Modeling Transformations

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Announcements

Midterm is October 10th

Overview

- Ray-Tracing so far
- Modeling transformations

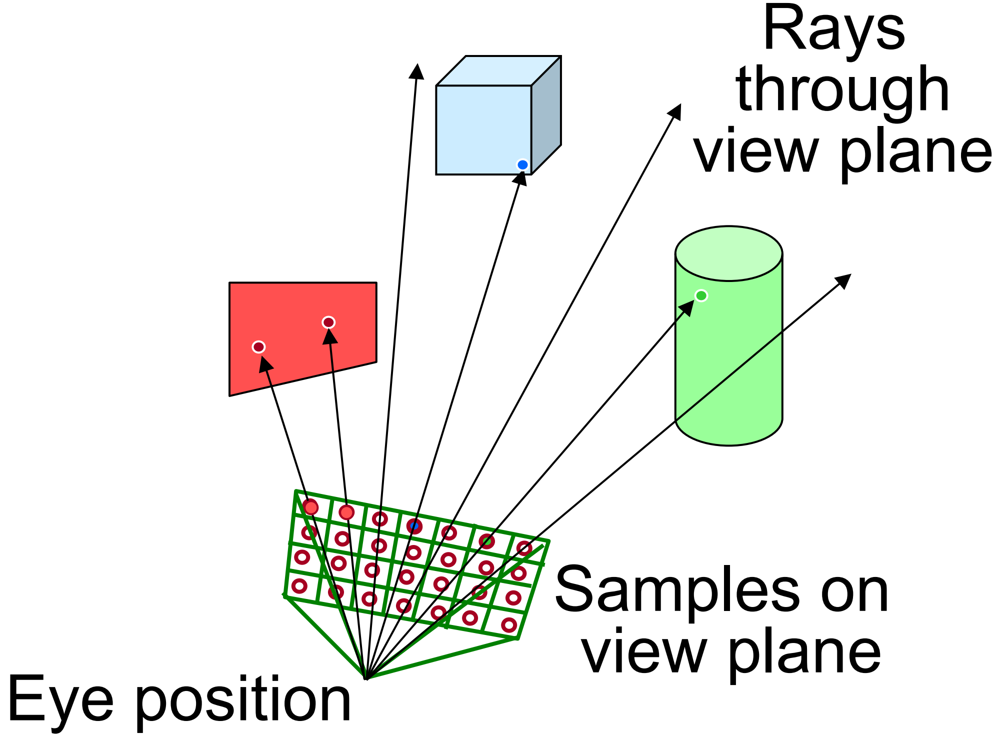
Ray Tracing

```
Image RayTrace( Camera camera , Scene scene , int width , int height , int depth , float cutoff )
{
	Image image( width , height );
	for( int i=0 ; i<width ; i++ ) for( int j=0 ; j<height ; j++ )
	{
		Ray< 3 > ray = ConstructRayThroughPixel( camera , i , j );
		image[i][j] = GetColor( scene , ray , 1. , depth , Color( cutOff , cutOff , cutOff ) );
	}
	return image;
}
```

Ray Tracing

For each sample …

- Construct ray from eye position through view plane
- Compute color contribution of the ray



Ray Tracing

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```

Constructing Ray Through a Pixel

2D Example: Side view of camera

- Where is the i-th pixel, $p[i]$, with $i \in [0, \text{height})$?

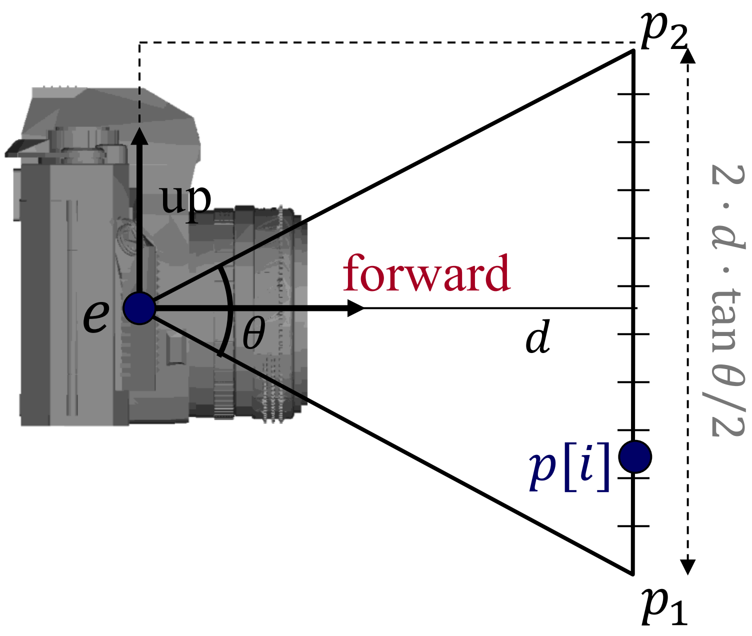
θ = field of view angle (given)

d = distance to view plane (arbitrary)

$$p_1 = e + d \cdot \text{forward} - d \cdot \tan\frac{\theta}{2} \cdot \text{up}$$

$$p_2 = e + d \cdot \text{forward} + d \cdot \tan\frac{\theta}{2} \cdot \text{up}$$

$$p[i] = p_1 + \left(\frac{i + 0.5}{\text{height}}\right) \cdot (p_2 - p_1)$$



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Ray Tracing

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Image RayTrace( Camera camera , Scene scene , int width , int height , int depth , float cutoff )
{
        Image image( width , height );
```

```
Color GetColor( Scene scene , Ray< 3 > ray , float ir , int rDepth , Color cutOff )
{
        Color c(0,0,0);
        Ray< 3 > reflect , refract;
        if( !rDepth || ( cutOff[0]>1 && cutOff[1]>1 && cutOff[2]>1 ) ) return c;
        HitInformation hit;

