Parametric Surfaces

Michael Kazhdan

(601.457/657)

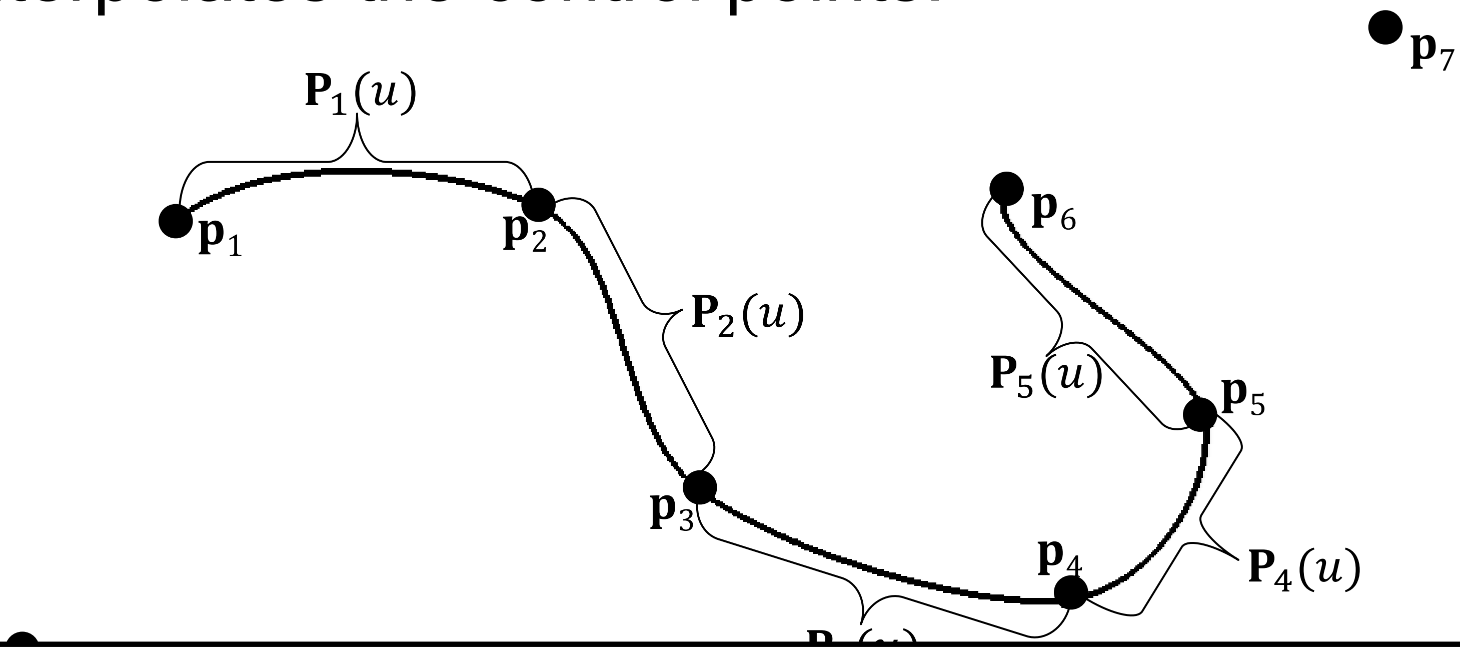
Outline

Spline Surfaces

Sweep Surfaces

Cubic Splines

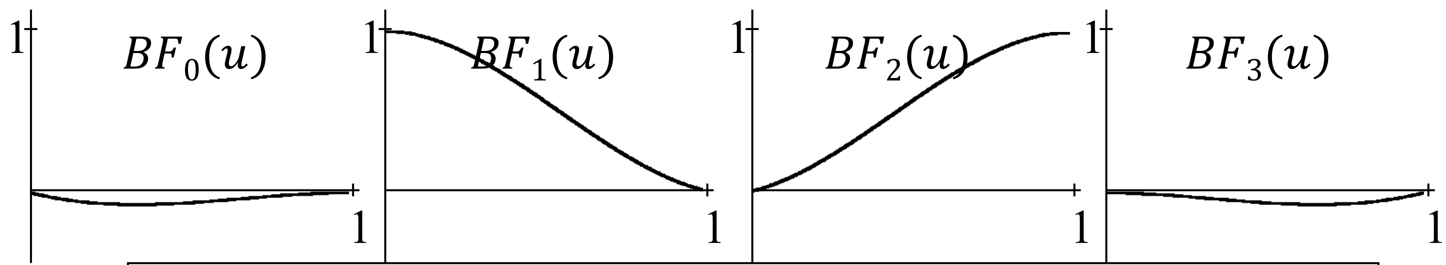
Given $n+1$ control points, $\{\mathbf{p}_0, \ldots, \mathbf{p}_n\}$, we define $n-2$ cubic polynomial functions $\{\mathbf{P}_1(u), \ldots, \mathbf{P}_{n-2}(u)\}$ that jointly describe a curve that approximates / interpolates the control points.



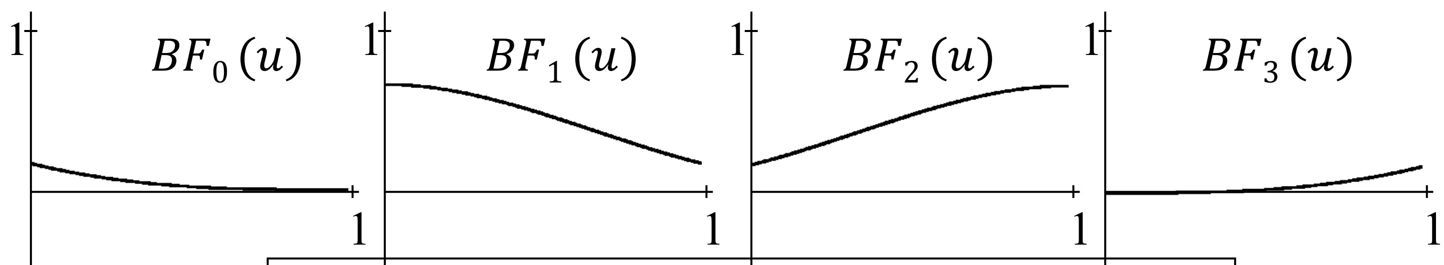
Each cubic function $\mathbf{P}_k(u)$ is defined on the interval $0 \leq u \leq 1$ and is determined by the points $\mathbf{p}_{k-1}$, $\mathbf{p}_k$, $\mathbf{p}_{k+1}$, and $\mathbf{p}_{k+2}$.

Cubic Blending Functions

Blending functions provide a way for expressing the functions $\mathbf{P}_k(u)$ as a weighted sum of the four control points $\mathbf{p}_{k-1}$, $\mathbf{p}_k$, $\mathbf{p}_{k+1}$, and $\mathbf{p}_{k+2}$:



Catmull-Rom Blending Functions (Cardinal with $\tau = 1/2$)



Uniform Cubic B-Spline Blending Functions

$$\mathbf{P}_k(u) = BF_0(u)\cdot\mathbf{p}_{k-1} + BF_1(u)\cdot\mathbf{p}_k + BF_2(u)\cdot\mathbf{p}_{k+1} + BF_3(u)\cdot\mathbf{p}_{k+2}$$

Blending Functions

For spline curves, we need/want:

- Translation Equivariance:

  $BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1$ for all $0 \le u \le 1$.

- n-th Order Continuity:

$$\begin{aligned} 0 &= BF_0^n(1) \\ BF_0^n(0) &= BF_1^n(1) \\ BF_1^n(0) &= BF_2^n(1) \\ BF_2^n(0) &= BF_3^n(1) \\ BF_3^n(0) &= 0 \end{aligned}$$

- Convex Hull Containment:

  $BF_0(u), BF_1(u), BF_2(u), BF_3(u) \ge 0$, for all $0 \le u \le 1$.

- Interpolation:

$$\begin{pmatrix} BF_0(0) \\ BF_1(0) \\ BF_2(0) \\ BF_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} BF_0(1) \\ BF_1(1) \\ BF_2(1) \\ BF_3(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{P}_k(u) = BF_0(u) \cdot \mathbf{p}_{k-1} + BF_1(u) \cdot \mathbf{p}_k + BF_2(u) \cdot \mathbf{p}_{k+1} + BF_3(u) \cdot \mathbf{p}_{k+2}$$

Overview

From Curves to surfaces

- Spline Curves and Blending Functions
- Weighted Averaging
- Spline Surfaces
- Spline Surface Properties

Weighted Averaging

Suppose we have values:

$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$,

and (averaging) weights:

α_1, α_2, α_3, and α_4, with $\alpha_1+\alpha_2+\alpha_3+\alpha_4=1$,

β_1, β_2, β_3, and β_4, with $\beta_1+\beta_2+\beta_3+\beta_4=1$.

We can express the weighted average of the $\mathbf{v}_i$ as:

$$\sum_{i=1}^{4}\alpha_i\mathbf{v}_i=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{pmatrix}\begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\\ \mathbf{v}_3\\ \mathbf{v}_4\end{pmatrix} \quad \sum_{i=1}^{4}\beta_i\mathbf{v}_i=\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\end{pmatrix}$$

Weighted Averaging

If we have a matrix of values:

$$\begin{pmatrix} \mathbf{v}_{11} & \mathbf{v}_{21} & \mathbf{v}_{31} & \mathbf{v}_{41} \\ \mathbf{v}_{12} & \mathbf{v}_{22} & \mathbf{v}_{32} & \mathbf{v}_{42} \\ \mathbf{v}_{13} & \mathbf{v}_{23} & \mathbf{v}_{33} & \mathbf{v}_{43} \\ \mathbf{v}_{14} & \mathbf{v}_{24} & \mathbf{v}_{34} & \mathbf{v}_{44} \end{pmatrix}$$

multiplying on the **left** by $(\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4)$ gives:

$$(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4)\begin{pmatrix} \mathbf{v}_{11} & \mathbf{v}_{21} & \mathbf{v}_{31} & \mathbf{v}_{41} \\ \mathbf{v}_{12} & \mathbf{v}_{22} & \mathbf{v}_{32} & \mathbf{v}_{42} \\ \mathbf{v}_{13} & \mathbf{v}_{23} & \mathbf{v}_{33} & \mathbf{v}_{43} \\ \mathbf{v}_{14} & \mathbf{v}_{24} & \mathbf{v}_{34} & \mathbf{v}_{44} \end{pmatrix}$$

Weighted Averaging

If we have a matrix of values:

$$\begin{pmatrix} \mathbf{v}_{11} & \mathbf{v}_{21} & \mathbf{v}_{31} & \mathbf{v}_{41} \\ \mathbf{v}_{12} & \mathbf{v}_{22} & \mathbf{v}_{32} & \mathbf{v}_{42} \\ \mathbf{v}_{13} & \mathbf{v}_{23} & \mathbf{v}_{33} & \mathbf{v}_{43} \\ \mathbf{v}_{14} & \mathbf{v}_{24} & \mathbf{v}_{34} & \mathbf{v}_{44} \end{pmatrix}$$

multiplying on the **left** by $(\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4)$ gives:

$$(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4)\begin{pmatrix} \mathbf{v}_{11} & \mathbf{v}_{21} & \mathbf{v}_{31} & \mathbf{v}_{41} \\ \mathbf{v}_{12} & \mathbf{v}_{22} & \mathbf{v}_{32} & \mathbf{v}_{42} \\ \mathbf{v}_{13} & \mathbf{v}_{23} & \mathbf{v}_{33} & \mathbf{v}_{43} \\ \mathbf{v}_{14} & \mathbf{v}_{24} & \mathbf{v}_{34} & \mathbf{v}_{44} \end{pmatrix} = \begin{pmatrix} \sum \alpha_i \mathbf{v}_{1i} \\ \sum \alpha_i \mathbf{v}_{2i} \\ \sum \alpha_i \mathbf{v}_{3i} \\ \sum \alpha_i \mathbf{v}_{4i} \end{pmatrix}^{\mathrm{T}}$$

… A **row** vector whose entries are the weighted average of the matrix’s columns.

