Animating Transformations

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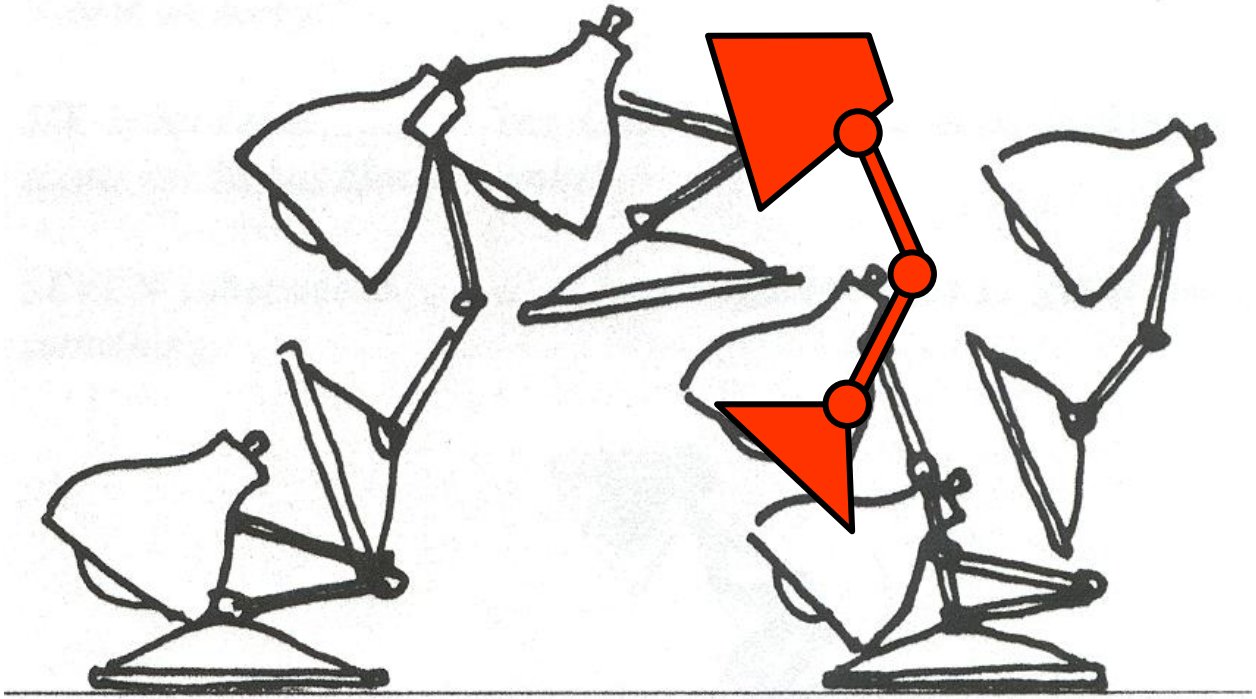
(601.457/657)

Recall

Keyframe Animation:

Interpolate variables describing keyframes to determine poses for character “in-between”



Lasseter `87

Keyframe Animation

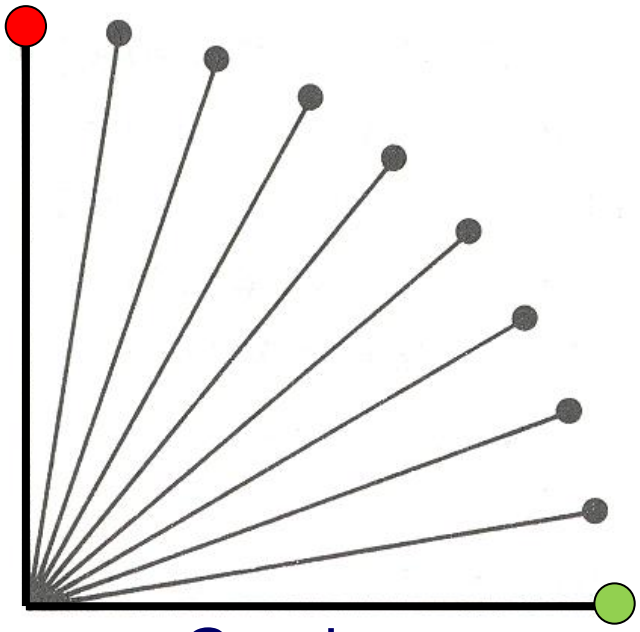
Q: Why interpolate/blend joint parameters rather than vertex positions?

A: For translations, it doesn't make a difference (assuming the blend is translation equivariant).

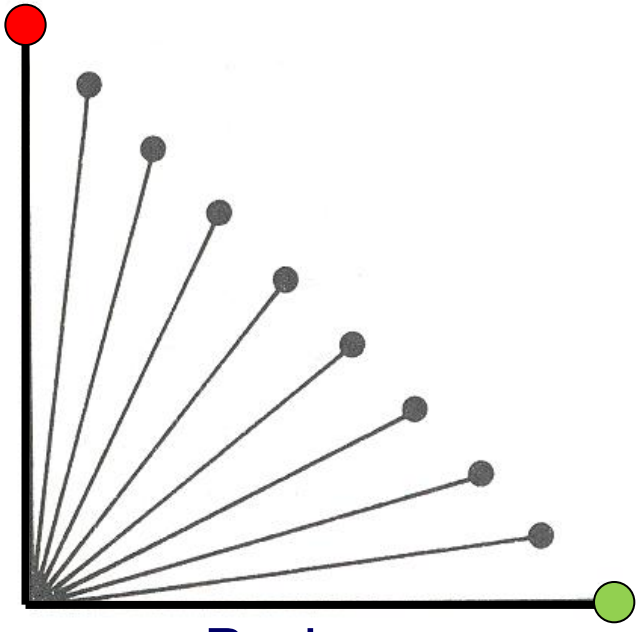
Keyframe Animation

Q: Why interpolate/blend joint parameters rather than vertex positions?

A: For rotations, it could lead to geometric distortion.



Good arm



Bad arm

Watt & Watt

Keyframe Animation

Q: Why interpolate/blend joint parameters rather than vertex positions?