        if( FindIntersection( ray , scene , hit ) )
        {
                c += GetSurfaceColor( hit.position );
                if( Dot( ray.direction , hit.normal )<0 )
                {
                        reflect.direction = Reflect( ray.direction , hit.normal );
                        reflect.position = hit.position + reflect.direction*ε;
                        c += GetColor( scene , reflect , ir , rDepth-1 , cutOff/hit.kSpec )*hit.kSpec;
                }
                refract.direction = Refract( ray.direction , hit.normal , ir , hit.ir );
                refract.position = hit.position + refract.direction*ε;
                c += GetColor( scene , refract , hit.ir , rDepth-1 , cutOff/hit.kTran )*hit.kTran;
        }
        return c;
}
```

Ray-Scene Intersection

Intersections with geometric primitives

- Sphere
- Triangle
- Groups of primitives (scene)

Acceleration techniques

- Bounding volume hierarchies
- Spatial partitions
  - » Uniform grids
  - » Octrees
  - » BSP trees

Ray Tracing

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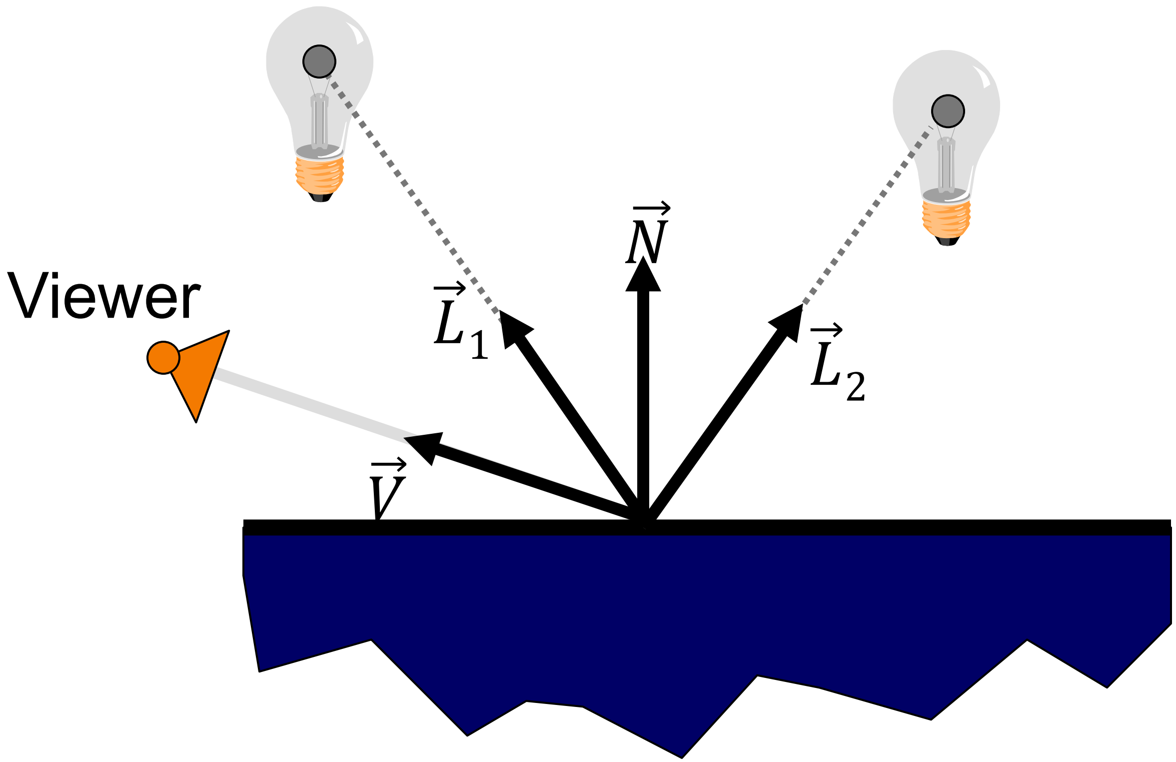
```

Surface Illumination Calculation



$$I = K_E + \sum_L \left[K_A \cdot I_L^A + \left(K_D \cdot \langle \vec{N}, \vec{L} \rangle + K_S \cdot \langle \vec{V}, \vec{R}(\vec{L}) \rangle^{K_n}\right) \cdot I_L \cdot S_L\right] + K_s \cdot I_R + K_T \cdot I_T$$