Weighted Averaging

If we have a matrix of values:

$$\begin{pmatrix} \mathbf{v}_{11} & \mathbf{v}_{21} & \mathbf{v}_{31} & \mathbf{v}_{41} \\ \mathbf{v}_{12} & \mathbf{v}_{22} & \mathbf{v}_{32} & \mathbf{v}_{42} \\ \mathbf{v}_{13} & \mathbf{v}_{23} & \mathbf{v}_{33} & \mathbf{v}_{43} \\ \mathbf{v}_{14} & \mathbf{v}_{24} & \mathbf{v}_{34} & \mathbf{v}_{44} \end{pmatrix}$$

multiplying on the **right** by $(\beta_1\ \beta_2\ \beta_3\ \beta_4)^\top$ gives:

$$\begin{pmatrix} \mathbf{v}_{11} & \mathbf{v}_{21} & \mathbf{v}_{31} & \mathbf{v}_{41} \\ \mathbf{v}_{12} & \mathbf{v}_{22} & \mathbf{v}_{32} & \mathbf{v}_{42} \\ \mathbf{v}_{13} & \mathbf{v}_{23} & \mathbf{v}_{33} & \mathbf{v}_{43} \\ \mathbf{v}_{14} & \mathbf{v}_{24} & \mathbf{v}_{34} & \mathbf{v}_{44} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} \sum \beta_j \mathbf{v}_{j1} \\ \sum \beta_j \mathbf{v}_{j2} \\ \sum \beta_j \mathbf{v}_{j3} \\ \sum \beta_j \mathbf{v}_{j4} \end{pmatrix}$$

… A **column** vector with entries that are the weighted average of the matrix's rows.

Weighted Averaging

Simultaneously multiplying on the left by $(\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4)$ and on the right by $(\beta_1\ \beta_2\ \ \beta_3\ \beta_4)^\top$ gives:

$$(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4)\begin{pmatrix} \mathbf{v}_{11} & \mathbf{v}_{21} & \mathbf{v}_{31} & \mathbf{v}_{41} \\ \mathbf{v}_{12} & \mathbf{v}_{22} & \mathbf{v}_{32} & \mathbf{v}_{42} \\ \mathbf{v}_{13} & \mathbf{v}_{23} & \mathbf{v}_{33} & \mathbf{v}_{43} \\ \mathbf{v}_{14} & \mathbf{v}_{24} & \mathbf{v}_{34} & \mathbf{v}_{44} \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

Weighted Averaging

Simultaneously multiplying on the left by $(\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4)$ and on the right by $(\beta_1\ \beta_2\ \ \beta_3\ \beta_4)^\top$ gives:

$$(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4)\begin{pmatrix} \mathbf{v}_{11} & \mathbf{v}_{21} & \mathbf{v}_{31} & \mathbf{v}_{41} \\ \mathbf{v}_{12} & \mathbf{v}_{22} & \mathbf{v}_{32} & \mathbf{v}_{42} \\ \mathbf{v}_{13} & \mathbf{v}_{23} & \mathbf{v}_{33} & \mathbf{v}_{43} \\ \mathbf{v}_{14} & \mathbf{v}_{24} & \mathbf{v}_{34} & \mathbf{v}_{44} \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \sum_{i,j=1}^{4} \alpha_j \beta_i \mathbf{v}_{ij}$$

$\Rightarrow$ The weighted sum of the $\mathbf{v}_{ij}$, weighted by $\alpha_j\beta_i$.

<u>Claim</u>: This is a weighted average of the $\mathbf{v}_{ij}$:

To show this, we show that the total sum of the weights $\alpha_i\beta_j$ is equal to 1.

Weighted Averaging

Simultaneously multiplying on the left by $(\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4)$ and on the right by $(\beta_1\ \beta_2\ \ \beta_3\ \beta_4)^\top$ gives:

$$(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4)\begin{pmatrix} \mathbf{v}_{11} & \mathbf{v}_{21} & \mathbf{v}_{31} & \mathbf{v}_{41} \\ \mathbf{v}_{12} & \mathbf{v}_{22} & \mathbf{v}_{32} & \mathbf{v}_{42} \\ \mathbf{v}_{13} & \mathbf{v}_{23} & \mathbf{v}_{33} & \mathbf{v}_{43} \\ \mathbf{v}_{14} & \mathbf{v}_{24} & \mathbf{v}_{34} & \mathbf{v}_{44} \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \sum_{i,j=1}^{4} \alpha_j \beta_i \mathbf{v}_{ij}$$

$\Rightarrow$ The weighted sum of the $\mathbf{v}_{ij}$, weighted by $\alpha_j\beta_i$.

<u>Claim</u>: This is a weighted average of the $\mathbf{v}_{ij}$:

$$\sum_{i,j=1}^{4} \alpha_i \beta_j = \sum_{i=1}^{4} \alpha_i \left(\sum_{j=1}^{4} \beta_j \right)$$

$$= \sum_{i=1}^{4} \alpha_i = 1$$

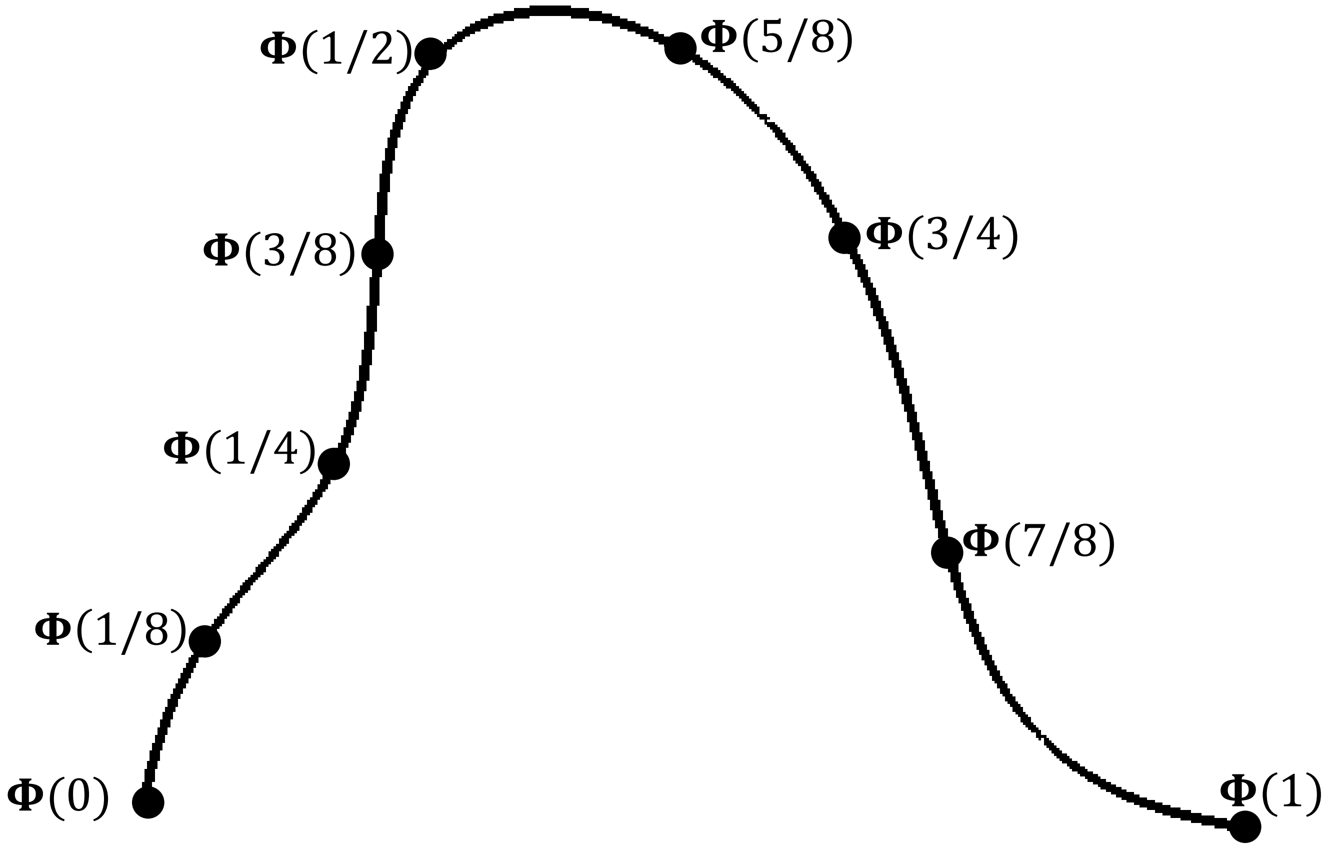
Overview

From Curves to surfaces

- Spline Curves and Blending Functions
- Weighted Averaging
- Spline Surfaces
- Spline Surface Properties

Spline Surfaces

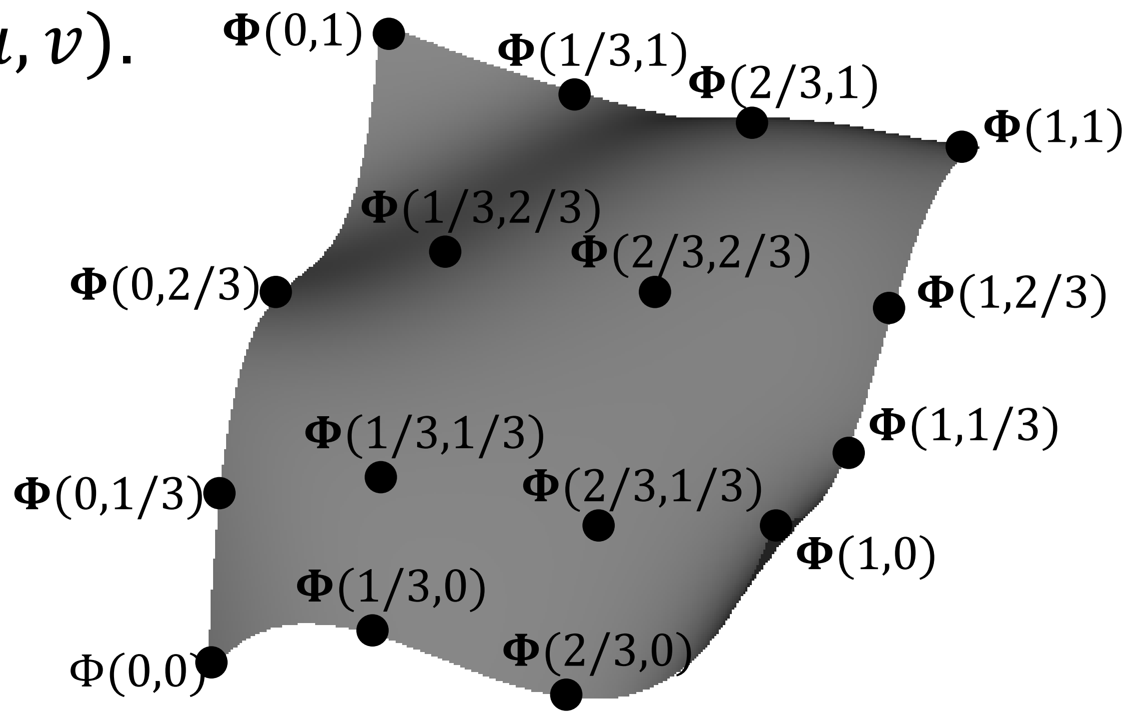
A *parametric curve* is a function in one variable $\mathbf{\Phi}(u)$ associating a position to every value of u.



Spline Surfaces

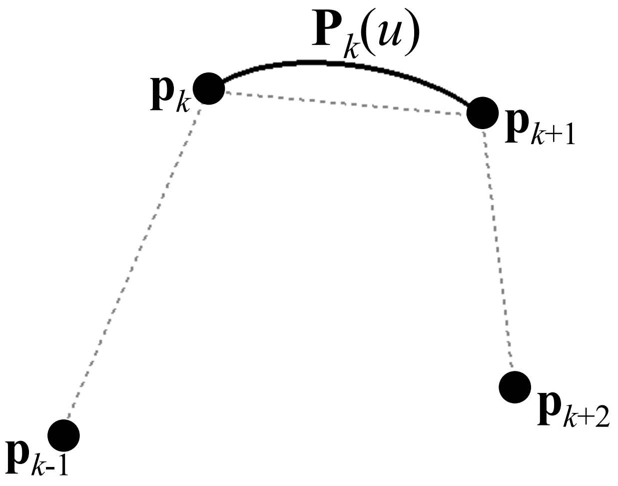
A *parametric curve* is a function in one variable $\mathbf{\Phi}(u)$ associating a position to every value of u.

A *parametric patch/surface* is a function in two variables $\mathbf{\Phi}(u,v)$ associating a position to every pair of values of (u,v).



Spline Surfaces

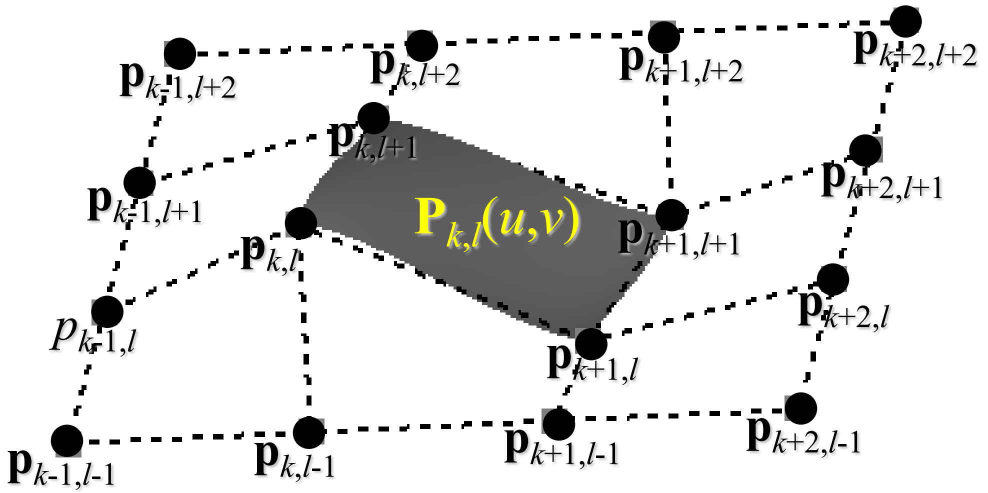
We use **four** control points to define a **cubic** polynomial $\mathbf{P}_k(u)$ in one variable $(0 \leq u \leq 1)$.