A: For rotations, it could lead to geometric distortion.

For articulating objects, transformations are a combination of translation and rotation

- Translations are straight-forward:
  - Use favorite spline to fit the translation
- How do we interpolate/approximate rotations?

Overview

Orthogonal Transformations, Rotations, and SVD

Interpolating/Approximating Points

- Vectors
- Unit-Vectors

Interpolating/Approximating Transformations

- Matrices
- Rotations
  - SVD Factorization
  - Euler Angles

Orthogonal Transformations

What are orthogonal transformations?

An *orthogonal matrix* $\mathbf{O} \in \mathbb{R}^{n \times n}$ is a linear transformation that preserves the dot-product:

$$\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{O}\mathbf{v}, \mathbf{O}\mathbf{w} \rangle$$

Recall that the (standard) dot-product between two vectors can be expressed as a matrix multiplication:

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^\top \mathbf{w}$$

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$$\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{O}\mathbf{v}, \mathbf{O}\mathbf{w} \rangle$$

$\Rightarrow$ If $\mathbf{O}$ is an orthogonal matrix:

$$\begin{aligned} \mathbf{v}^\top \mathbf{w} &= (\mathbf{O}\mathbf{v})^\top (\mathbf{O}\mathbf{w}) \\ &= \mathbf{v}^\top \mathbf{O}^\top \mathbf{O}\mathbf{w} \end{aligned}$$

Since this is true for all $\mathbf{v}$ and $\mathbf{w}$, this means that:

$$\mathbf{O}^\top \cdot \mathbf{O} = \text{identity} \quad \Leftrightarrow \quad \mathbf{O}^\top = \mathbf{O}^{-1}$$

Orthogonal Transformations

What are orthogonal transformations?

$$\mathbf{O}^{\mathsf{T}} \cdot \mathbf{O} = \text{identity}$$

For matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$:

The determinant of the product is:

$$\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$$

The determinant of the transpose is:

$$\det(\mathbf{A}) = \det(\mathbf{A}^{\mathsf{T}})$$

It follows that:

$$\det(\mathbf{O}^{\mathsf{T}} \cdot \mathbf{O}) = \det(\text{identity})$$

$$\det(\mathbf{O}) \cdot \det(\mathbf{O}) = 1$$

$$\Downarrow$$

$$\det(\mathbf{O}) = \pm 1$$

Rotations

What are rotations?

A *rotation matrix* $\mathbf{O} \in \mathbb{R}^{n \times n}$ is:

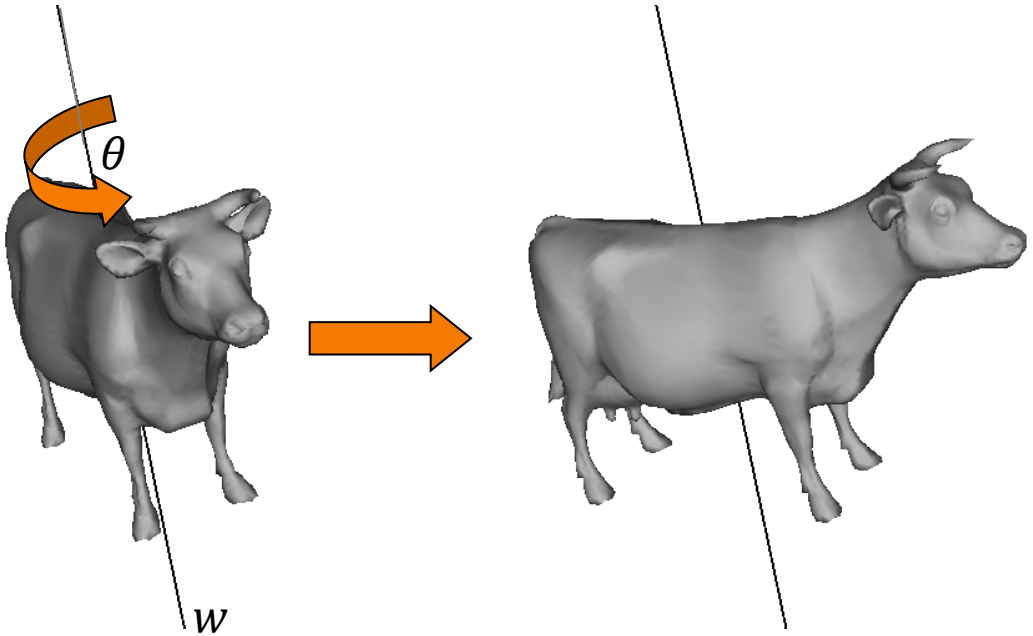
1. An orthogonal matrix
2. That preserves orientation (i.e. $\det(\mathbf{O}) = 1$).

Rotations

What are rotations in 3D?

A rotation in 3D can also be specified by:

- Its axis of rotation w (with $\|w\| = 1$) and
- Its angle of rotation θ



Rotations

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A rotation in 3D can also be specified by:

- Its axis of rotation w (with $\|w\| = 1$) and
- Its angle of rotation θ

Properties:

- The rotation corresponding to (θ, w) is the same as the rotation corresponding to $(-\theta, -w)$.
- The inverse of a rotation corresponding to (θ, w) is $(-\theta, w)$.
- Given rotations corresponding to (θ_1, w) and (θ_2, w), the product of the rotations corresponds to $(\theta_1 + \theta_2, w)$.

How do we define the product of rotations corresponding to (θ_1, w_1) and (θ_2, w_2)?

SVD

Any matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$ can be expressed in terms of its Singular Value Decomposition as:

$$\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}^{\mathsf{T}}$$

with:

- $\mathbf{U}, \mathbf{V} \in \mathbb{R}^{n \times n}$ orthogonal
- $\mathbf{D} \in \mathbb{R}^{n \times n}$ diagonal (i.e. off-diagonals are 0)
  - The diagonal entries are:
    - Non-negative
    - Decreasing

SVD

Applications:

- Procrustes method
- Finding the (pseudo-)inverse of a matrix
- Compression

SVD

Finding the Inverse of a Matrix:

If we have an invertible matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$, we can use the SVD to compute the inverse of $\mathbf{M}$.