Ray Tracing

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Image RayTrace( Camera camera , Scene scene , int width , int height , int depth , float cutoff )
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		c += GetColor( scene , refract , hit.ir , rDepth-1 , cutOff/hit.kTran )*hit.kTran;
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	return c;
}
```
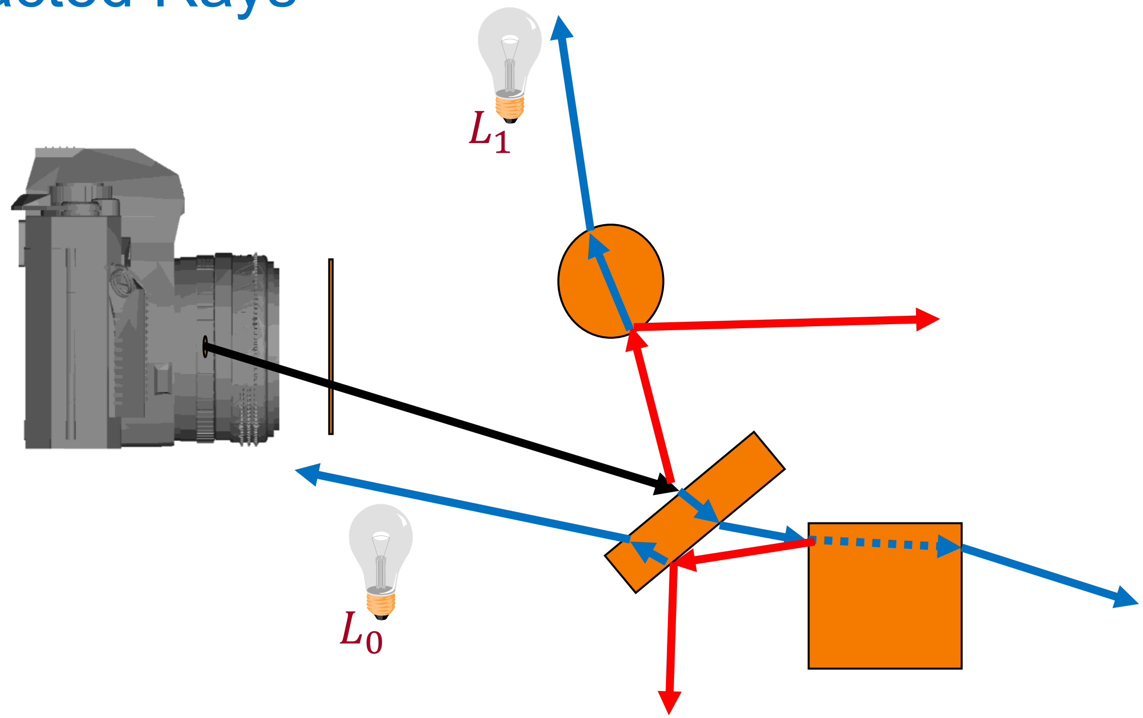

Ray Tracing (Recursive)

Consider the contribution of:

- Reflected Rays
- Refracted Rays



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Overview

- Raytracing so far
- Modeling transformations

Modeling Transformations

Specify transformations for objects allows:

- Defining objects in their own coordinate systems
- Using one object definition multiple times in a scene

Overview

2D Transformations

- Basic 2D transformations
- Matrix representation
- Matrix composition

3D Transformations

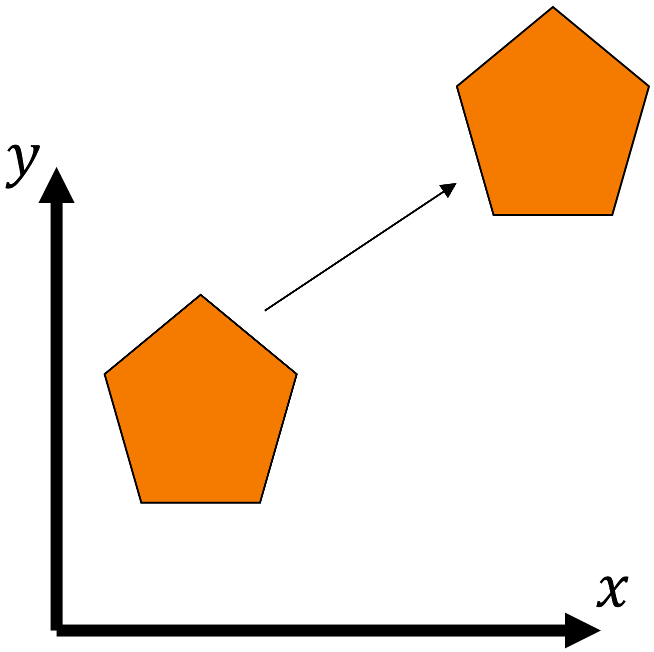
- Basic 3D transformations
- Same as 2D

Simple 2D Transformations

Translation

$$p' = p + t$$

$$\begin{bmatrix} p'_x \\ p'_y \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

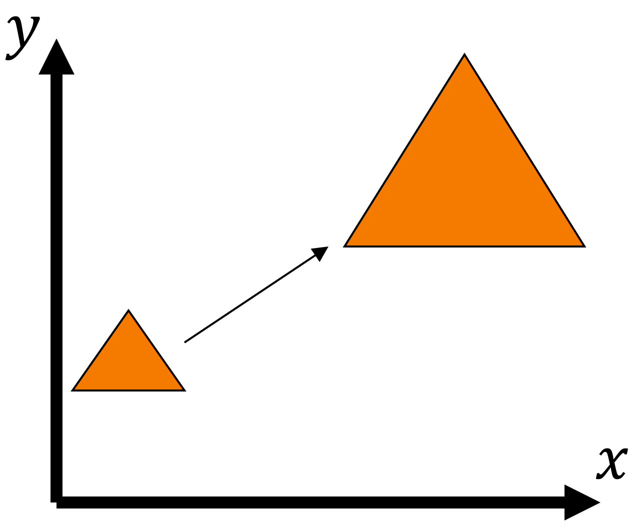


Simple 2D Transformations

Scale

$$p' = S \cdot p$$

$$\begin{bmatrix} p'_x \\ p'_y \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

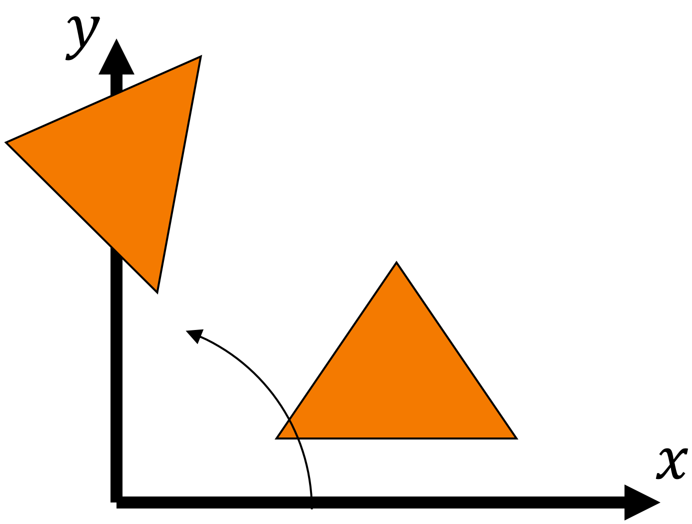


Simple 2D Transformation

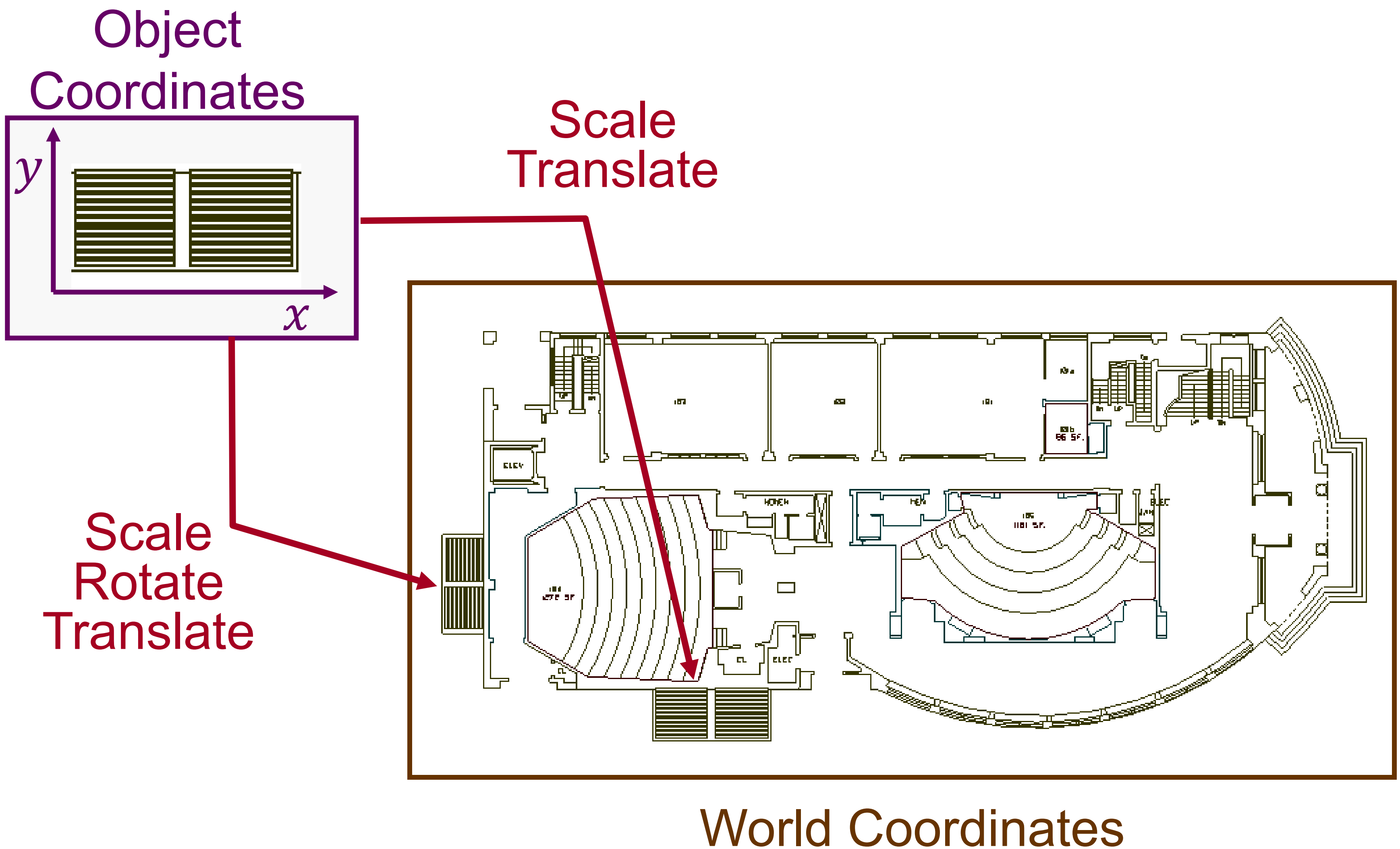
Rotation

$$p' = R \cdot p$$

$$\begin{bmatrix} p'_x \\ p'_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

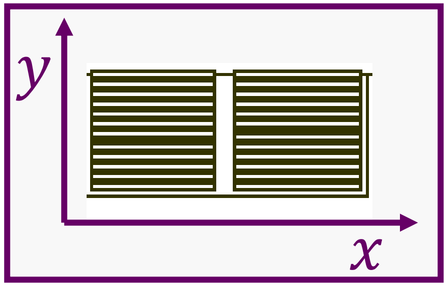


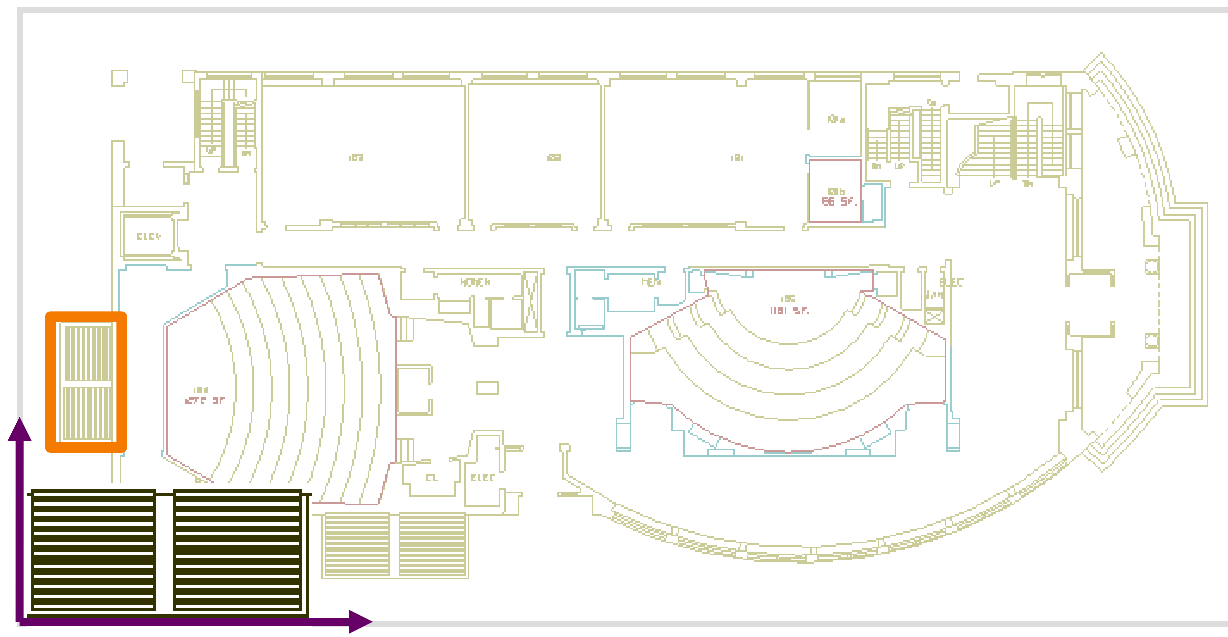
2D Modeling Transformations



2D Modeling Transformations

Object Coordinates





2D Modeling Transformations