Spline Surfaces

We use **four** control points to define a **cubic** polynomial $\mathbf{P}_k(u)$ in one variable $(0 \le u \le 1)$.

We use $\mathbf{4} \times \mathbf{4}$ control points to define a **bi-cubic** polynomial $\mathbf{P}_{k,l}(u, v)$ in two variables $(0 \le u, v \le 1)$.



Spline Surfaces

We use **four** control points to define a **cubic** polynomial $\mathbf{P}_k(u)$ in one variable ($0 \leq u \leq 1$).

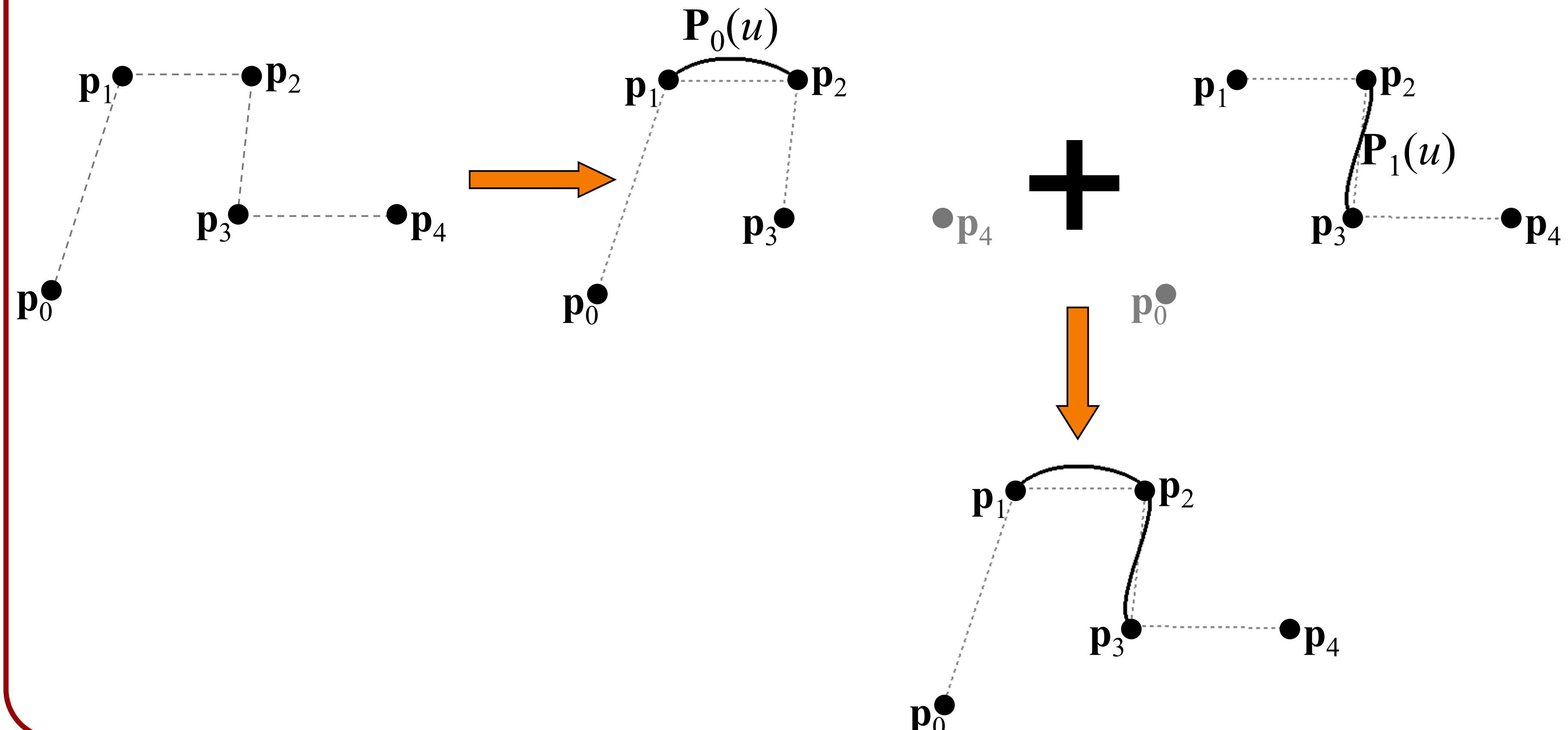
We use $\mathbf{4} \times \mathbf{4}$ control points to define a **bi-cubic** polynomial $\mathbf{P}_{k,l}(u,v)$ in two variables ($0 \leq u, v \leq 1$).

A *bi-cubic polynomial* is a polynomial which is cubic in each variable:

$$\begin{aligned}\mathbf{P}(u,v) = {} & \mathbf{a}u^3v^3 + \\ & +\mathbf{b}u^3v^2 + \mathbf{c}u^2v^3 + \\ & +\mathbf{d}u^2v^2 + \mathbf{e}u^1v^3 + \mathbf{f}u^3v^1 + \\ & +\cdots\end{aligned}$$

Spline Surfaces

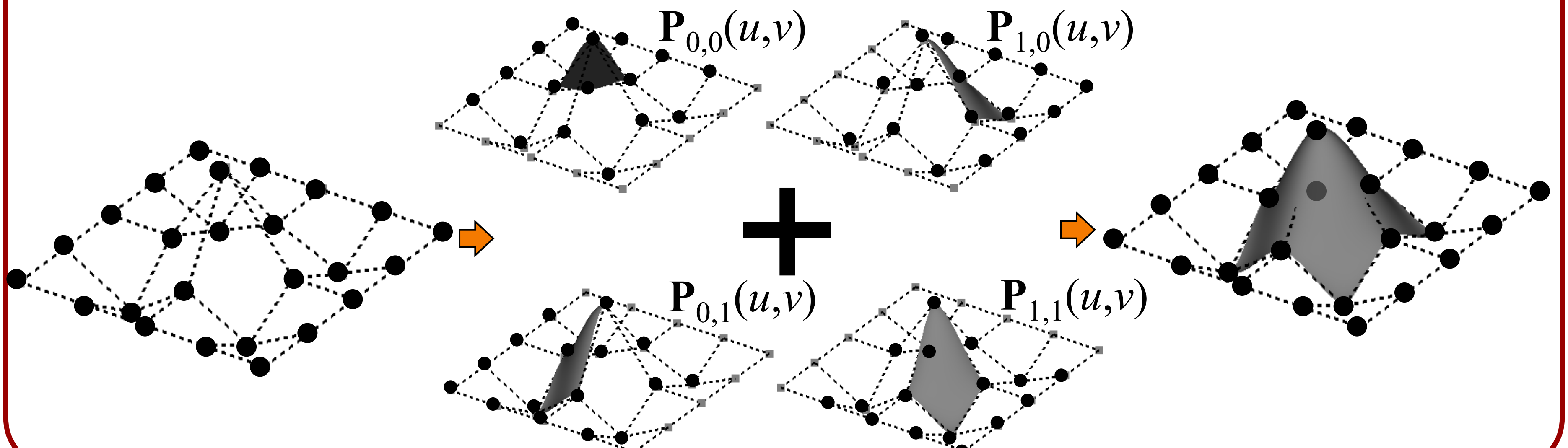
Given n points, we fit a **piecewise** cubic curve consisting of $n-3$ segments to the points.



Spline Surfaces

Given n points, we fit a **piecewise** cubic curve consisting of $n-3$ segments to the points.

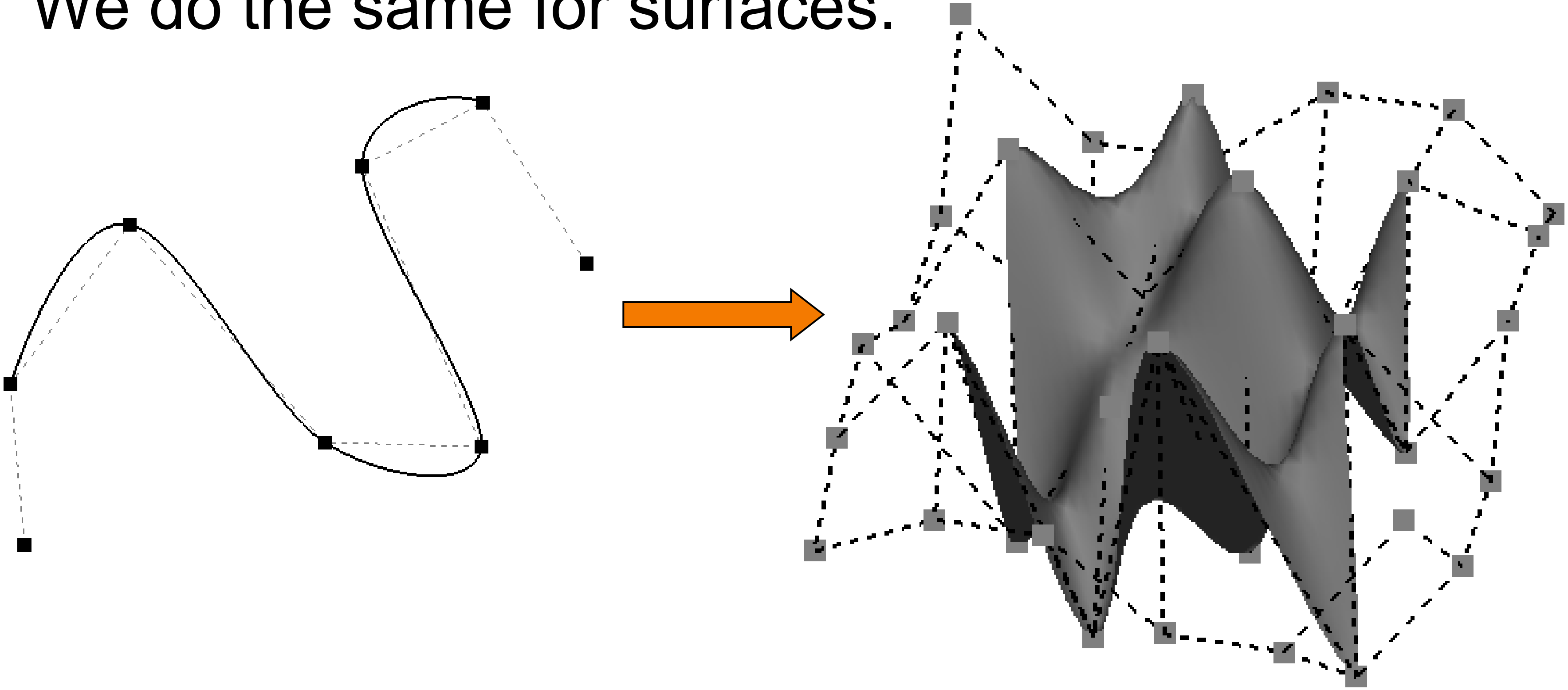
Given $n \times m$ points, we fit a **piecewise** bi-cubic surface, consisting of $(n-3) \times (m-3)$ patches to the points.



Spline Surfaces

We generate spline curves by using the blending function to compute the weighted average of the control points.

We do the same for surfaces.



