}_k$ and $\mathbf{O}_k$, and set Φ_{k+1} to minimize:

$$E(\Phi_{k+1}, \boldsymbol{\delta}_k, \mathbf{O}_k) = \sum_{i=1}^{n} \left\| \mathbf{O}_k(\mathbf{p}_i + \boldsymbol{\delta}_k) - \mathbf{q}_{\Phi_{k+1}(i)} \right\|^2$$

$\Rightarrow \Phi_{k+1}$ is the nearest-neighbor map:

$$\Phi_{k+1}(i) = \underset{j \in \{1,\ldots,m\}}{\arg\min} \left\| \mathbf{O}_k(\mathbf{p}_i + \boldsymbol{\delta}_k) - \mathbf{q}_j \right\|^2$$

Iterative Closest Point (ICP)

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2. Fix Φ_{k+1}, and set $\boldsymbol{\delta}_{k+1}$ and $\mathbf{O}_{k+1}$ to minimize:

$$E(\Phi_{k+1}, \boldsymbol{\delta}_{k+1}, \mathbf{O}_{k+1}) = \sum_{i=1}^{n} \left\| \mathbf{O}_{k+1}(\mathbf{p}_i + \boldsymbol{\delta}_{k+1}) - \mathbf{q}_{\Phi_{k+1}(i)} \right\|^2$$

3. Update $k = k + 1$. Goto step 1.

Iterative Closest Point (ICP)

Algorithm:

0. Initialize $k = 0, \boldsymbol{\delta}_k = \vec{0}$, and $\mathbf{O}_k = \mathbf{Id}$.

1. Fix $\boldsymbol{\delta}_k$ and $\mathbf{O}_k$, and set Φ_{k+1} to minimize:

$$E(\Phi_{k+1}, \boldsymbol{\delta}_k, \mathbf{O}_k) = \sum_{i=1}^{n} \left\| \mathbf{O}_k(\mathbf{p}_i + \boldsymbol{\delta}_k) - \mathbf{q}_{\Phi_{k+1}(i)} \right\|^2$$

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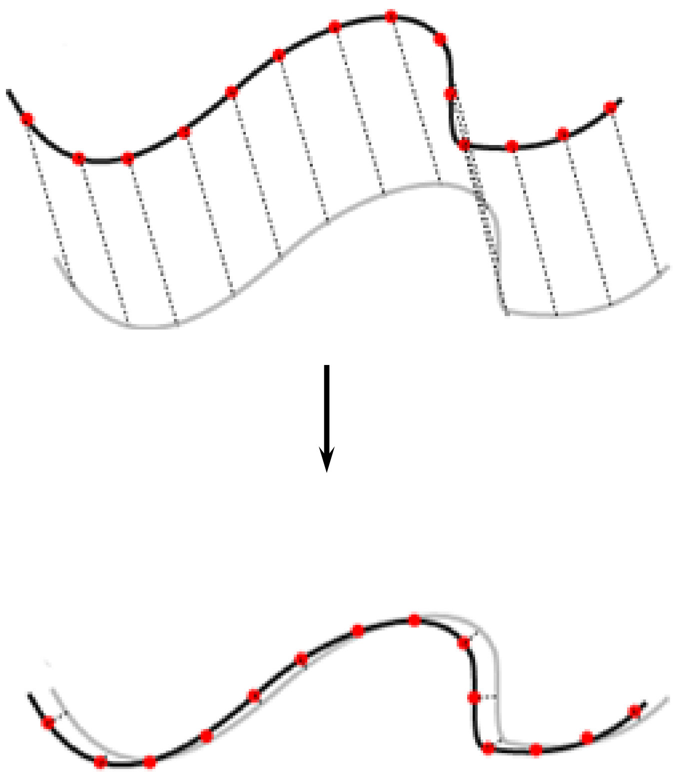
3.

Since the two steps reduce the same energy, the sum of squared distances reduces monotonically.

Iterative Closest Point (ICP)



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ICP(*Scan1* , *Scan2*) :

1. For each point on *Scan1*, find the nearest point on *Scan2*.
2. Translate and rootate *Scan1* to minimize the distance between corresponding points.
3. Go to step 1

Range Processing Pipeline

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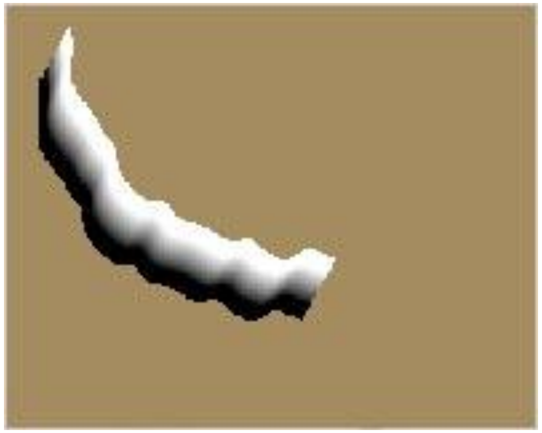
Range Processing Pipeline