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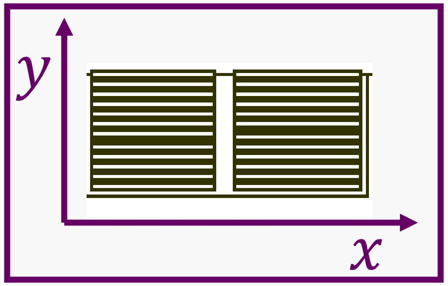
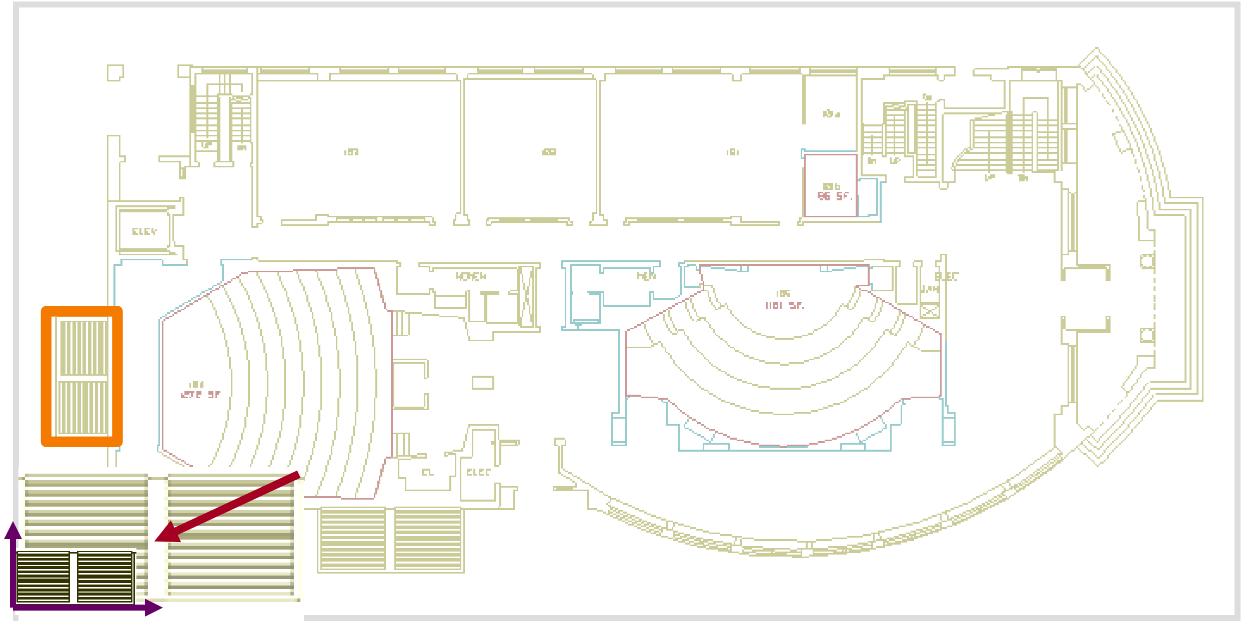
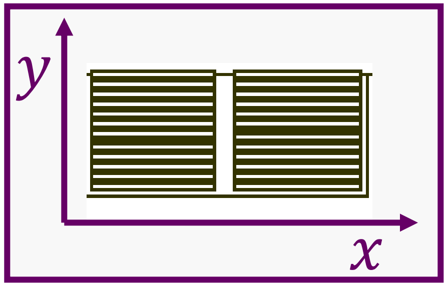


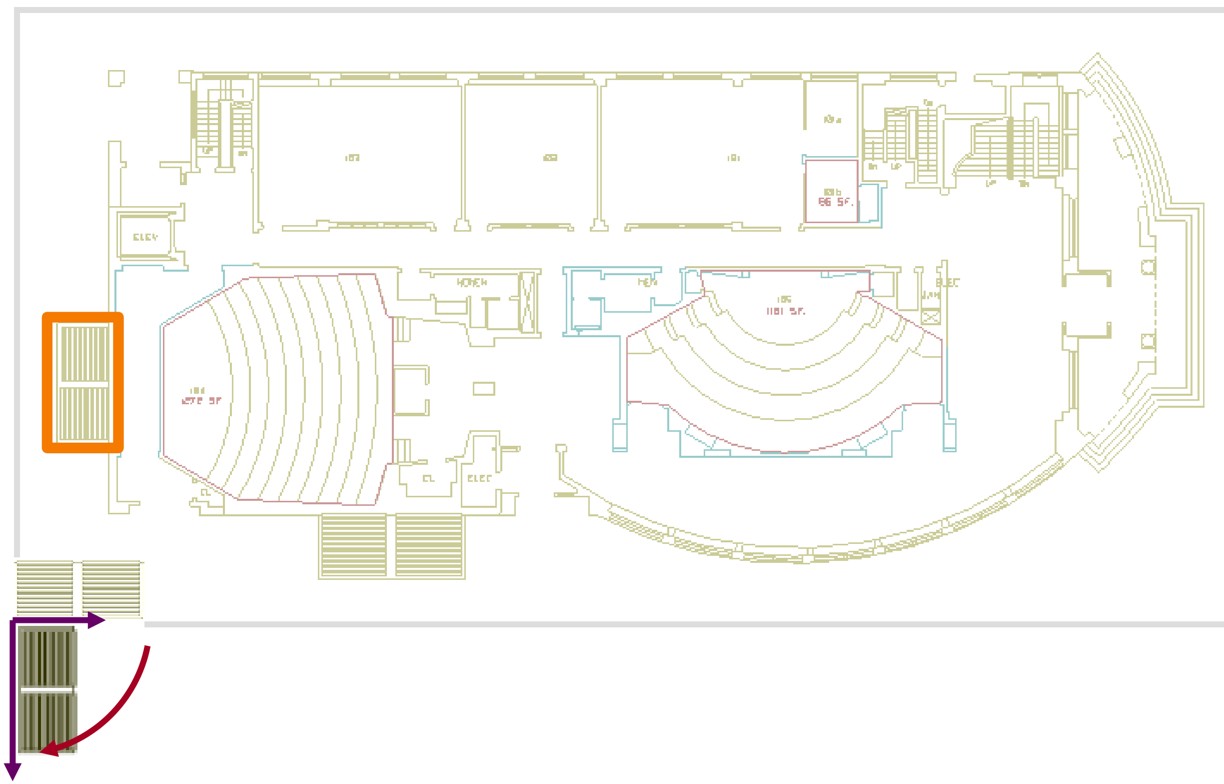
Scale .3, .3



2D Modeling Transformations

Object
Coordinates

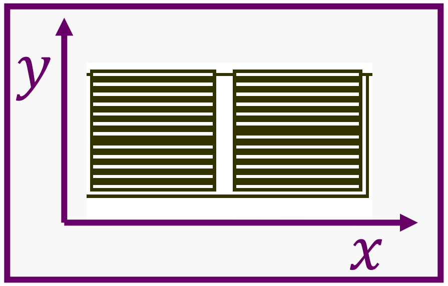




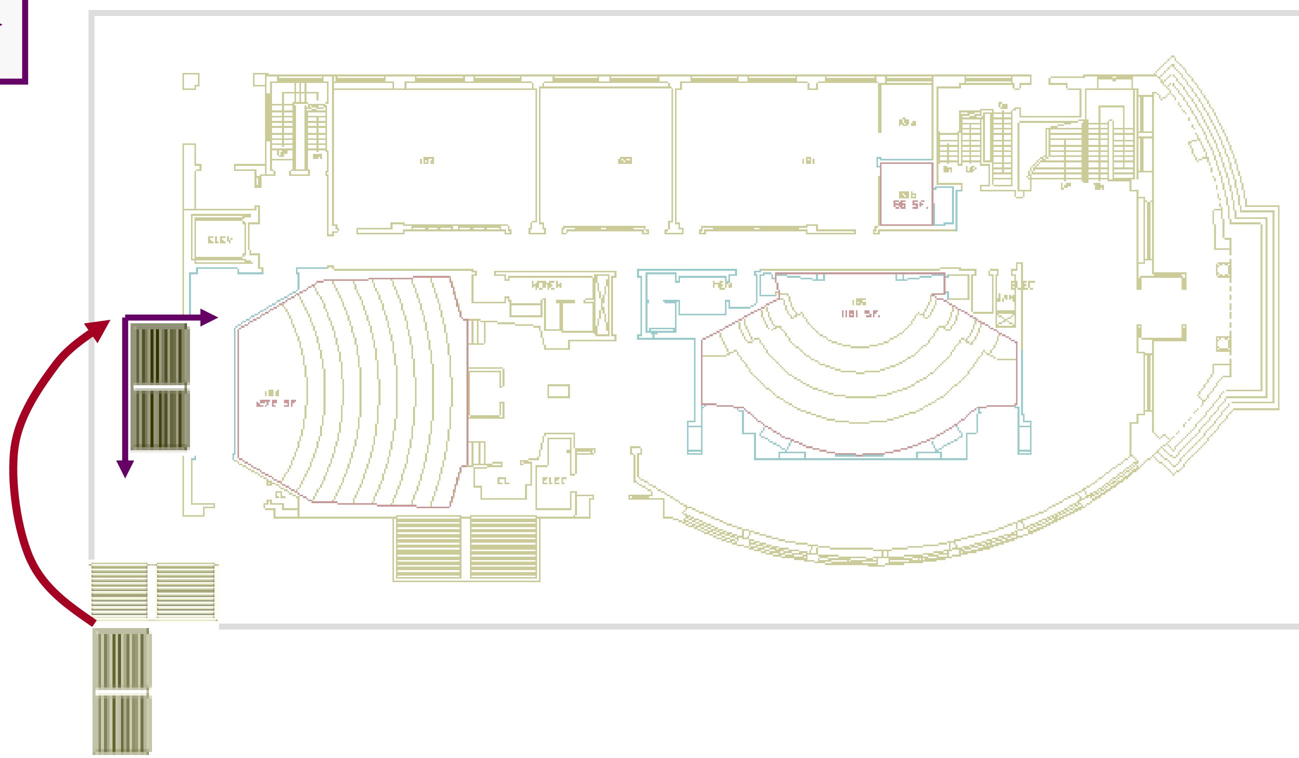
Scale .3, .3
Rotate -90

2D Modeling Transformations

Object Coordinates

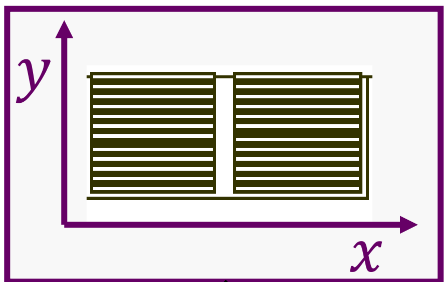


Scale .3, .3
Rotate -90
Translate 3, 5

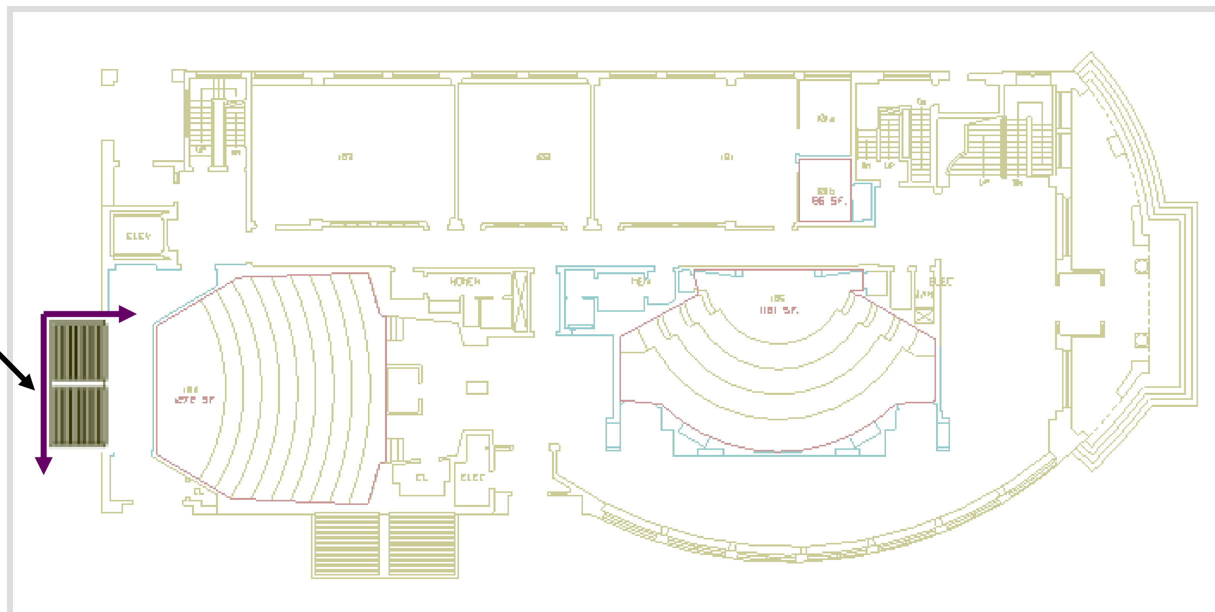


2D Modeling Transformations

Object Coordinates



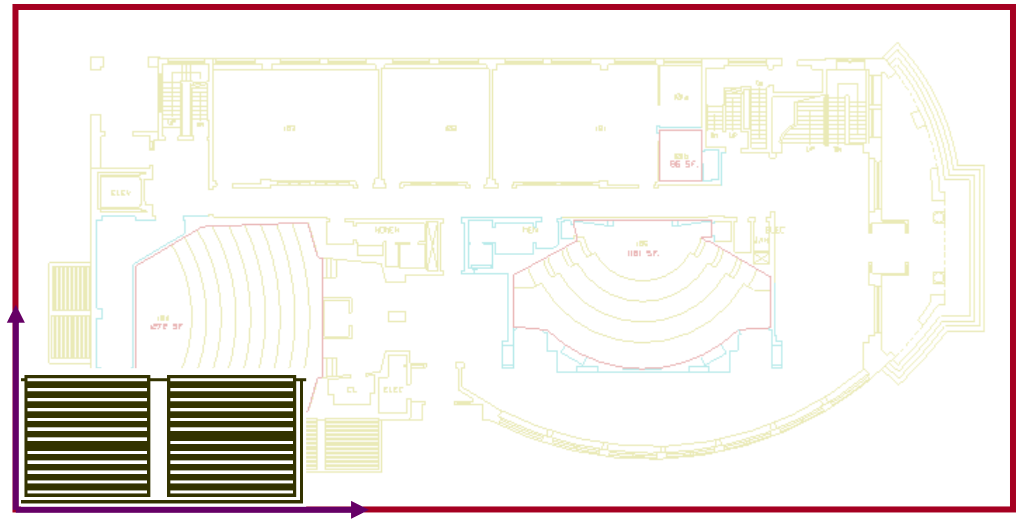
The composition take us from object to world coordinates



Scale .3, .3
Rotate -90
Translate 3, 5

Composing 2D Transformations

- Translation:
  - $x' = x + t_x$
  - $y' = y + t_y$
- Scale:
  - $x' = x \cdot s_x$
  - $y' = y \cdot s_y$
- Rotation:
  - $x' = x \cdot \cos\theta - y \cdot \sin\theta$
  - $y' = x \cdot \sin\theta + y \cdot \cos\theta$



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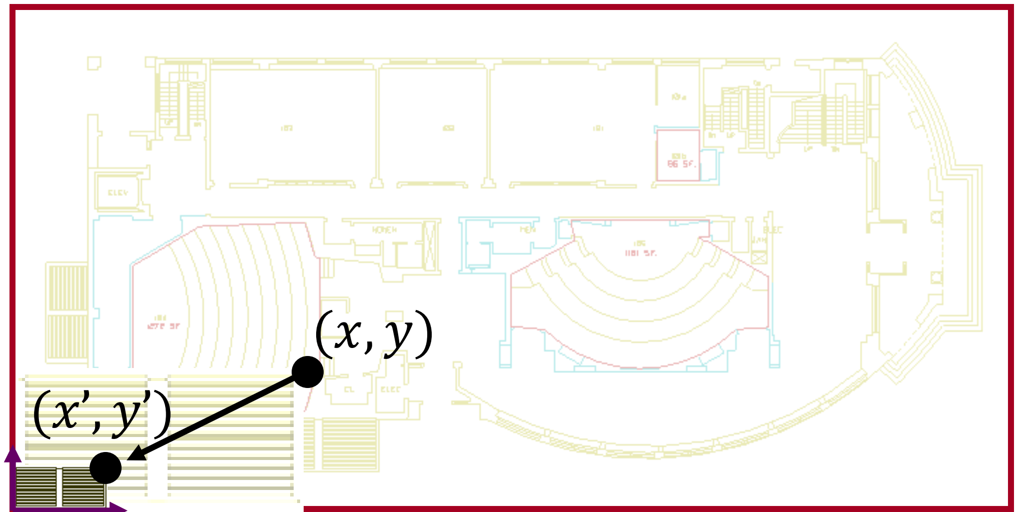

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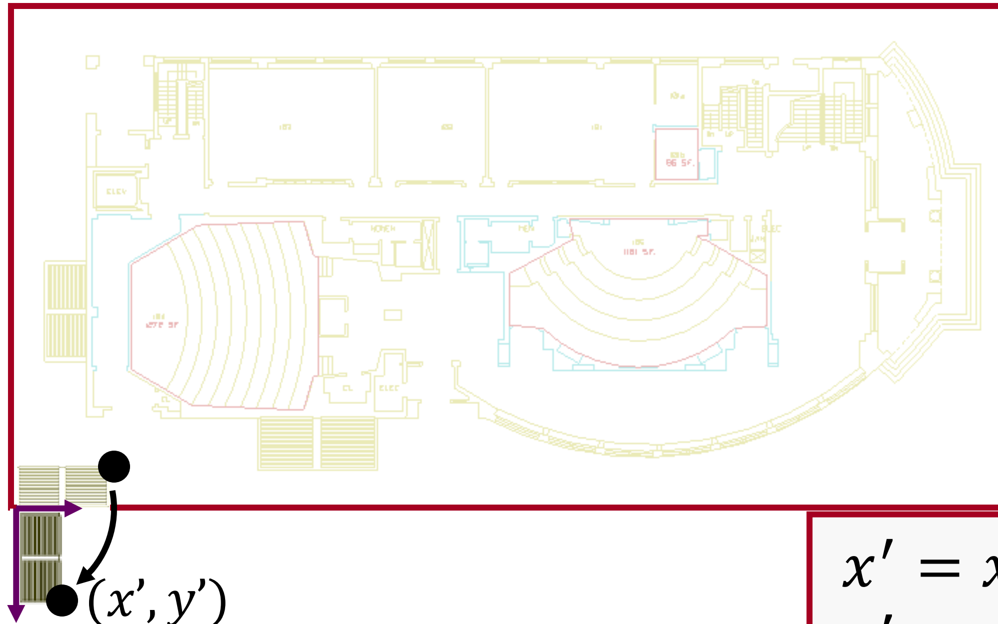

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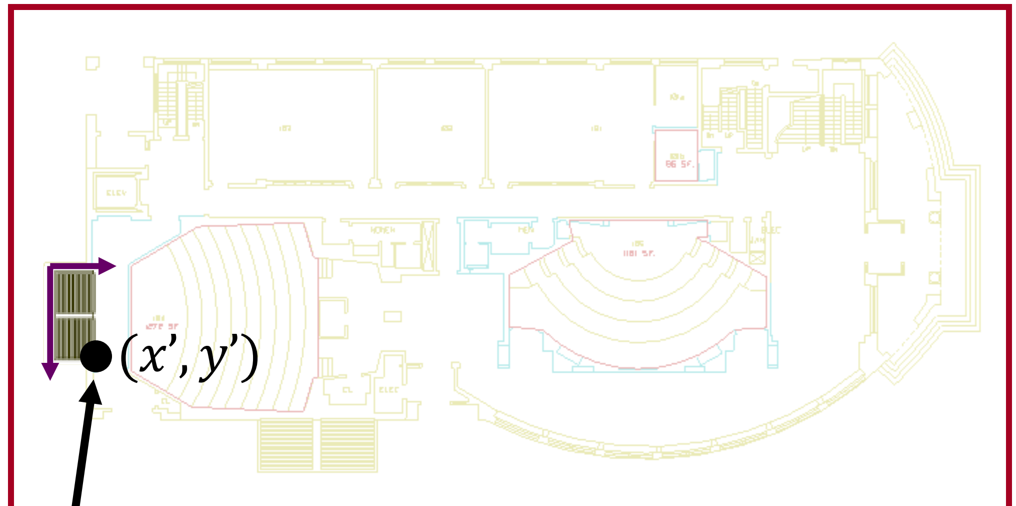

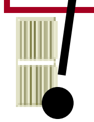

$$x' = x \cdot s_x$$
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$$x' = (x \cdot s_x) \cdot \cos\theta - (y \cdot s_y) \cdot \sin\theta + t_x$$
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Composing 2D Transformations

Naïve composition makes the expression more complicated as more transformations are applied!

- Tra
  - $x' = x + t_x$
  - $y' = y + t_y$
- Scale:
  - $x' = x \cdot s_x$
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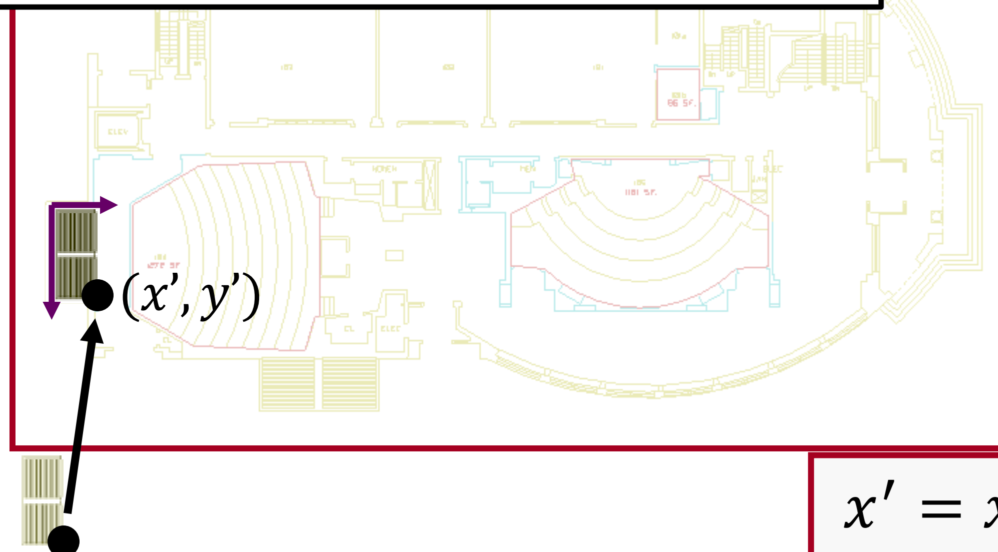