Cubic Blending Functions

Recall

For a cubic segment of a spline curve, we express the spline curve in matrix form as:

$$\mathbf{P}_k(u) = \begin{pmatrix} BF_0(u) & BF_1(u) & BF_2(u) & BF_3(u) \end{pmatrix} \begin{pmatrix} \mathbf{p}_{k-1} \\ \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \mathbf{p}_{k+2} \end{pmatrix}$$

$$\mathbf{P}_k(u) = BF_0(u) \cdot \mathbf{p}_{k-1} + BF_1(u) \cdot \mathbf{p}_k + BF_2(u) \cdot \mathbf{p}_{k+1} + BF_3(u) \cdot \mathbf{p}_{k+2}$$

Cubic Blending Functions

Recall

For a cubic segment of a spline curve, we express the spline curve in matrix form as:

$$\mathbf{P}_k(u) = \begin{pmatrix} BF_0(u) & BF_1(u) & BF_2(u) & BF_3(u) \end{pmatrix} \begin{pmatrix} \mathbf{p}_{k-1} \\ \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \mathbf{p}_{k+2} \end{pmatrix} = \begin{pmatrix} BF_0(u) \\ BF_1(u) \\ BF_2(u) \\ BF_3(u) \end{pmatrix}^\top \begin{pmatrix} \mathbf{p}_{k-1} \\ \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \mathbf{p}_{k+2} \end{pmatrix}$$

Since the sum of the $BF_i(u)$ equals 1, this is a weighted average of the control points.

$$\mathbf{P}_k(u) = BF_0(u) \cdot \mathbf{p}_{k-1} + BF_1(u) \cdot \mathbf{p}_k + BF_2(u) \cdot \mathbf{p}_{k+1} + BF_3(u) \cdot \mathbf{p}_{k+2}$$

Cubic Blending Functions

If we are given a 4×4 array of control points, we can define a bi-cubic spline patch similarly:

$$\mathbf{P}_{k,l}(u,v) = \begin{pmatrix} BF_0(v) \\ BF_1(v) \\ BF_2(v) \\ BF_3(v) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathbf{p}_{k-1,l-1} & \mathbf{p}_{k,l-1} & \mathbf{p}_{k+1,l-1} & \mathbf{p}_{k+2,l-1} \\ \mathbf{p}_{k-1,l} & \mathbf{p}_{k,l} & \mathbf{p}_{k+1,l} & \mathbf{p}_{k+2,l} \\ \mathbf{p}_{k-1,l+1} & \mathbf{p}_{k,l+1} & \mathbf{p}_{k+1,l+1} & \mathbf{p}_{k+2,l+1} \\ \mathbf{p}_{k-1,l+2} & \mathbf{p}_{k,l+2} & \mathbf{p}_{k+1,l+2} & \mathbf{p}_{k+2,l+2} \end{pmatrix} \begin{pmatrix} BF_0(u) \\ BF_1(u) \\ BF_2(u) \\ BF_3(u) \end{pmatrix}$$

Since, the sum of the $BF_i(u)$ equals 1, $\mathbf{P}_{k,l}(u,v)$ is a weighted **average** of the control points.

Cubic Spline Patches

Computing the value of the patch at a point (u_0, v_0) amounts to:

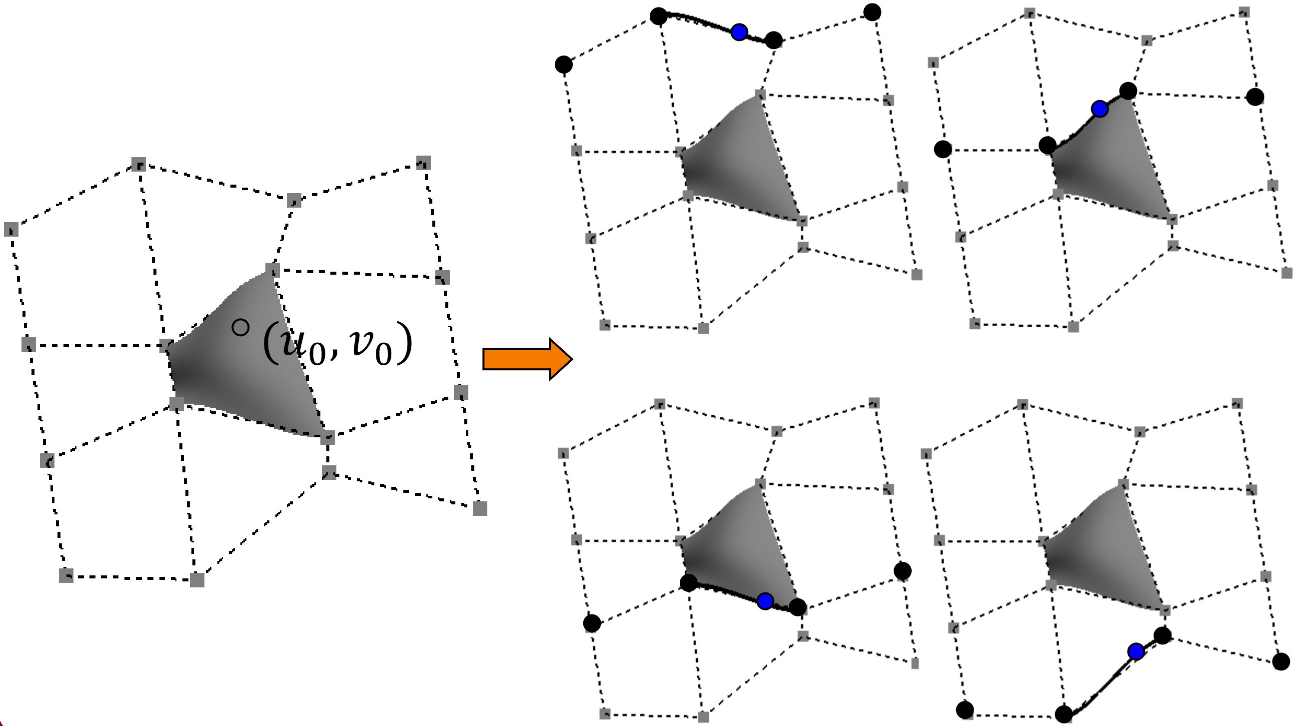
1. Averaging the rows using the weights $BF_i(u_0)$
2. Averaging the result using the weights $BF_i(v_0)$.

$$\mathbf{P}_{k,l}(u,v) = \begin{pmatrix} BF_0(v) \\ BF_1(v) \\ BF_2(v) \\ BF_3(v) \end{pmatrix}^{\mathsf{T}} \left(\begin{pmatrix} \mathbf{p}_{k-1,l-1} & \mathbf{p}_{k,l-1} & \mathbf{p}_{k+1,l-1} & \mathbf{p}_{k+2,l-1} \\ \mathbf{p}_{k-1,l} & \mathbf{p}_{k,l} & \mathbf{p}_{k+1,l} & \mathbf{p}_{k+2,l} \\ \mathbf{p}_{k-1,l+1} & \mathbf{p}_{k,l+1} & \mathbf{p}_{k+1,l+1} & \mathbf{p}_{k+2,l+1} \\ \mathbf{p}_{k-1,l+2} & \mathbf{p}_{k,l+2} & \mathbf{p}_{k+1,l+2} & \mathbf{p}_{k+2,l+2} \end{pmatrix} \begin{pmatrix} BF_0(u) \\ BF_1(u) \\ BF_2(u) \\ BF_3(u) \end{pmatrix} \right)$$

Cubic Spline Patches

Computing the value of the patch at a point (u_0, v_0) amounts to:

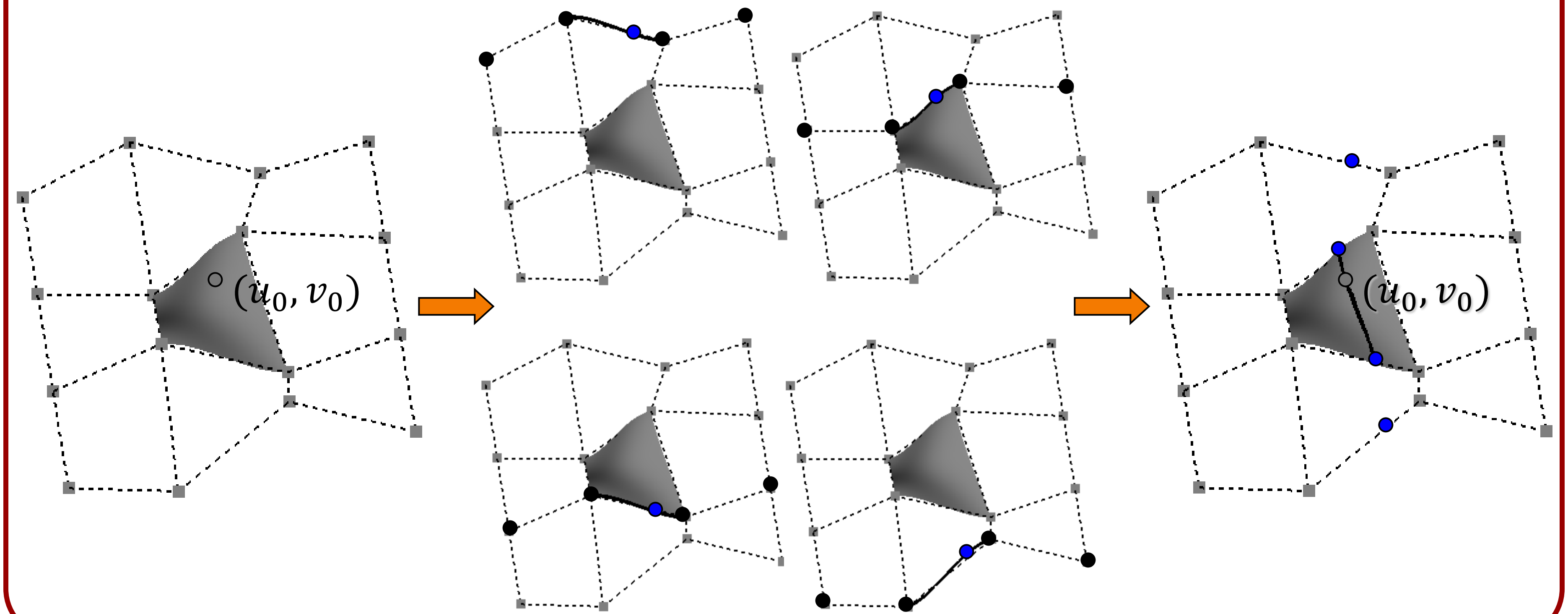
1. Averaging the rows using the weights $BF_i(u_0)$
2. Averaging the result using the weights $BF_i(v_0)$.



Cubic Spline Patches

Computing the value of the patch at a point (u_0, v_0) amounts to:

1. Averaging the rows using the weights $BF_i(u_0)$
2. Averaging the result using the weights $BF_i(v_0)$.



Cubic Spline Patches

$$\mathbf{P}_{k,l}(u,v)=\begin{pmatrix}BF_0(v)\\BF_1(v)\\BF_2(v)\\BF_3(v)\end{pmatrix}^{\top}\begin{pmatrix}\mathbf{p}_{k-1,l-1} & \mathbf{p}_{k,l-1} & \mathbf{p}_{k+1,l-1} & \mathbf{p}_{k+2,l-1}\\ \mathbf{p}_{k-1,l} & \mathbf{p}_{k,l} & \mathbf{p}_{k+1,l} & \mathbf{p}_{k+2,l}\\ \mathbf{p}_{k-1,l+1} & \mathbf{p}_{k,l+1} & \mathbf{p}_{k+1,l+1} & \mathbf{p}_{k+2,l+1}\\ \mathbf{p}_{k-1,l+2} & \mathbf{p}_{k,l+2} & \mathbf{p}_{k+1,l+2} & \mathbf{p}_{k+2,l+2}\end{pmatrix}\begin{pmatrix}BF_0(u)\\BF_1(u)\\BF_2(u)\\BF_3(u)\end{pmatrix}$$

Multiplying out the matrices we get:

$$\begin{aligned}\mathbf{P}_{k,l}(u,v) &= BF_0(u)\cdot BF_0(v)\cdot\mathbf{p}_{k-1,l-1}+BF_1(u)\cdot BF_0(v)\cdot\mathbf{p}_{k,l-1}+\cdots\\ &+BF_0(u)\cdot BF_1(v)\cdot\mathbf{p}_{k-1,l}+BF_1(u)\cdot BF_1(v)\cdot\mathbf{p}_{k,l}+\cdots\\ &+\cdots\end{aligned}$$

⇒ Setting $BF_{i,j}(u,v)=BF_i(u)\cdot BF_j(v)$ we get:

$$\begin{aligned}\mathbf{P}_{k,l}(u,v) &= BF_{0,0}(u,v)\cdot\mathbf{p}_{k-1,l-1}+BF_{1,0}(u,v)\cdot\mathbf{p}_{k,l-1}+\cdots\\ &+BF_{0,1}(u,v)\cdot\mathbf{p}_{k-1,l}+BF_{1,1}(u,v)\cdot\mathbf{p}_{k,l}+\cdots\\ &+\cdots\end{aligned}$$

Cubic Spline Patches

$$\mathbf{P}_{k,l}(u,v)=\begin{pmatrix}BF_0(v)\\BF_1(v)\\BF_2(v)\\BF_3(v)\end{pmatrix}^{\mathsf{T}}\begin{pmatrix}\mathbf{p}_{k-1,l-1} & \mathbf{p}_{k,l-1} & \mathbf{p}_{k+1,l-1} & \mathbf{p}_{k+2,l-1}\\\mathbf{p}_{k-1,l} & \mathbf{p}_{k,l} & \mathbf{p}_{k+1,l} & \mathbf{p}_{k+2,l}\\\mathbf{p}_{k-1,l+1} & \mathbf{p}_{k,l+1} & \mathbf{p}_{k+1,l+1} & \mathbf{p}_{k+2,l+1}\\\mathbf{p}_{k-1,l+2} & \mathbf{p}_{k,l+2} & \mathbf{p}_{k+1,l+2} & \mathbf{p}_{k+2,l+2}\end{pmatrix}\begin{pmatrix}BF_0(u)\\BF_1(u)\\BF_2(u)\\BF_3(u)\end{pmatrix}$$