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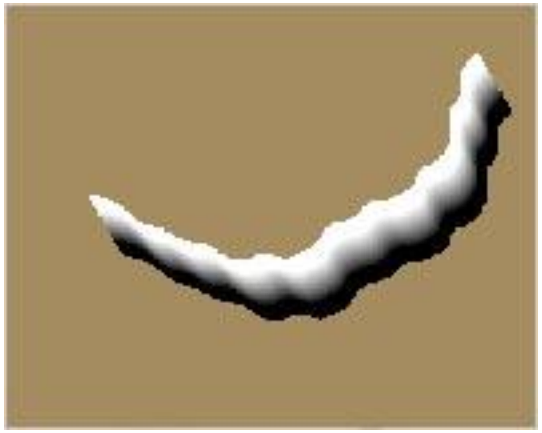

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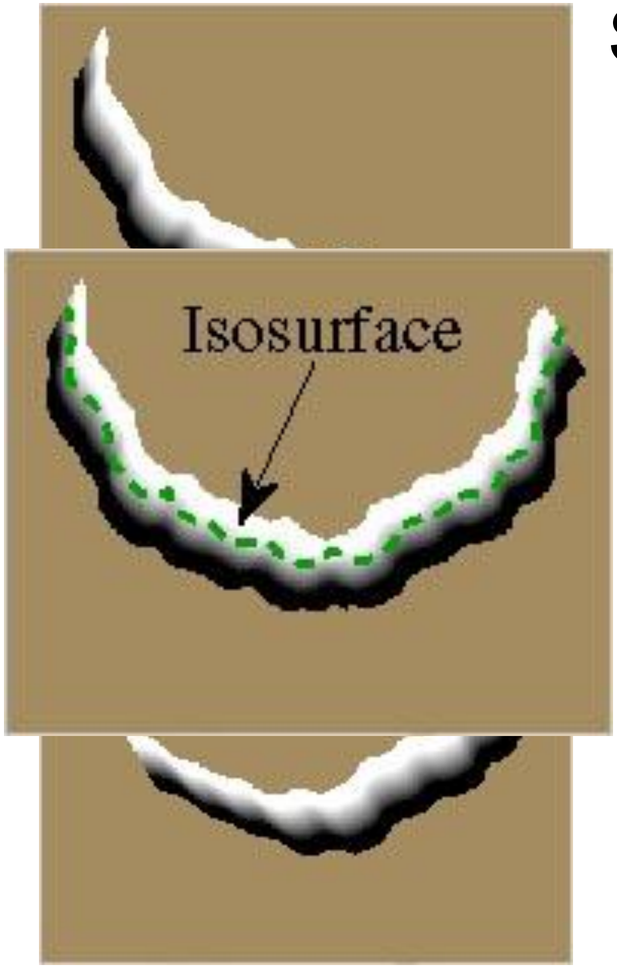

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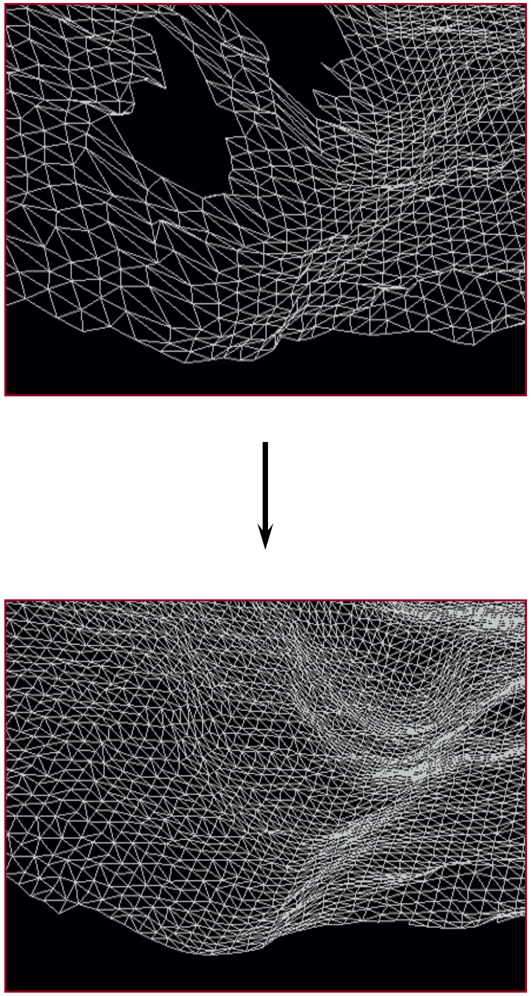

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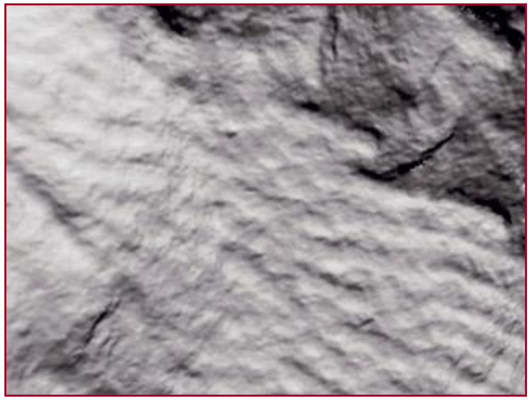