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Overview

- 2D Transformations
  - Basic 2D transformations
  - Matrix representation
  - Matrix composition
- 3D Transformations
  - Basic 3D transformations
  - Same as 2D

Matrix Representation

Represent 2D transformation by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Multiply matrix by column vector

$\Updownarrow$

apply transformation to point

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \Leftrightarrow \quad \begin{aligned} x' &= a \cdot x + b \cdot y \\ y' &= c \cdot x + d \cdot y \end{aligned}$$

Matrix Representation

Transformation composition is matrix multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix}$$

$\Rightarrow$ The composition is still represented by matrix

2x2 Matrices

What transforms can we represent with a matrix?

2D Scale around (0,0)?

$$x' = s_x \cdot x \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$y' = s_y \cdot y$$

2D Rotate around (0,0)?

$$x' = \cos\theta \cdot x - \sin\theta \cdot y \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$y' = \sin\theta \cdot x + \cos\theta \cdot y$$

2D Mirror over Y axis?

$$x' = -x \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$y' = \ \ y$$

2x2 Matrices

What transforms can we represent with a matrix?

2D Scale around (0,0)?

$$x' = s_x \cdot x$$
$$y' = s_y \cdot y$$

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Like scale with negative scale values

2D Mirror over Y axis?

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2D Translation?

$$x' = x + t_x$$
$$y' = y + t_y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

What transforms can we represent with a matrix?

2D Translation?

$$x' = x + t_x$$
$$y' = y + t_y$$

NO!