Multiplying out the matrices we get:

$$\begin{aligned}\mathbf{P}_{k,l}(u,v)=\ & BF_0(u)\cdot BF_0(v)\cdot\mathbf{p}_{k-1,l-1}+BF_1(u)\cdot BF_0(v)\cdot\mathbf{p}_{k,l-1}+\cdots\\&+BF_0(u)\cdot BF_1(v)\cdot\mathbf{p}_{k-1,l}+BF_1(u)\cdot BF_1(v)\cdot\mathbf{p}_{k,l}+\cdots\\&+\cdots\end{aligned}$$

⇒ Setting $BF_{i,j}(u,v)=BF_i(u)\cdot BF_j(v)$ we get:

$$\begin{aligned}\mathbf{P}_{k,l}(u,v)=\ & BF_{0,0}(u,v)\cdot\mathbf{p}_{k-1,l-1}+BF_{1,0}(u,v)\cdot\mathbf{p}_{k,l-1}+\cdots\\&+BF_{0,1}(u,v)\cdot\mathbf{p}_{k-1,l}+BF_{1,1}(u,v)\cdot\mathbf{p}_{k,l}+\cdots\\&+\cdots\end{aligned}$$

Cubic Spline Patches

$$\mathbf{P}_{k,l}(u,v) = \begin{pmatrix} BF_0(v) \\ BF_1(v) \\ BF_2(v) \\ BF_3(v) \end{pmatrix}^{\top} \begin{pmatrix} \mathbf{p}_{k-1,l-1} & \mathbf{p}_{k,l-1} & \mathbf{p}_{k+1,l-1} & \mathbf{p}_{k+2,l-1} \\ \mathbf{p}_{k-1,l} & \mathbf{p}_{k,l} & \mathbf{p}_{k+1,l} & \mathbf{p}_{k+2,l} \\ \mathbf{p}_{k-1,l+1} & \mathbf{p}_{k,l+1} & \mathbf{p}_{k+1,l+1} & \mathbf{p}_{k+2,l+1} \\ \mathbf{p}_{k-1,l+2} & \mathbf{p}_{k,l+2} & \mathbf{p}_{k+1,l+2} & \mathbf{p}_{k+2,l+2} \end{pmatrix} \begin{pmatrix} BF_0(u) \\ BF_1(u) \\ BF_2(u) \\ BF_3(u) \end{pmatrix}$$

Multiplying out the matrices we get:

$$\begin{aligned}\mathbf{P}_{k,l}(u,v) = &\; BF_0(u)\cdot BF_0(v)\cdot \mathbf{p}_{k-1,l-1} + BF_1(u)\cdot BF_0(v)\cdot \mathbf{p}_{k,l-1} + \cdots \\ &+ BF_0(u)\cdot BF_1(v)\cdot \mathbf{p}_{k-1,l} + BF_1(u)\cdot BF_1(v)\cdot \mathbf{p}_{k,l} + \cdots \\ &+ \cdots\end{aligned}$$

⇒ Setting $BF_{i,j}(u,v) = BF_i(u)\cdot BF_j(v)$ we get:

$$\begin{aligned}\mathbf{P}_{k,l}(u,v) = &\; BF_{0,0}(u,v)\cdot \mathbf{p}_{k-1,l-1} + BF_{1,0}(u,v)\cdot \mathbf{p}_{k,l-1} + \cdots \\ &+ BF_{0,1}(u,v)\cdot \mathbf{p}_{k-1,l} + BF_{1,1}(u,v)\cdot \mathbf{p}_{k,l} + \cdots \\ &+ \cdots\end{aligned}$$

Cubic Spline Patches

$$\mathbf{P}_{k,l}(u,v)=\begin{pmatrix}BF_0(v)\\BF_1(v)\\BF_2(v)\\BF_3(v)\end{pmatrix}^{\mathsf{T}}\begin{pmatrix}\mathbf{p}_{k-1,l-1} & \mathbf{p}_{k,l-1} & \mathbf{p}_{k+1,l-1} & \mathbf{p}_{k+2,l-1}\\ \mathbf{p}_{k-1,l} & \mathbf{p}_{k,l} & \mathbf{p}_{k+1,l} & \mathbf{p}_{k+2,l}\\ \mathbf{p}_{k-1,l+1} & \mathbf{p}_{k,l+1} & \mathbf{p}_{k+1,l+1} & \mathbf{p}_{k+2,l+1}\\ \mathbf{p}_{k-1,l+2} & \mathbf{p}_{k,l+2} & \mathbf{p}_{k+1,l+2} & \mathbf{p}_{k+2,l+2}\end{pmatrix}\begin{pmatrix}BF_0(u)\\BF_1(u)\\BF_2(u)\\BF_3(u)\end{pmatrix}$$

Multiplying out the matrices we get:

$$\begin{aligned}\mathbf{P}_{k,l}(u,v)=\;& BF_0(u)\cdot BF_0(v)\cdot\mathbf{p}_{k-1,l-1}+BF_1(u)\cdot BF_0(v)\cdot\mathbf{p}_{k,l-1}+\cdots\\ &+BF_0(u)\cdot BF_1(v)\cdot\mathbf{p}_{k-1,l}+BF_1(u)\cdot BF_1(v)\cdot\mathbf{p}_{k,l}+\cdots\\ &+\cdots\end{aligned}$$

⇒ Setting $BF_{i,j}(u,v)=BF_i(u)\cdot BF_j(v)$ we get:

$$\begin{aligned}\mathbf{P}_{k,l}(u,v)=\;& BF_{0,0}(u,v)\cdot\mathbf{p}_{k-1,l-1}+BF_{1,0}(u,v)\cdot\mathbf{p}_{k,l-1}+\cdots\\ &+BF_{0,1}(u,v)\cdot\mathbf{p}_{k-1,l}+BF_{1,1}(u,v)\cdot\mathbf{p}_{k,l}+\cdots\\ &+\cdots\end{aligned}$$

Cubic Spline Patches

$$\mathbf{P}_{k,l}(u,v) = \begin{pmatrix} BF_0(v) \\ BF_1(v) \\ BF_2(v) \\ BF_3(v) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathbf{p}_{k-1,l-1} & \mathbf{p}_{k,l-1} & \mathbf{p}_{k+1,l-1} & \mathbf{p}_{k+2,l-1} \\ \mathbf{p}_{k-1,l} & \mathbf{p}_{k,l} & \mathbf{p}_{k+1,l} & \mathbf{p}_{k+2,l} \\ \mathbf{p}_{k-1,l+1} & \mathbf{p}_{k,l+1} & \mathbf{p}_{k+1,l+1} & \mathbf{p}_{k+2,l+1} \\ \mathbf{p}_{k-1,l+2} & \mathbf{p}_{k,l+2} & \mathbf{p}_{k+1,l+2} & \mathbf{p}_{k+2,l+2} \end{pmatrix} \begin{pmatrix} BF_0(u) \\ BF_1(u) \\ BF_2(u) \\ BF_3(u) \end{pmatrix}$$

Multiplying out the matrices we get:

$$\begin{aligned} \mathbf{P}_{k,l}(u,v) = & BF_0(u) \cdot BF_0(v) \cdot \mathbf{p}_{k-1,l-1} + BF_1(u) \cdot BF_0(v) \cdot \mathbf{p}_{k,l-1} + \cdots \\ & + BF_0(u) \cdot BF_1(v) \cdot \mathbf{p}_{k-1,l} + BF_1(u) \cdot BF_1(v) \cdot \mathbf{p}_{k,l} + \cdots \\ & + \cdots \end{aligned}$$

$\Rightarrow$ Setting $BF_{i,j}(u,v) = BF_i(u) \cdot BF_j(v)$ we get:

$$\begin{aligned} \mathbf{P}_{k,l}(u,v) = & BF_{0,0}(u,v) \cdot \mathbf{p}_{k-1,l-1} + BF_{1,0}(u,v) \cdot \mathbf{p}_{k,l-1} + \cdots \\ & + BF_{0,1}(u,v) \cdot \mathbf{p}_{k-1,l} + BF_{1,1}(u,v) \cdot \mathbf{p}_{k,l} + \cdots \\ & + \cdots \end{aligned}$$

Cubic Spline Patches

$$\mathbf{P}_{k,l}(u,v) = \begin{pmatrix} BF_0(v) \\ BF_1(v) \\ BF_2(v) \\ BF_3(v) \end{pmatrix}^\top \begin{pmatrix} \mathbf{p}_{k-1,l-1} & \mathbf{p}_{k,l-1} & \mathbf{p}_{k+1,l-1} & \mathbf{p}_{k+2,l-1} \\ \mathbf{p}_{k-1,l} & \mathbf{p}_{k,l} & \mathbf{p}_{k+1,l} & \mathbf{p}_{k+2,l} \\ \mathbf{p}_{k-1,l+1} & \mathbf{p}_{k,l+1} & \mathbf{p}_{k+1,l+1} & \mathbf{p}_{k+2,l+1} \\ \mathbf{p}_{k-1,l+2} & \mathbf{p}_{k,l+2} & \mathbf{p}_{k+1,l+2} & \mathbf{p}_{k+2,l+2} \end{pmatrix} \begin{pmatrix} BF_0(u) \\ BF_1(u) \\ BF_2(u) \\ BF_3(u) \end{pmatrix}$$

Recall that we can write out blending functions as:

$$\begin{pmatrix} BF_0(u) & BF_1(u) & BF_2(u) & BF_3(u) \end{pmatrix}^\top = \mathbf{M}_{\text{Spline}} U$$

with $U^\top = (u^3\ u^2\ u\ \ 1)$ and $\mathbf{M}_{\text{Spline}}$ the spline matrix.

This gives:

$$\mathbf{P}_{k,l}(u,v) = V^\top \mathbf{M}_{\text{Spline}}^\top \begin{pmatrix} \mathbf{p}_{k-1,l-1} & \mathbf{p}_{k,l-1} & \mathbf{p}_{k+1,l-1} & \mathbf{p}_{k+2,l-1} \\ \mathbf{p}_{k-1,l} & \mathbf{p}_{k,l} & \mathbf{p}_{k+1,l} & \mathbf{p}_{k+2,l} \\ \mathbf{p}_{k-1,l+1} & \mathbf{p}_{k,l+1} & \mathbf{p}_{k+1,l+1} & \mathbf{p}_{k+2,l+1} \\ \mathbf{p}_{k-1,l+2} & \mathbf{p}_{k,l+2} & \mathbf{p}_{k+1,l+2} & \mathbf{p}_{k+2,l+2} \end{pmatrix} \mathbf{M}_{\text{Spline}} U$$

with $V^\top = (v^3\ v^2\ v\ 1)$.

Cubic Spline Patches

$$\mathbf{P}_{k,l}(u,v) = \begin{pmatrix} BF_0(v) \\ BF_1(v) \\ BF_2(v) \\ BF_3(v) \end{pmatrix}^\top \begin{pmatrix} \mathbf{p}_{k-1,l-1} & \mathbf{p}_{k,l-1} & \mathbf{p}_{k+1,l-1} & \mathbf{p}_{k+2,l-1} \\ \mathbf{p}_{k-1,l} & \mathbf{p}_{k,l} & \mathbf{p}_{k+1,l} & \mathbf{p}_{k+2,l} \\ \mathbf{p}_{k-1,l+1} & \mathbf{p}_{k,l+1} & \mathbf{p}_{k+1,l+1} & \mathbf{p}_{k+2,l+1} \\ \mathbf{p}_{k-1,l+2} & \mathbf{p}_{k,l+2} & \mathbf{p}_{k+1,l+2} & \mathbf{p}_{k+2,l+2} \end{pmatrix} \begin{pmatrix} BF_0(u) \\ BF_1(u) \\ BF_2(u) \\ BF_3(u) \end{pmatrix}$$

Recall that we can write out blending functions as:

$$\begin{pmatrix} BF_0(u) & BF_1(u) & BF_2(u) & BF_3(u) \end{pmatrix}^\top = \mathbf{M}_{\text{Spline}} U$$

> Surface splines that are obtained from curve splines in this way are referred to as *tensor product splines*.

$$\mathbf{P}_{k,l}(u,v) = V^\top \mathbf{M}_{\text{Spline}}^\top \begin{pmatrix} \mathbf{p}_{k-1,l-1} & \mathbf{p}_{k,l-1} & \mathbf{p}_{k+1,l-1} & \mathbf{p}_{k+2,l-1} \\ \mathbf{p}_{k-1,l} & \mathbf{p}_{k,l} & \mathbf{p}_{k+1,l} & \mathbf{p}_{k+2,l} \\ \mathbf{p}_{k-1,l+1} & \mathbf{p}_{k,l+1} & \mathbf{p}_{k+1,l+1} & \mathbf{p}_{k+2,l+1} \\ \mathbf{p}_{k-1,l+2} & \mathbf{p}_{k,l+2} & \mathbf{p}_{k+1,l+2} & \mathbf{p}_{k+2,l+2} \end{pmatrix} \mathbf{M}_{\text{Spline}} U$$

with $V^\top = (v^3\ v^2\ v\ 1)$.