Expressing $\mathbf{M}$ in terms of its SVD gives:

$$\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}^{\mathsf{T}}$$

with:

$\mathbf{U}, \mathbf{V} \in \mathbb{R}^{n \times n}$ orthogonal,

$\mathbf{D} \in \mathbb{R}^{n \times n}$ diagonal

SVD

Finding the Inverse of a Matrix :

$$\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}^{\mathsf{T}}$$

We can express $\mathbf{M}^{-1}$ as:

$$\mathbf{M}^{-1} = (\mathbf{U}\mathbf{D}\mathbf{V}^{\mathsf{T}})^{-1} = (\mathbf{V}^{\mathsf{T}})^{-1}\mathbf{D}^{-1}\mathbf{U}^{-1}$$
$$= \mathbf{V}\mathbf{D}^{-1}\mathbf{U}^{\mathsf{T}}$$

Since:

- $\mathbf{U}$ is an orthogonal transformation, $\mathbf{U}^{-1} = \mathbf{U}^{\mathsf{T}}$.
- $\mathbf{V}$ is an orthogonal transformation, $\mathbf{V}^{-1} = \mathbf{V}^{\mathsf{T}}$.

SVD

Solving Linear Systems:

$$\mathbf{M}^{-1} = \mathbf{V}\mathbf{D}^{-1}\mathbf{U}^{\top}$$

Since $\mathbf{D}$ is a diagonal matrix:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix} \quad \Rightarrow \quad \mathbf{D}^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda_{n-1}} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{\lambda_n} \end{pmatrix}$$

This is not necessarily an *efficient* way to invert a matrix!

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Vectors

Given n control points $\{\mathbf{p}_0, \ldots, \mathbf{p}_{n-1}\}$, define a curve $\Phi(t)$ that approximates/interpolates the points.

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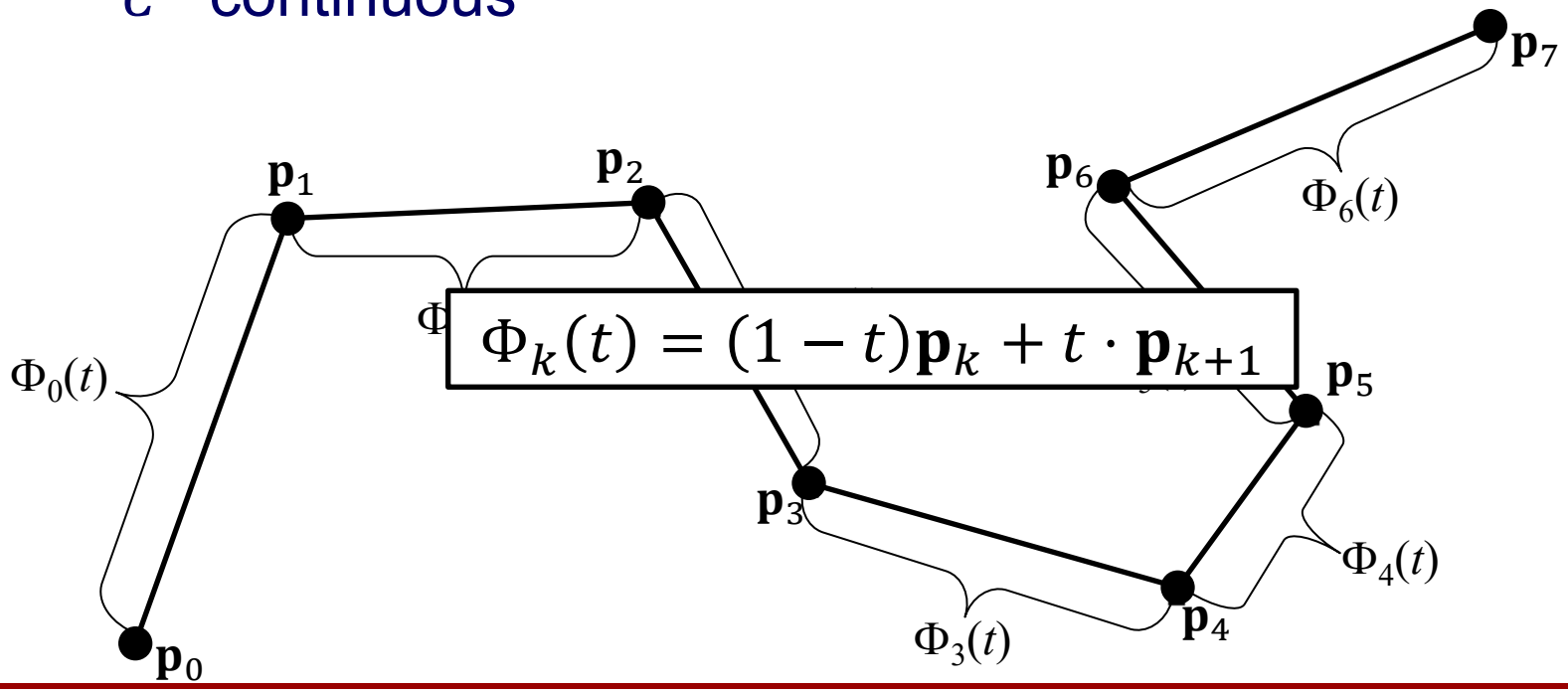
Linear Interpolation:

Interpolating

C^0 continuous



Vectors

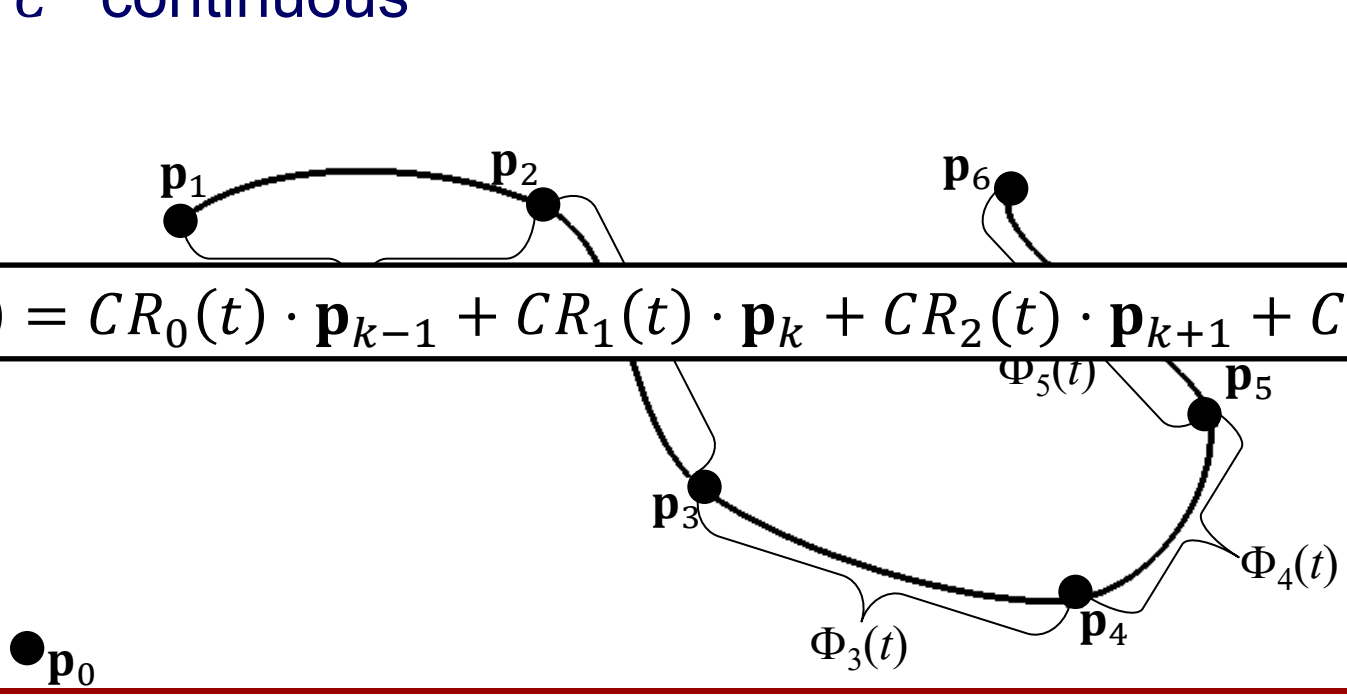
Given n control points $\{\mathbf{p}_0, \ldots, \mathbf{p}_{n-1}\}$, define a curve $\Phi(t)$ that approximates/interpolates the points.

Catmull-Rom Splines (Cardinal Splines w/ $\tau = {}^1/_2$):

Interpolating

C^1 continuous

$$\Phi_k(t) = CR_0(t) \cdot \mathbf{p}_{k-1} + CR_1(t) \cdot \mathbf{p}_k + CR_2(t) \cdot \mathbf{p}_{k+1} + CR_3(t) \cdot \mathbf{p}_{k+2}$$



Vectors

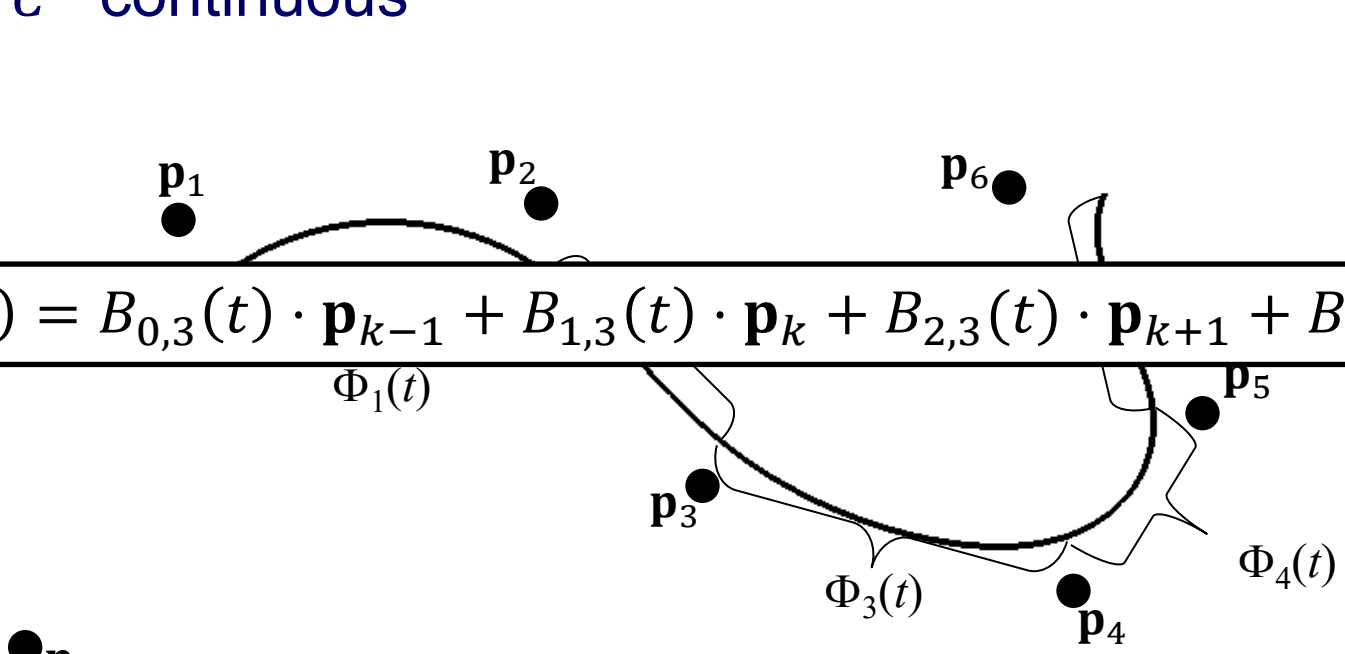
Given n control points $\{\mathbf{p}_0, \ldots, \mathbf{p}_{n-1}\}$, define a curve $\Phi(t)$ that approximates/interpolates the points.