Only <u>linear</u> 2D transformations can be represented with a 2×2 matrix

Linear Transformations

Linear transformations are combinations of …

- Scale, and
- Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Properties of linear transformations:

- Satisfies: $T(s_1 \cdot p_1 + s_2 \cdot p_2) = s_1 \cdot T(p_1) + s_2 \cdot T(p_2)$

⇒ Origin maps to origin

⇒ Lines map to lines

⇒ Preserves (weighted) average

⇒ Parallel lines remain parallel

⇒ Closed under composition

Linear Transformations

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⇒ Closed under composition

Translations do not map the origin to the origin

2D Translation

Treat 2D positions as 3D positions by adding a third, *homogenous*, coordinate with fixed value "1":

$$(x, y) \to (x, y, 1)$$

- Represent translations using a 3x3 matrix:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow \begin{aligned} x' &= x + t_x \\ y' &= y + t_y \end{aligned}$$

Basic 2D Transformations

Basic 2D transformations as 3×3 matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

Homog. Coordinates: $(x, y, 1) \leftrightarrow (x, y)$

More generally:

- For $w \neq 0$ we associate $(x, y, w) \leftrightarrow \left(\frac{x}{w}, \frac{y}{w}\right)$

What about when $w = 0$?

Consider the limit:

$$\lim_{w \to 0} (x, y, w) \leftrightarrow \lim_{w \to 0} \left(\frac{x}{w}, \frac{y}{w}\right)$$

$\Rightarrow$ In the limit this is the *ideal point* at infinity in direction (x, y)…
… also the ideal point in direction $(-x, -y)$

Homog. Coordinates: $(x, y, 1) \leftrightarrow (x, y)$

More generally:

- For $w \neq 0$ we associate $(x, y, w) \leftrightarrow \left(\frac{x}{w}, \frac{y}{w}\right)$
- We associate $\lim_{w \to 0} (x, y, w) \leftrightarrow \lim_{w \to 0} \left(\frac{x}{w}, \frac{y}{w}\right)$