Cubic Spline Patches

We can choose our favorite spline curve (Cardinal, uniform cubic-B, etc.) and use its blending functions to define a spline patch:

$$\mathbf{P}_{k,l}(u,v) = V^\top \boxed{\mathbf{M}_{\text{Spline}}^\top} \begin{pmatrix} \mathbf{p}_{k-1,l-1} & \mathbf{p}_{k,l-1} & \mathbf{p}_{k+1,l-1} & \mathbf{p}_{k+2,l-1} \\ \mathbf{p}_{k-1,l} & \mathbf{p}_{k,l} & \mathbf{p}_{k+1,l} & \mathbf{p}_{k+2,l} \\ \mathbf{p}_{k-1,l+1} & \mathbf{p}_{k,l+1} & \mathbf{p}_{k+1,l+1} & \mathbf{p}_{k+2,l+1} \\ \mathbf{p}_{k-1,l+2} & \mathbf{p}_{k,l+2} & \mathbf{p}_{k+1,l+2} & \mathbf{p}_{k+2,l+2} \end{pmatrix} \boxed{\mathbf{M}_{\text{Spline}}} U$$

Overview

From Curves to surfaces

- Spline Curves and Blending Functions
- Weighted Averaging
- Spline Surfaces
- Spline Surface Properties

Blending Functions

For spline curves, we want:

- Translation Equivariance:

  $BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1$ for all $0 \leq u \leq 1$.

- n-th Order Continuity:

$$
\begin{aligned}
0 &= BF_0^n(1) \\
BF_0^n(0) &= BF_1^n(1) \\
BF_1^n(0) &= BF_2^n(1) \\
BF_2^n(0) &= BF_3^n(1) \\
BF_3^n(0) &= 0
\end{aligned}
$$

- Convex Hull Containment:

  $BF_0(u), BF_1(u), BF_2(u), BF_3(u) \geq 0$, for all $0 \leq u \leq 1$.

- Interpolation:

$$
\begin{matrix} BF_0(0) \\ BF_1(0) \\ BF_2(0) \\ BF_3(0) \end{matrix} = \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \end{matrix} \quad \text{and} \quad \begin{matrix} BF_0(1) \\ BF_1(1) \\ BF_2(1) \\ BF_3(1) \end{matrix} = \begin{matrix} 0 \\ 0 \\ 1 \\ 0 \end{matrix}
$$

Do tensor product splines satisfy these conditions?

Surface Spline Properties

Translation equivariance:

As with curves, we need the sum of the blending functions $BF_{i,j}(u,v)$ to equal 1.

But since

$$BF_{i,j}(u,v) = BF_i(u) \cdot BF_j(v)$$

if the $BF_i(u)$ are weighting functions that sum to 1, then the tensor product functions $BF_{i,j}(u,v)$ also sum to 1.

Surface Spline Properties

Continuity:

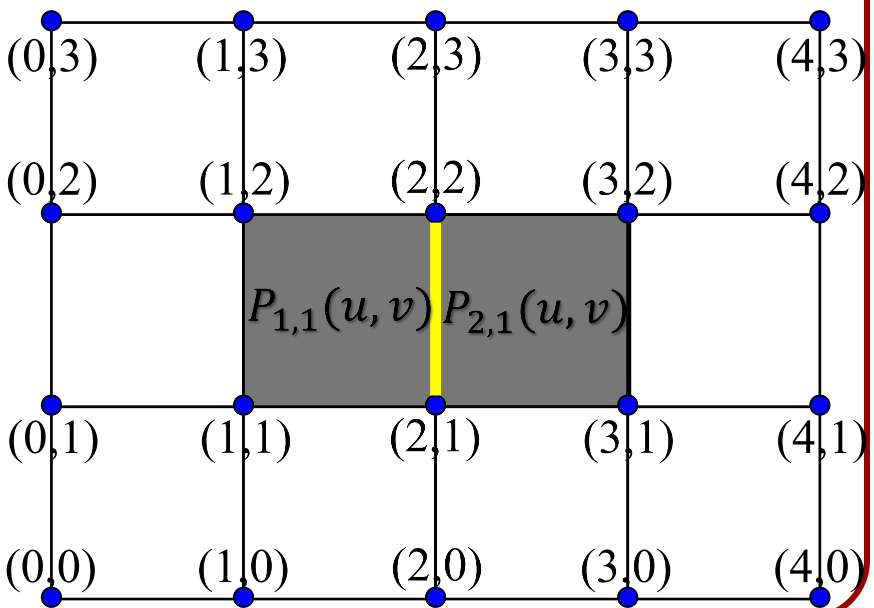
W.L.O.G. consider continuity along the yellow edge:

$$0 = \mathbf{P}_{1,1}(1, v) - \mathbf{P}_{2,1}(0, v) \quad \forall 0 \le v \le 1$$

$$\Downarrow$$

$$0 = \sum_{i=0}^{3}\sum_{j=0}^{3} B_{i,j}(1, v) \cdot \mathbf{p}_{i,j} - \sum_{i=0}^{3}\sum_{j=0}^{3} B_{i,j}(0, v) \cdot \mathbf{p}_{i+1,j}$$

Re-index the second term so that the control point indices match.



Surface Spline Properties

Continuity:

W.L.O.G. consider continuity along the yellow edge:

$$0 = \mathbf{P}_{1,1}(1,v) - \mathbf{P}_{2,1}(0,v) \qquad \forall 0 \le v \le 1$$

$$\Downarrow$$

$$0 = \sum_{i=0}^{3}\sum_{j=0}^{3} B_{i,j}(1,v)\cdot\mathbf{p}_{i,j} - \sum_{i=0}^{3}\sum_{j=0}^{3} B_{i,j}(0,v)\cdot\mathbf{p}_{i+1,j}$$

$$\Downarrow$$

$$0 = \sum_{i=0}^{3}\sum_{j=0}^{3} B_{i,j}(1,v)\cdot\mathbf{p}_{i,j} - \sum_{i=1}^{4}\sum_{j=0}^{3} B_{i-1,j}(0,v)\cdot\mathbf{p}_{i,j}$$

Decompose the equation in terms of the control points shared by both patches.

$\mathbf{p}_{i,j}$ w/ $1 \le i \le 3$ and $0 \le j \le 3$

Surface Spline Properties

Continuity:

W.L.O.G. consider continuity along the yellow edge:

$$0 = \mathbf{P}_{1,1}(1,v) - \mathbf{P}_{2,1}(0,v) \quad \forall 0 \le v \le 1$$

$$\Downarrow$$

$$0 = \sum_{i=0}^{3}\sum_{j=0}^{3} B_{i,j}(1,v)\cdot \mathbf{p}_{i,j} - \sum_{i=0}^{3}\sum_{j=0}^{3} B_{i,j}(0,v)\cdot \mathbf{p}_{i+1,j}$$

$$\Downarrow$$

$$0 = \sum_{i=0}^{3}\sum_{j=0}^{3} B_{i,j}(1,v)\cdot \mathbf{p}_{i,j} - \sum_{i=1}^{4}\sum_{j=0}^{3} B_{i-1,j}(0,v)\cdot \mathbf{p}_{i,j}$$

$$\Downarrow$$

$$0 = \sum_{j=0}^{3} B_{0,j}(1,v)\cdot \mathbf{p}_{0,j} + \sum_{i=1}^{3}\sum_{j=0}^{3} B_{i,j}(1,v)\cdot \mathbf{p}_{i,j} - \sum_{i=1}^{3}\sum_{j=0}^{3} B_{i-1,j}(0,v)\cdot \mathbf{p}_{i,j} - \sum_{j=0}^{3} B_{3,j}(0,v)\cdot \mathbf{p}_{4,j}$$

Combine terms using the same control points.

Surface Spline Properties

<u>Continuity</u>:

W.L.O.G. consider continuity along the yellow edge:

$$0 = \mathbf{P}_{1,1}(1,v) - \mathbf{P}_{2,1}(0,v) \qquad \forall 0 \le v \le 1$$

$$\Downarrow$$

$$0 = \sum_{i=0}^{3}\sum_{j=0}^{3} B_{i,j}(1,v)\cdot \mathbf{p}_{i,j} - \sum_{i=0}^{3}\sum_{j=0}^{3} B_{i,j}(0,v)\cdot \mathbf{p}_{i+1,j}$$

$$\Downarrow$$

$$0 = \sum_{i=0}^{3}\sum_{j=0}^{3} B_{i,j}(1,v)\cdot \mathbf{p}_{i,j} - \sum_{i=1}^{4}\sum_{j=0}^{3} B_{i-1,j}(0,v)\cdot \mathbf{p}_{i,j}$$

$$\Downarrow$$

$$0 = \sum_{j=0}^{3} B_{0,j}(1,v)\cdot \mathbf{p}_{0,j} + \sum_{i=1}^{3}\sum_{j=0}^{3} B_{i,j}(1,v)\cdot \mathbf{p}_{i,j} - \sum_{i=1}^{3}\sum_{j=0}^{3} B_{i-1,j}(0,v)\cdot \mathbf{p}_{i,j} - \sum_{j=0}^{3} B_{3,j}(0,v)\cdot \mathbf{p}_{4,j}$$

$$\Downarrow$$

$$0 = \sum_{j=0}^{3} B_{0,j}(1,v)\cdot \mathbf{p}_{0,j} + \sum_{i=1}^{3}\sum_{j=0}^{3} \left(B_{i,j}(1,v) - B_{i-1,j}(0,v)\right)\cdot \mathbf{p}_{i,j} - \sum_{j=0}^{3} B_{3,j}(0,v)\cdot \mathbf{p}_{4,j}$$

Surface Spline Properties

Continuity:

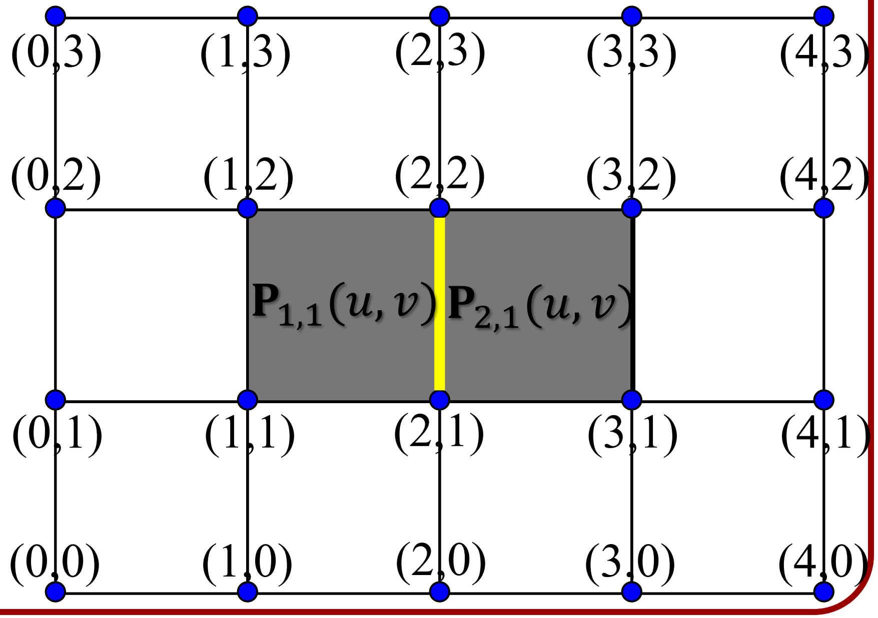
W.L.O.G. consider continuity along the yellow edge:

$$0 = \sum_{j=0}^{3} B_{0,j}(1,v) \cdot \mathbf{p}_{0,j} + \sum_{i=1}^{3}\sum_{j=0}^{3}\left(B_{i,j}(1,v) - B_{i-1,j}(0,v)\right) \cdot \mathbf{p}_{i,j} - \sum_{j=0}^{3} B_{3,j}(0,v) \cdot \mathbf{p}_{4,j}$$

For this to be true for all control points $\mathbf{p}_{ij}$, we need:

- $B_{0,j}(1,v) = B_{3,j}(0,v) = 0$
- $B_{1,j}(1,v) = B_{0,j}(0,v)$
- $B_{2,j}(1,v) = B_{1,j}(0,v)$
- $B_{3,j}(1,v) = B_{2,j}(0,v)$

for all $v \in [0,1]$



Surface Spline Properties

Continuity:

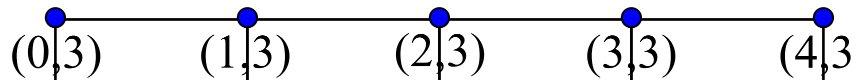
W.L.O.G. consider continuity along the yellow edge:

$$0 = \sum_{j=0}^{3} B_{0,j}(1,v)\cdot \mathbf{p}_{0,j} + \sum_{i=1}^{3}\sum_{j=0}^{3}\left(B_{i,j}(1,v) - B_{i-1,j}(0,v)\right)\cdot \mathbf{p}_{i,j} - \sum_{j=0}^{3} B_{3,j}(0,v)\cdot \mathbf{p}_{4,j}$$

For this to be true for all control points $\mathbf{p}_{ij}$, we need:

- $\boxed{B_{0,j}(1,v) = B_{3,j}(0,v) = 0}$
- $B_{1,j}(1,v) = B_{0,j}(0,v)$
- $B_{2,j}$ ...
- $B_{3,j}$ ...

for all ...



$$B_{0,j}(1,v) = B_{3,j}(0,v) = 0$$
$$\Updownarrow$$
$$B_0(1)\cdot B_j(v) = B_3(0)\cdot B_j(v) = 0$$
$$\Uparrow$$
$$B_0(1) = B_3(0) = 0$$

Which is satisfied if the 1D B-spline is continuous!