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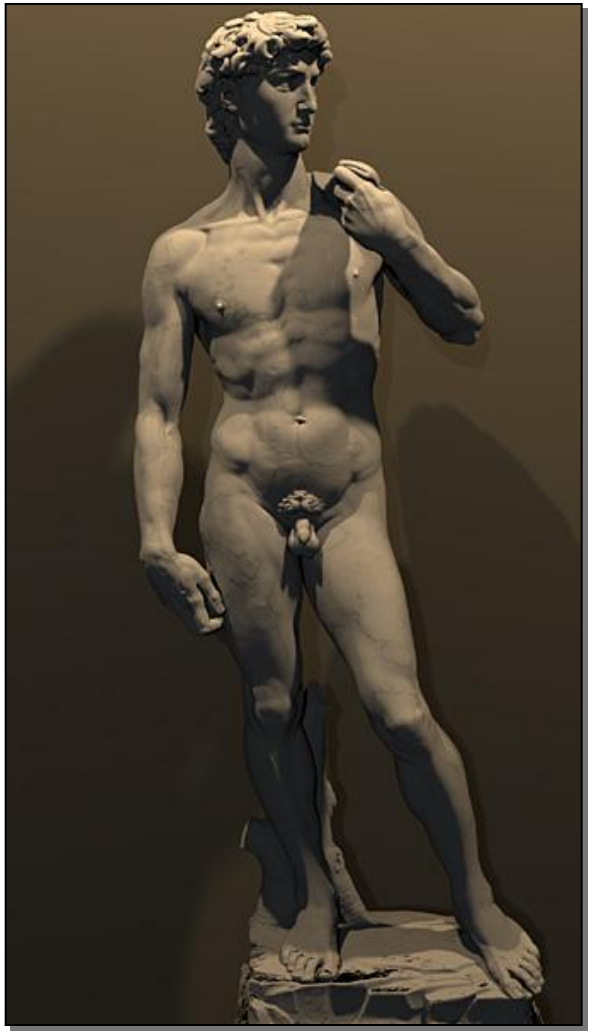
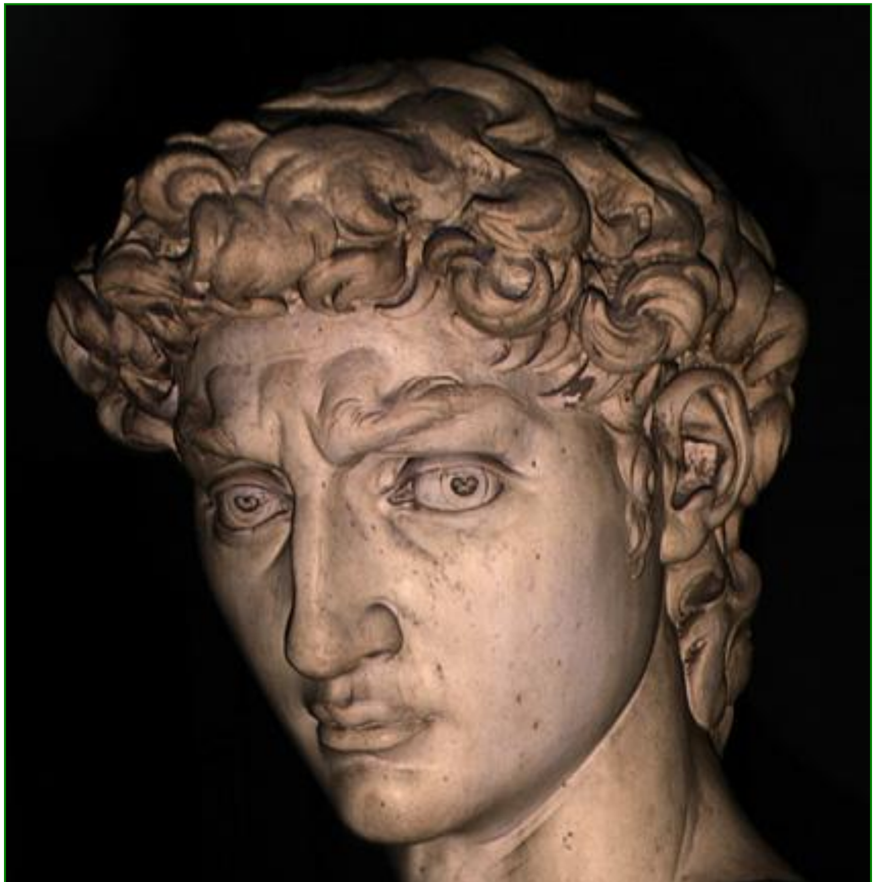
Statistics About the Scan of David



- 480 individually aimed scans
- 0.3 mm sample spacing
- 2 billion polygons
- 7,000 color images
- 32 gigabytes
- 30 nights of scanning
- 22 people

Head of Michelangelo's David



Photograph

1.0 mm computer model