- $(0, 0, 0)$ is not allowed

$\Rightarrow$ In addition to supporting translation, homogenous coordinates describe geometry at infinity.

Homog. Coordinates: $(x, y, 1) \leftrightarrow (x, y)$

More generally:

- For $w \neq 0$ we associate $(x, y, w) \leftrightarrow \left(\frac{x}{w}, \frac{y}{w}\right)$
- We associate $\lim_{w \to 0} (x, y, w) \leftrightarrow \lim_{w \to 0} \left(\frac{x}{w}, \frac{y}{w}\right)$

Note:
As defined, the points $(a, b, 0)$ and $(-a, -b, 0)$ represent the same point at infinity.

Warning:
OpenGL distinguishes these as the two end-points of the line with equation $bx = ay$, allowing for representation of directional light sources.

Affine Transformations

Affine transformations are combinations of …

- Linear transformations, and
- Translations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Properties of affine transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Preserves (weighted) average
- Parallel lines remain parallel
- Closed under composition

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Note that with affine transformations $(x, y, 1)$ has to map to $(x', y', 1)$

Properti

- Orig
- Lines map to lines
- Preserves (weighted) average
- Parallel lines remain parallel
- Closed under composition

Projective Transformations

Projective transformations …

- Affine transformations, and
- Projective warps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of projective transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- (Weighted) average is not necessarily preserved
- Parallel lines do not necessarily remain parallel
- Closed under composition

Projective Transformations

Projective transformations …

- Affine transformations, and