Uniform Cubic B-Splines:

Approximating

C^2 continuous

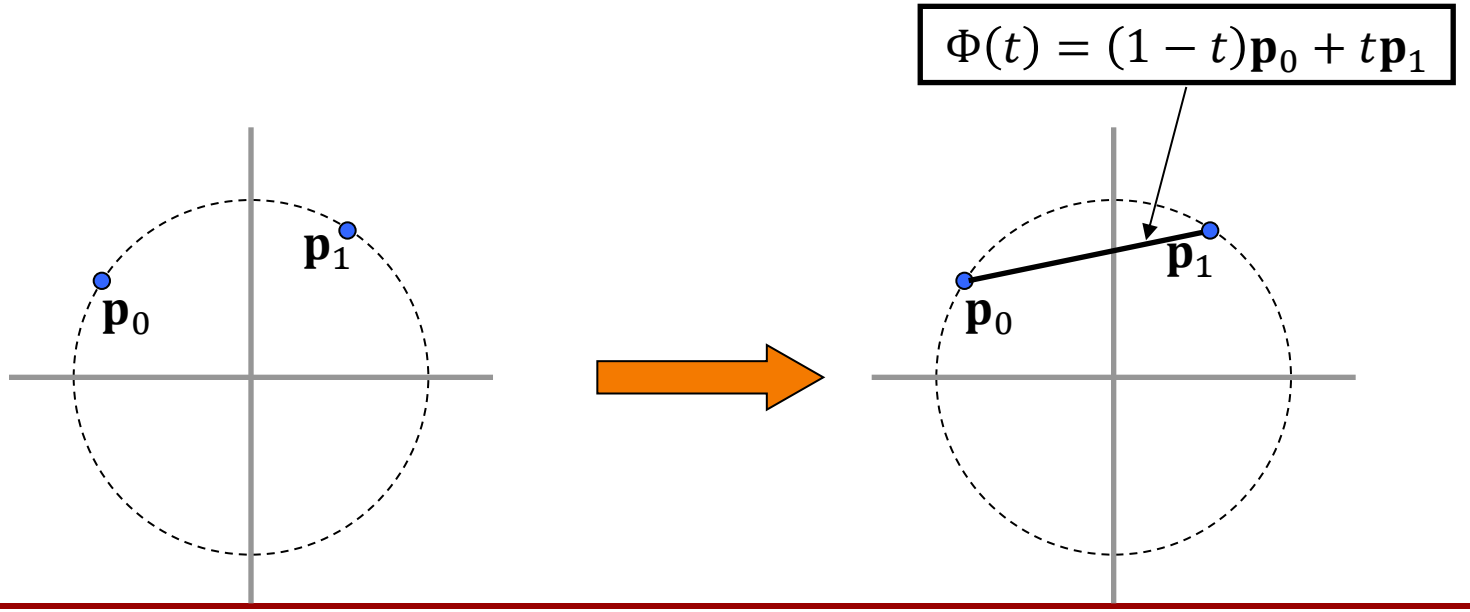
$$\Phi_k(t) = B_{0,3}(t) \cdot \mathbf{p}_{k-1} + B_{1,3}(t) \cdot \mathbf{p}_k + B_{2,3}(t) \cdot \mathbf{p}_{k+1} + B_{3,3}(t) \cdot \mathbf{p}_{k+2}$$



Unit-Vectors

What if we add the constraint that the curve $\Phi(t)$ lie on the unit circle/sphere ($\|\Phi(t)\| = 1$)?

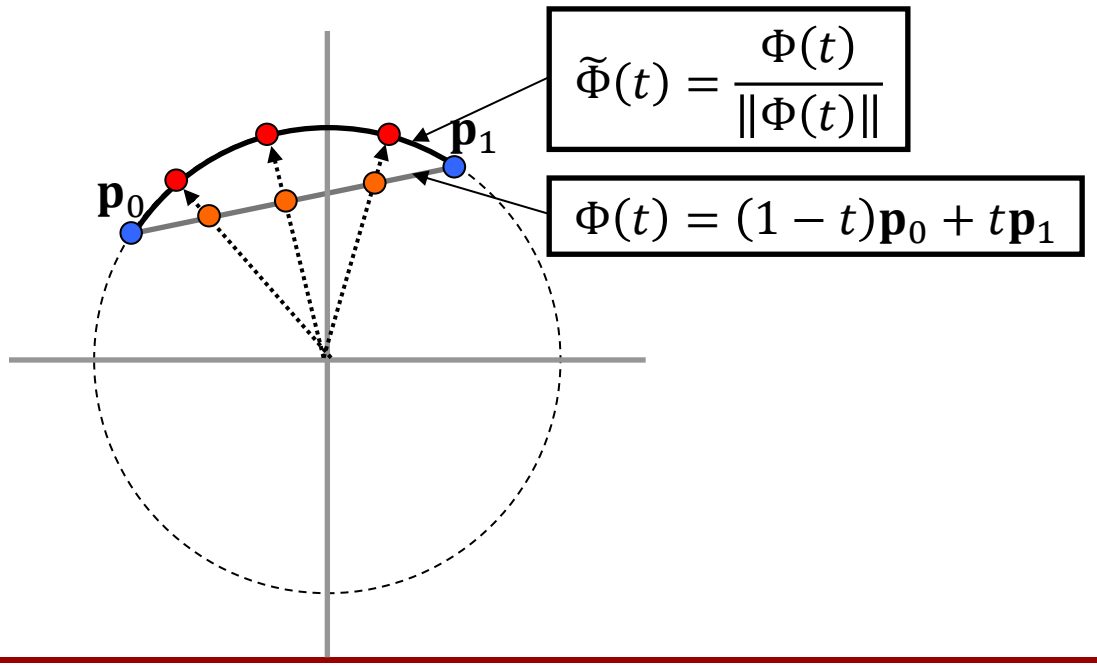
Note: Even if the control points lie on the circle/sphere, in-between points don't have to!