Surface Spline Properties

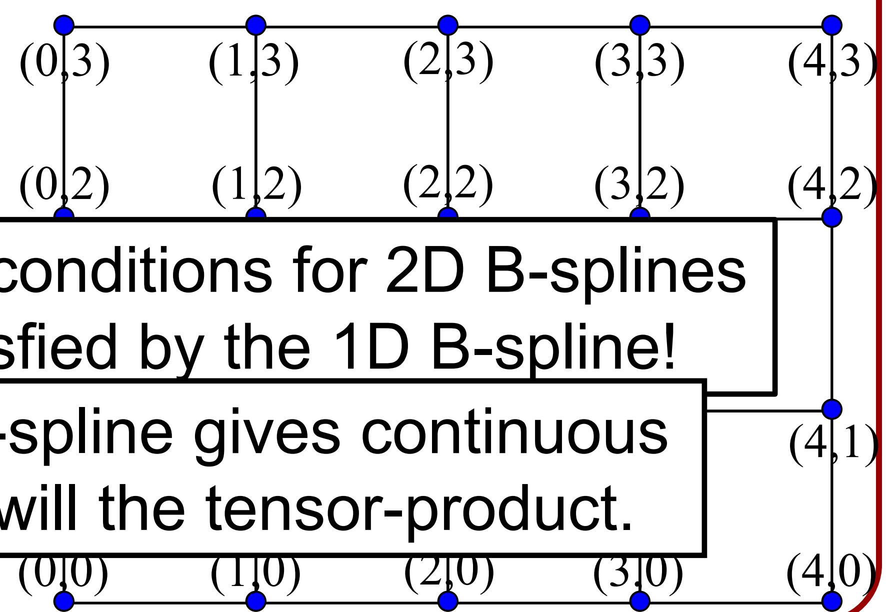
Continuity:

W.L.O.G. consider continuity along the yellow edge:

$$0 = \sum_{j=0}^{3} B_{0,j}(1,v) \cdot \mathbf{p}_{0,j} + \sum_{i=1}^{3}\sum_{j=0}^{3}\left(B_{i,j}(1,v) - B_{i-1,j}(0,v)\right) \cdot \mathbf{p}_{i,j} - \sum_{j=0}^{3} B_{3,j}(0,v) \cdot \mathbf{p}_{4,j}$$

For this to be true for all control points $\mathbf{p}_{ij}$, we need:

- $B_{0,j}(1,v) = B_{3,j}(0,v) = 0$
- $B_{1,j}(1,v) = B_{0,j}(0,v)$
- $B_{2,j}(1,v) = B_{1,j}(0,v)$
- $B_{3,j}(1,v) = B_{2,j}(0,v)$



Similarly, the other continuity conditions for 2D B-splines are satisfied if they are satisfied by the 1D B-spline!

More generally, if the 1D B-spline gives continuous n-th order derivatives, so will the tensor-product.

Surface Spline Properties

Convex hull containment:

For convex hull containment we need the weights of the blending function to be non-negative.

If the $BF_i(u)$ are non-negative, then since

$$BF_{i,j}(u,v) = BF_i(u) \cdot BF_j(v)$$

the $BF_{i,j}(u,v)$ will also be non-negative.

Surface Spline Properties

Interpolation:

For the spline surface to interpolate, it must satisfy:

» $BF_{1,1}(0,0) = BF_{1,2}(0,1) = BF_{2,1}(1,0) = BF_{2,2}(1,1) = 1$.

» All the other blending functions evaluate to 0 at the end-points.

Recall that the spline curve is interpolating if:

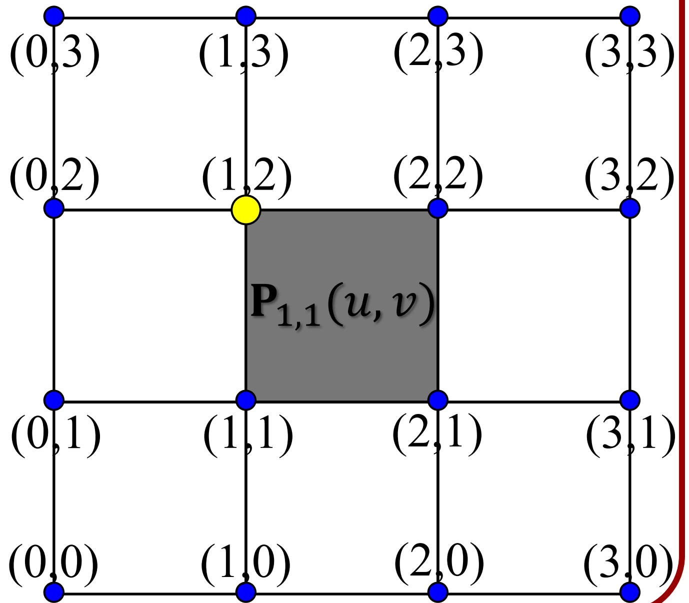
» $BF_0(0) = BF_2(0) = BF_3(0) = 0$

» $BF_0(1) = BF_1(1) = BF_3(1) = 0$

» $BF_1(0) = 1$

» $BF_2(1) = 1$

$$BF_{1,2}(0,1) = BF_1(0) \cdot BF_2(1)$$



Surface Spline Properties

We began by describing properties that we want spline curves to satisfy:

- Translation equivariance
- Continuity
- Convex hull containment
- Interpolation

If the spline curve satisfies these properties, then so will the tensor product spline surface!

Surface Spline Properties

We began by describing properties that we want spline curves to satisfy:

- Translation equivariance
- Continuity
- Convex hull containment
- Interpolation

If the … hen so will th…

As with curves, we can handle boundaries by:

- Ignoring them
- Doubling up
- Introducing cylindrical/toroidal periodicity

Surface Spline Demo

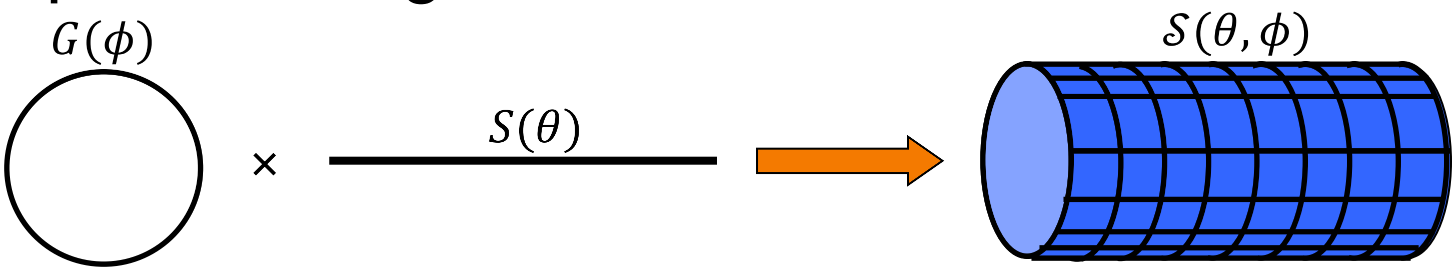
Outline

Spline Surfaces

Sweep Surfaces

Sweeps

Given a 3D *sweep curve* $S(\theta)$ and a 2D *generating curve* $G(\phi)$, define the *sweep surface* $\mathcal{S}(\theta,\phi)$ as the sweep of C along H:



In this example, the sweep curve is used to translate the generating curve:

$$\mathcal{S}(\theta,\phi)=S(\theta)+G(\phi)$$

We can define more complex sweep surfaces.

Example: Seashells

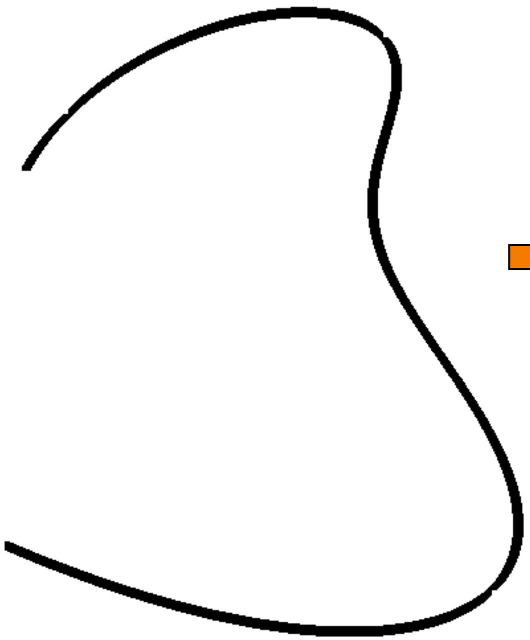
Create 3D polygonal surface models of seashells

"Modeling Seashells,"
Deborah Fowler, Hans Meinhardt,
and Przemyslaw Prusinkiewicz,
Computer Graphics (SIGGRAPH 92),
Chicago, Illinois, July, 1992, p 379-387.

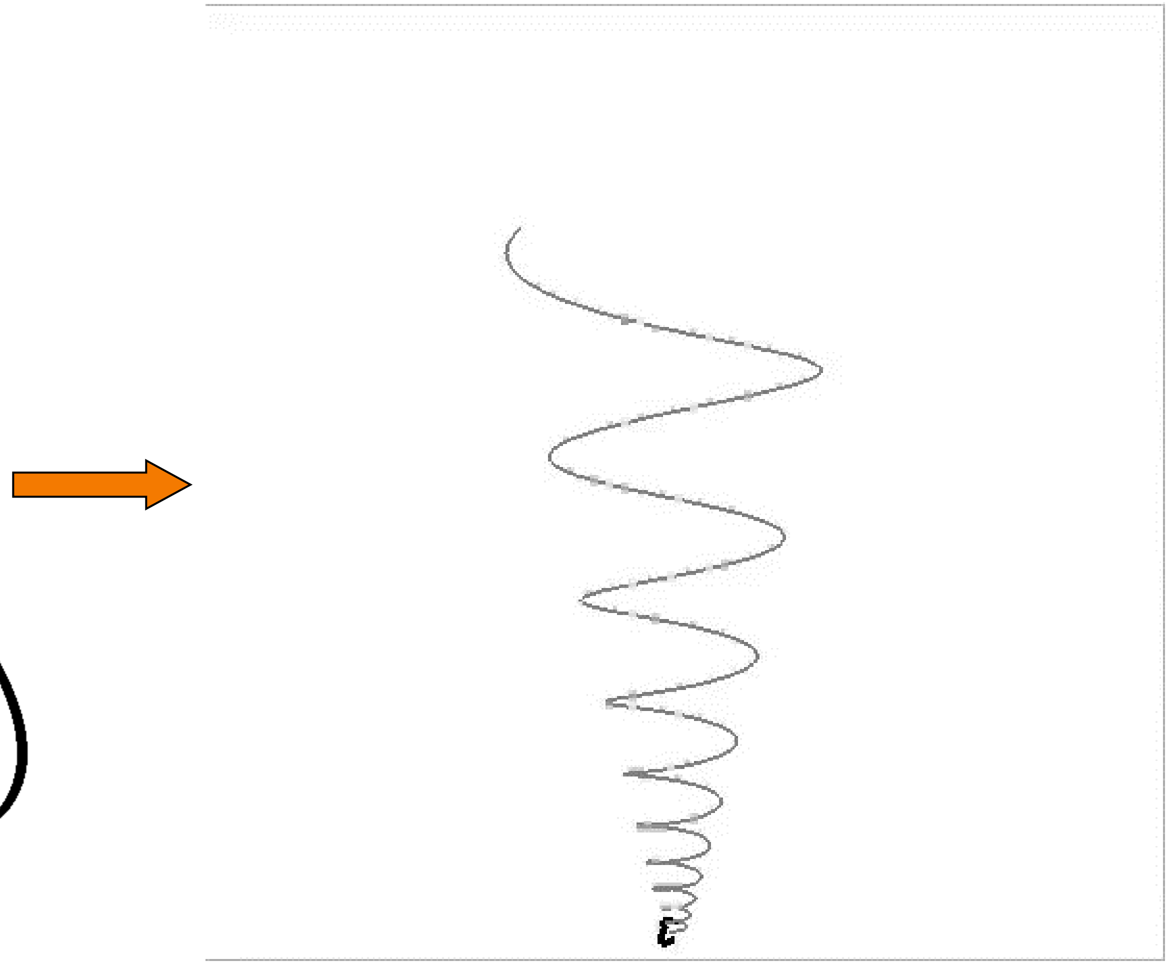
Fowler et al. Figure 7

Example: Seashells

Sweep generating curve around helico-spiral axis



Generating Curve



Example: Seashells

Sweep generating curve around helico-spiral axis

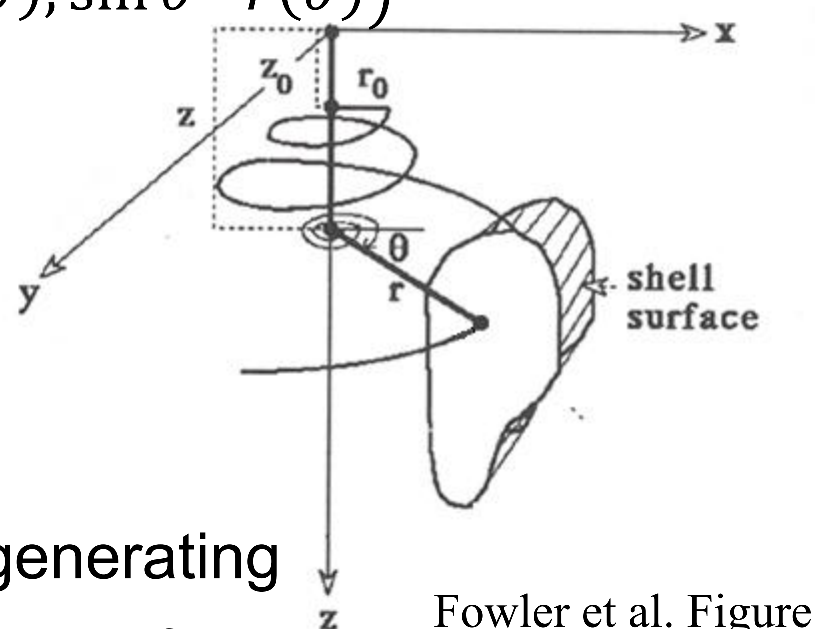
Helico-Spiral definition:

$$S(\theta) = \left(\cos\theta \cdot r(\theta), z(\theta), \sin\theta \cdot r(\theta)\right)$$

Angle: θ

Radius: $r(\theta) = e^{\lambda\theta}$

Height: $z(\theta) = e^{\mu\theta}$



Fowler et al. Figure 1

If $G(\phi) = \left(G_x(\phi), G_y(\phi)\right)$ is the generating curve, we can try to represent the surface as:

$$\mathcal{S}(\theta, \phi) = S(\theta) + \left(G_x(\phi), G_y(\phi), 0\right) \cdot r(\theta)$$

Example: Seashells

Sweep generating curve around helico-spiral axis

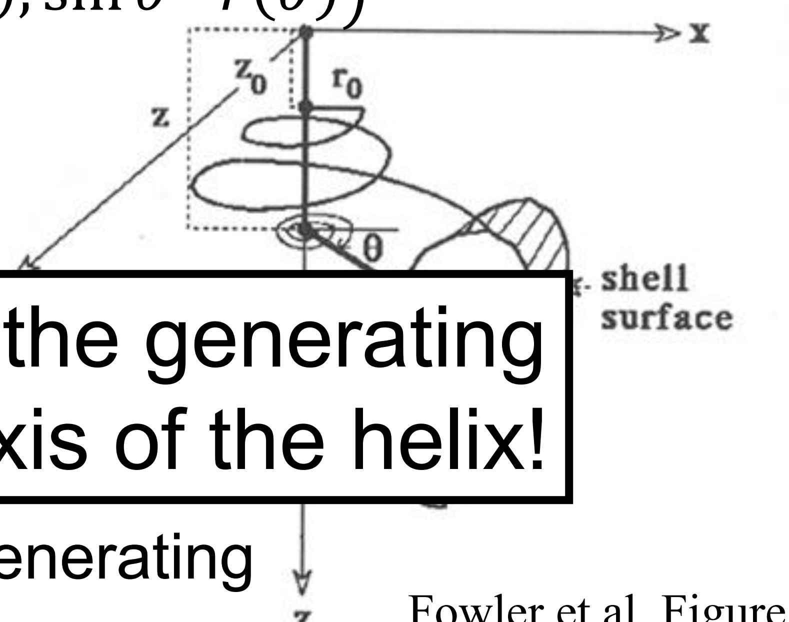
Helico-Spiral definition:

$$S(\theta) = \left(\cos\theta \cdot r(\theta), z(\theta), \sin\theta \cdot r(\theta)\right)$$

Angle: θ

Radius: $r(\theta) = e^{\lambda\theta}$

Height: $z(\theta) = e^{\mu\theta}$



Fowler et al. Figure 1

This doesn't rotate the generating curve around the axis of the helix!

If $G(\phi) = \left(G_x(\phi), G_y(\phi)\right)$ is the generating curve, we can try to represent the surface as:

$$\mathcal{S}(\theta, \phi) = S(\theta) + \left(G_x(\phi), G_y(\phi), 0\right) \cdot r(\theta)$$

Example: Seashells

Sweep generating curve around helico-spiral axis

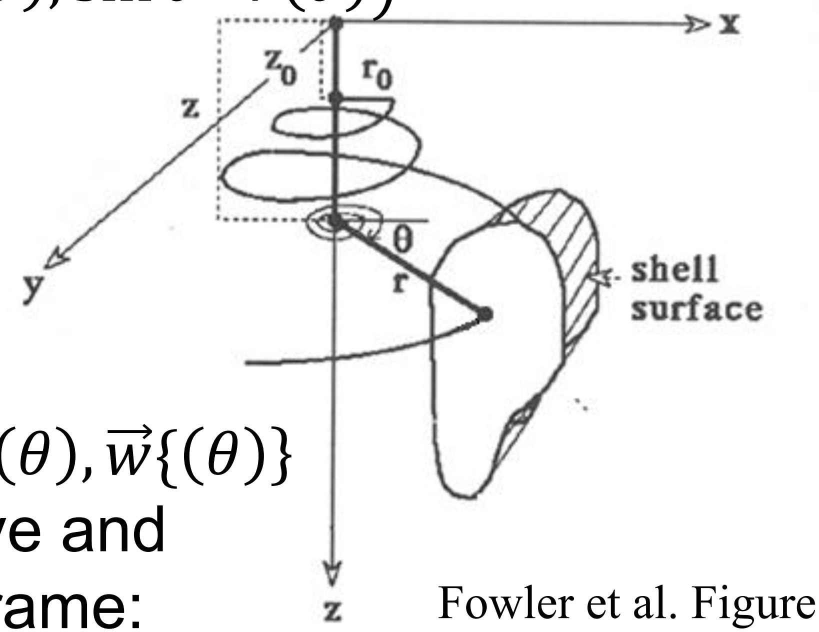
Helico-Spiral definition:

$$S(\theta) = \left(\cos\theta \cdot r(\theta), z(\theta), \sin\theta \cdot r(\theta)\right)$$

Angle: θ

Radius: $r(\theta) = e^{\lambda\theta}$

Height: $z(\theta) = e^{\mu\theta}$



Fowler et al. Figure 1

Compute a local **frame** $\{\vec{u}(\theta), \vec{v}(\theta), \vec{w}\{(\theta)\}$ at each point on the sweep curve and describe the surface w.r.t. this frame:

$$\mathcal{S}(\theta,\phi) = S(\theta) + \left(\vec{u}(\theta) \cdot G_x(\phi) + \vec{v}(\theta) \cdot G_y(\phi)\right) \cdot r(\theta)$$

Example: Seashells

Sweep ... axis

$\vec{u}(\theta)$ and $\vec{v}(\theta)$ define the plane that is perpendicular to the curve H at θ:

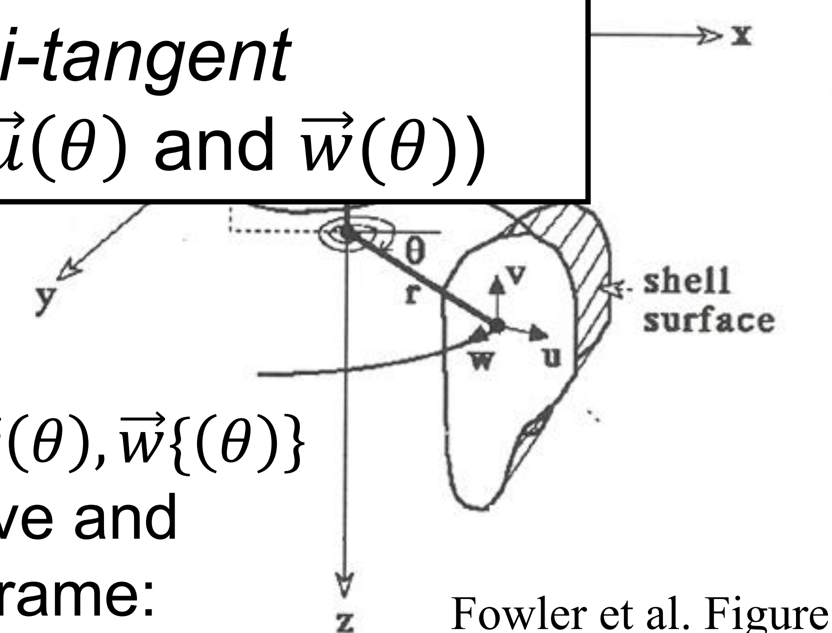
- $\vec{w}(\theta)$ is the curve *tangent*
- $\vec{u}(\theta)$ is the curve *normal*
- $\vec{v}(\theta)$ is the curve *bi-tangent* (perpendicular to $\vec{u}(\theta)$ and $\vec{w}(\theta)$)

Helico-

Angle:

Radius

Height: $z(\theta) = e^{\mu\theta}$



Fowler et al. Figure 1

Compute a local **frame** $\{\vec{u}(\theta), \vec{v}(\theta), \vec{w}\{(\theta)\}$ at each point on the sweep curve and describe the surface w.r.t. this frame:

$$\mathcal{S}(\theta, \phi) = S(\theta) + \left(\vec{u}(\theta) \cdot G_x(\phi) + \vec{v}(\theta) \cdot G_y(\phi)\right) \cdot r(\theta)$$

Example: Seashells

Generate different shells by varying parameters

Different helico-spirals

Fowler et al. Figure 2

Example: Seashells

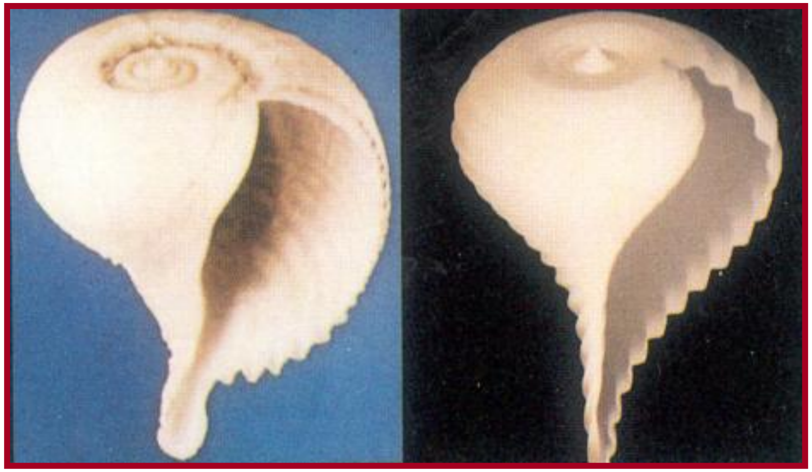
Generate different shells by varying parameters

Different generating curves

Fowler et al. Figure 3

Example: Seashells

Generate interesting shells with a simple procedural model!



Fowler et al. Figures 4,5,7