Unit-Vectors

What if we add the constraint that the curve $\Phi(t)$ lie on the unit circle/sphere ($\|\Phi(t)\| = 1$)?

We can normalize the in-between points by sending them to the *closest* circle/sphere point:



Curve Normalization

Limitations:

Curve Normalization

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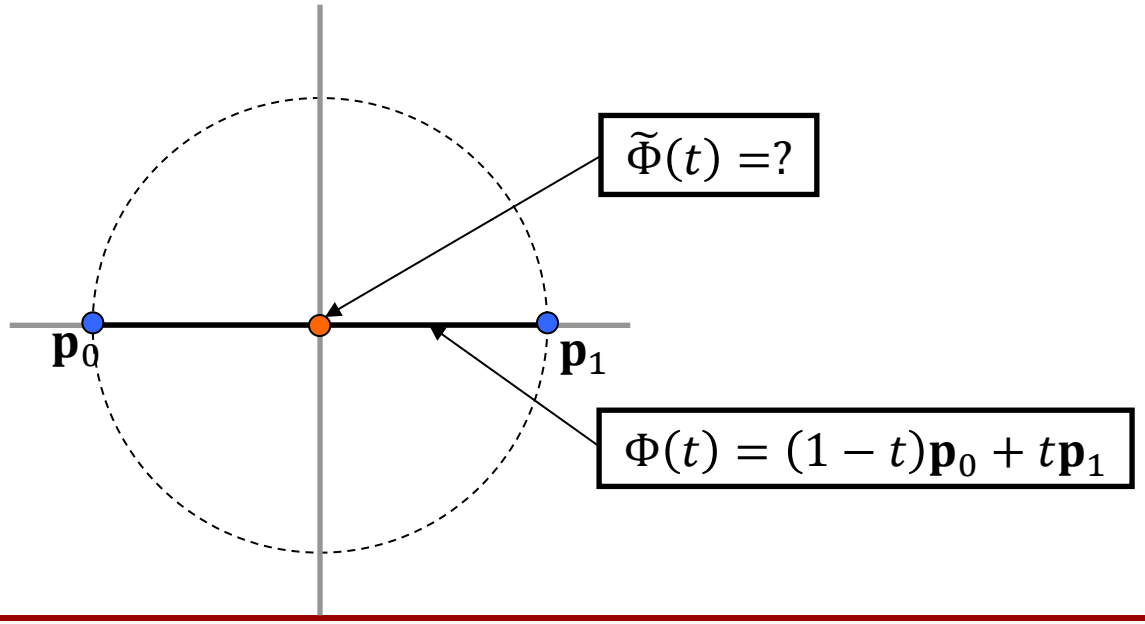
The normalized curve is not always well defined.

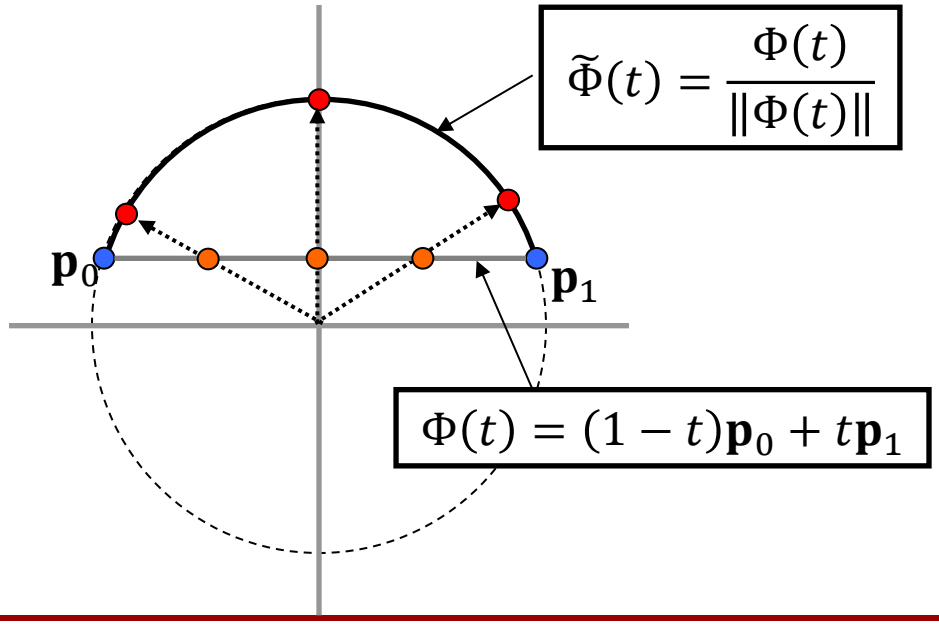


Curve Normalization

Limitations:

The normalized curve is not always well defined.

Having points uniformly distributed on the original curve, does not mean they will be uniformly distributed on the normalized one.



Curve Parameterization

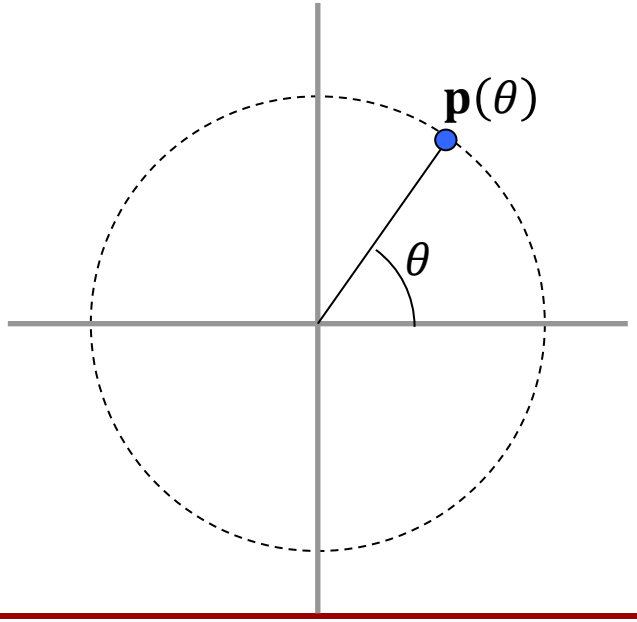
Define a parameterization of the circle/sphere.

Compute the parameters of the end-points.

Blend the parameters and evaluate.

SLERP (Spherical Linear Interpolation):

Parameterize: $(\cos\theta\,,\sin\theta)$



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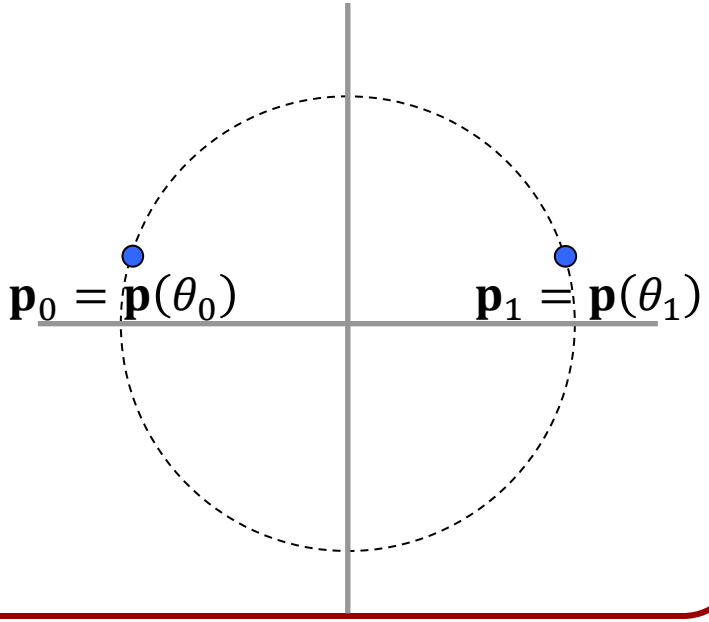
$$\mathbf{p}_0 = (\cos\theta_0, \sin\theta_0)$$
$$\mathbf{p}_1 = (\cos\theta_1\,, \sin\theta_1)$$



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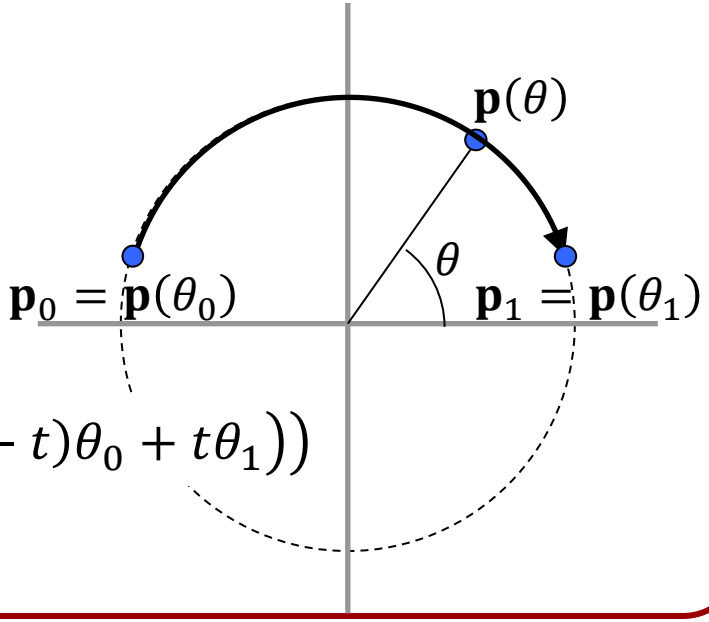
$\mathbf{p}_0 = (\cos\theta_0, \sin\theta_0)$

$\mathbf{p}_1 = (\cos\theta_1\,, \sin\theta_1)$

Set:

$$\Phi(t) = \big(\cos\big((1-t)\theta_0 + t\theta_1\big)\,, \sin\big((1-t)\theta_0 + t\theta_1\big)\big)$$



Curve Parameterization

Note:

Parameter may not be unique.

A shortest path in parameter space does not have to define a shortest path in the space of rotations.

In general, uniform paths in parameter space need not be mapped to uniform curves.

Compute:

$$\mathbf{p}_0 = (\cos\theta_0, \sin\theta_0)$$

$$\mathbf{p}_1 = (\cos\theta_1, \sin\theta_1)$$

Set:

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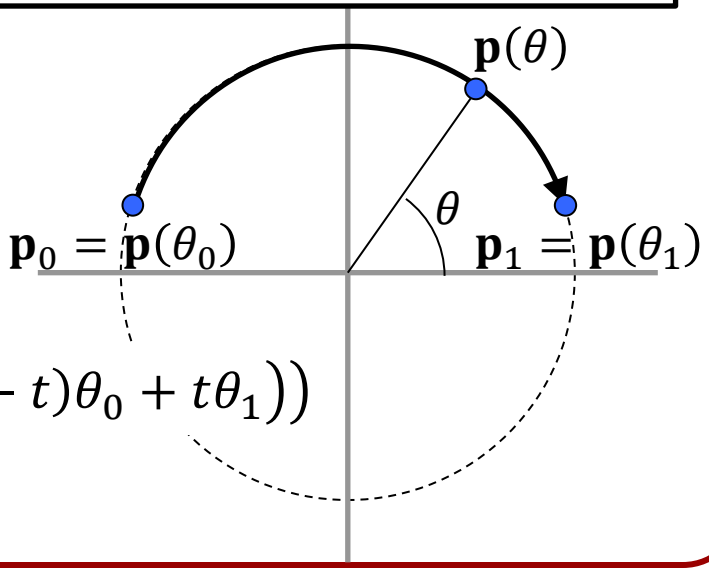

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As with vectors:

Linear Interpolation:

$$\Phi_k(t) = (1-t)\mathbf{M}_k + t \cdot \mathbf{M}_{k+1}$$

Catmull-Rom Interpolation:

$$\Phi_k(t) = CR_0(t) \cdot \mathbf{M}_{k-1} + CR_1(t) \cdot \mathbf{M}_k + CR_2(t) \cdot \mathbf{M}_{k+1} + CR_3(t) \cdot \mathbf{M}_{k+2}$$

Uniform Cubic B-Spline Approximation:

$$\Phi_k(t) = B_{0,3}(t) \cdot \mathbf{M}_{k-1} + B_{1,3}(t) \cdot \mathbf{M}_k + B_{2,3}(t) \cdot \mathbf{M}_{k+1} + B_{3,3}(t) \cdot \mathbf{M}_{k+2}$$

Rotations

What if we add the constraint that the values of the curve $\Phi(t)$ have to be rotations?

Note: In-between matrices won't be rotations!

Could try to normalize, by mapping every matrix $\Phi(t)$ to the nearest rotation.

Normalization: SVD Factorization

Given a matrix $\mathbf{M}$, what is the closest rotation $\mathbf{R}$?

Normalization: SVD Factorization

Given a matrix $\mathbf{M}$, what is the closest rotation $\mathbf{R}$?

Singular Value Decomposition (SVD) allows us to express $\mathbf{M}$ as a diagonal matrix, multiplied on the left/right by orthogonal transformations $\mathbf{O}_1/\mathbf{O}_2$:

$$\mathbf{M} = \mathbf{O}_1 \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \mathbf{O}_2$$

Because the λ_i are positive, the closest orthogonal transform $\mathbf{O}$ to $\mathbf{M}$ is:

$$\mathbf{O} = \mathbf{O}_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{O}_2$$

Normalization: SVD Factorization

Given a matrix $\mathbf{M}$, what is the closest rotation $\mathbf{R}$?

Singular Value Decomposition (SVD) allows us to ex… le… left…

In the SVD factorization, the diagonal values are non-negative, and ordered from largest to smallest.

The orthogonal transformations $\mathbf{O}_1$ and $\mathbf{O}_2$ are not necessarily rotations.

To get a rotation, we must make the product have determinant 1.

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The orthogonal transformations $\mathbf{O}_1$ and $\mathbf{O}_2$ are not necessarily rotations.

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Because the λ_i are positive and decreasing, the closest orthogonal transform $\mathbf{O}$ to $\mathbf{M}$ is:

$$\mathbf{R} = \mathbf{O}_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det(\mathbf{O}_1 \cdot \mathbf{O}_2) \end{pmatrix} \mathbf{O}_2$$

Parametrization: Euler Angles

Every rotation matrix $\mathbf{R}$ can be parameterized by:

- A rotation about the z-axis, multiplied by
- A rotation about the y-axis, multiplied by
- A rotation about the z-axis:

$$\mathbf{R}(\theta, \phi, \psi) = \mathbf{R}_z(\psi)\mathbf{R}_y(\phi)\mathbf{R}_z(\theta)$$

The angles $(\theta, \phi, \psi) \in [0, 2\pi) \times [0, \pi] \times [0, 2\pi)$ are called the Euler angles.

Parametrization: Euler Angles

Instead of blending matrices and then normalizing, we can blend the rotation parameters:

For each $\mathbf{M}_k$, compute the Euler angles $(\theta_k, \phi_k, \psi_k)$

Parametrization: Euler Angles

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- Interpolate/Approximate the Euler angles:

Parametrization: Euler Angles

Instead of blending matrices and then normalizing, we can blend the rotation parameters:

- For each $\mathbf{M}_k$, compute the Euler angles $(\theta_k, \phi_k, \psi_k)$
- Interpolate/Approximate the Euler angles:
  - Linear Interpolation:
    - $\theta_k(t) = (1-t)\theta_k + t \cdot \theta_{k+1}$
    - $\phi_k(t) = (1-t)\phi_k + t \cdot \phi_{k+1}$
    - $\psi_k(t) = (1-t)\psi_k + t \cdot \psi_{k+1}$

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- Interpolate/Approximate the Euler angles:
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  - Catmull-Rom Interpolation:
    - $\theta_k(t) = CR_0(t) \cdot \theta_{k-1} + CR_1(t) \cdot \theta_k + CR_2(t) \cdot \theta_{k+1} + CR_3(t) \cdot \theta_{k+2}$
    - $\phi_k(t) = CR_0(t) \cdot \phi_{k-1} + CR_1(t) \cdot \phi_k + CR_2(t) \cdot \phi_{k+1} + CR_3(t) \cdot \phi_{k+2}$
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- Interpolate/Approximate the Euler angles:
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  - Uniform Cubic B-Spline Approximation:
    - $\theta_k(t) = B_{0,3}(t) \cdot \theta_{k-1} + B_{1,3}(t) \cdot \theta_k + B_{2,3}(t) \cdot \theta_{k+1} + B_{3,3}(t) \cdot \theta_{k+2}$
    - $\phi_k(t) = B_{0,3}(t) \cdot \phi_{k-1} + B_{1,3}(t) \cdot \phi_k + B_{2,3}(t) \cdot \phi_{k+1} + B_{3,3}(t) \cdot \phi_{k+2}$
    - $\psi_k(t) = B_{0,3}(t) \cdot \psi_{k-1} + B_{1,3}(t) \cdot \psi_k + B_{2,3}(t) \cdot \psi_{k+1} + B_{3,3}(t) \cdot \psi_{k+2}$

Parametrization: Euler Angles

Instead of blending matrices and then normalizing, we can blend the rotation parameters:

- For each $\mathbf{M}_k$, compute the Euler angles $(\theta_k, \phi_k, \psi_k)$
- Interpolate/Approximate the Euler angles:
  - Linear Interpolation
  - Catmull-Rom Interpolation
  - Uniform Cubic B-Spline Approximation
- Set the value of the in-between matrix to:

$$\Phi_k(t) = \mathbf{R}_z\big(\theta_k(t)\big)\mathbf{R}_y\big(\phi_k(t)\big)\mathbf{R}_z\big(\psi_k(t)\big)$$

Translations and Rotations

Summary:

- Represent translations and rotations independently:
  - Blend the translations with splines
  - Blend rotations with splines and
    - Normalization, or
    - Parametrization

There are other approaches that treat translations and rotations in a unified manner:

Dual Quaternions for Rigid Transformation Blending, [Kavan *et al*., 2006]