- Projective warps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Note that under projective transformations $(x, y, 1)$ does **not** have to map to $(x', y', 1)$

Properti

- Orig
- Lines map to lines
- (Weighted) average is not necessarily preserved
- Parallel lines do not necessarily remain parallel
- Closed under composition

Overview

2D Transformations

- Basic 2D transformations
- Matrix representation
- Matrix composition

3D Transformations

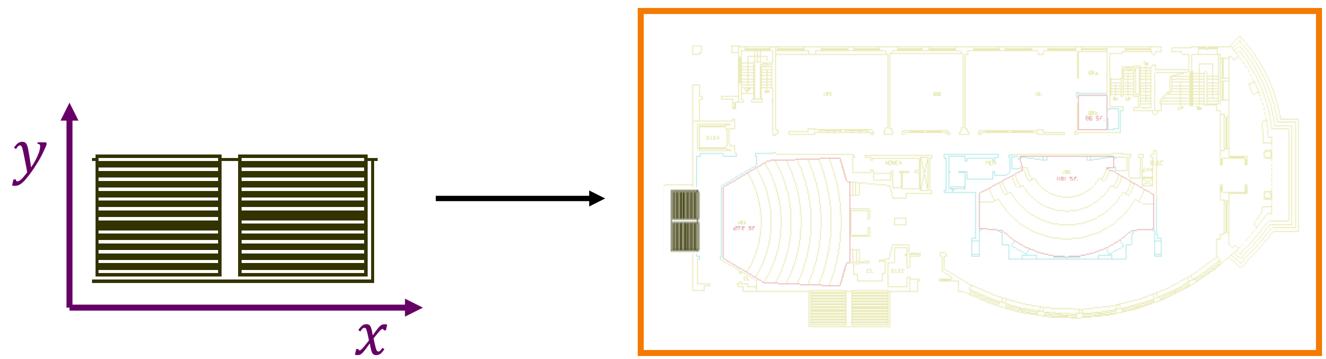
- Basic 3D transformations
- Same as 2D

Matrix Composition

Transformations combine with matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$p' = T(t_x, t_y) \circ R(\theta) \circ S(s_x, s_y) \; p$$



Matrix Composition

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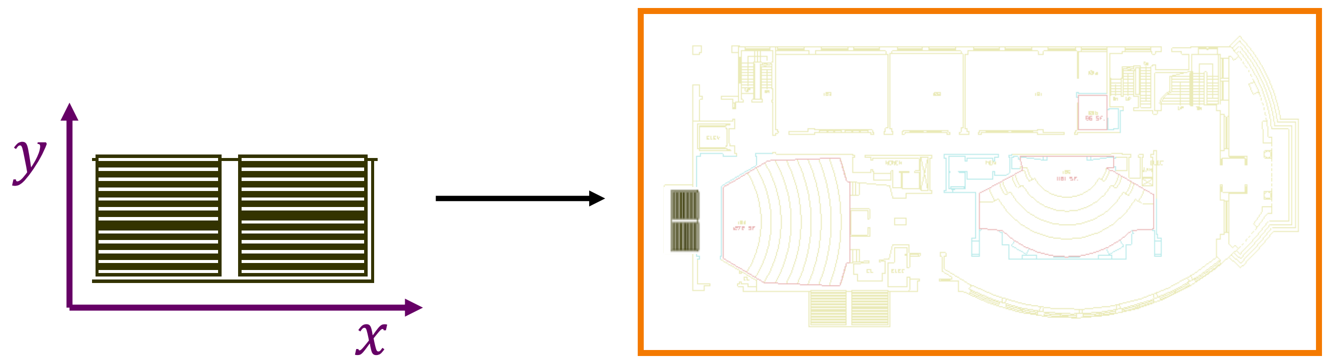
$$= \begin{bmatrix} s_x \cdot \cos\theta & -s_y \cdot \sin\theta & t_x \\ s_x \cdot \sin\theta & s_y \cdot \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Matrix Composition

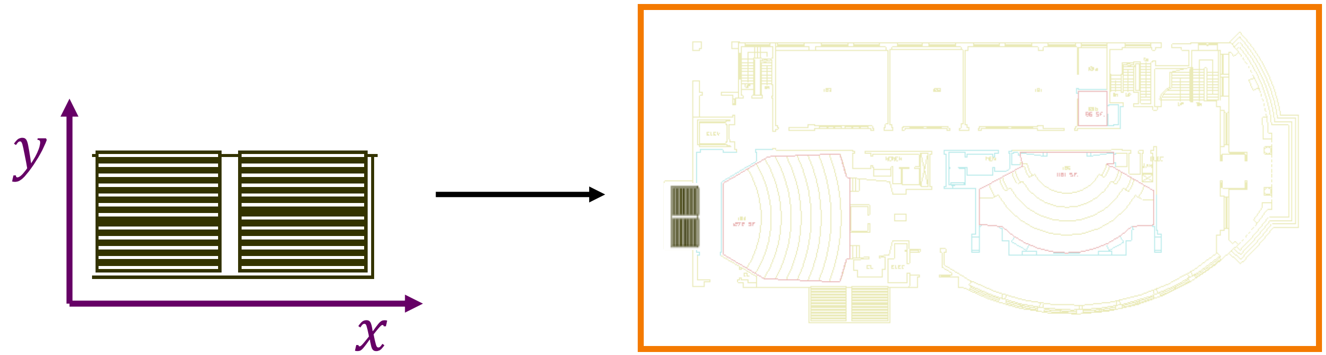
Transformations combine with matrix multiplication

- General purpose representation
- Efficiently implemented with matrix (pre-)multiplication

$$p' = T\left(R\left(S(p)\right)\right)$$

$$\Updownarrow$$

$$p' = (T \circ R \circ S)(p)$$



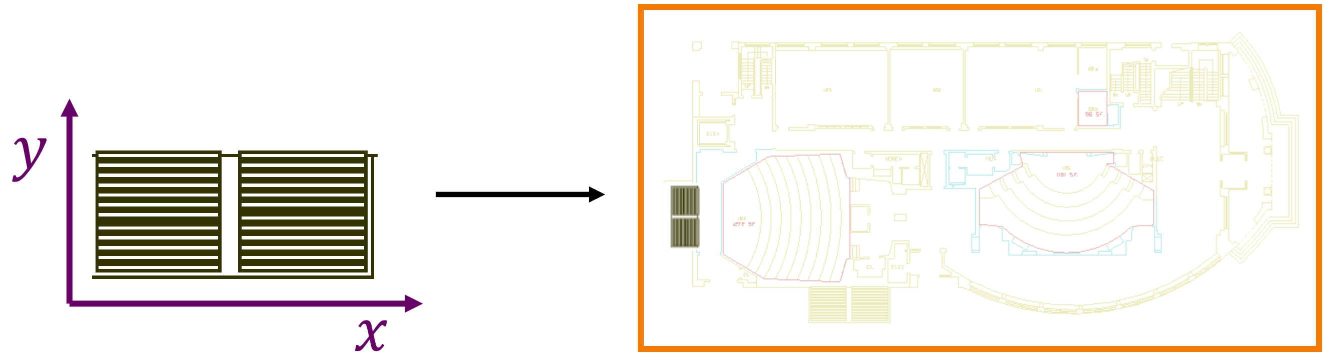
Matrix Composition

[NOTE] order of transformations matters

- Matrix multiplication is not commutative

$$p' = T \cdot R \cdot S \cdot p$$

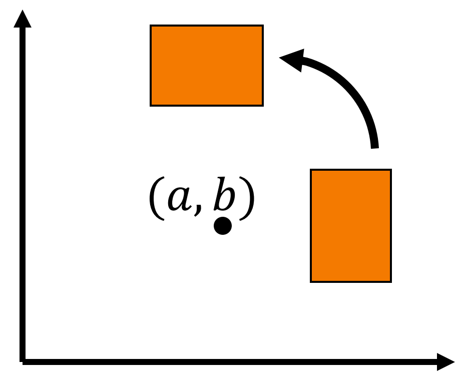
"Global" ⟷ "Local"



Matrix Composition

Rotate by θ around arbitrary point (a, b)

- $M = T(a, b) \circ R(\theta) \circ T(-a, -b)$

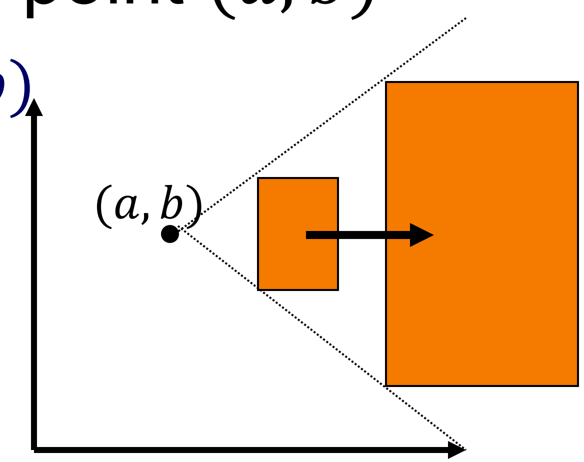


Approach:

1. Translate (a, b) to the origin.
2. Do the rotation about origin.
3. Translate back.

Scale by (s_x, s_y) around arbitrary point (a, b)

- $M = T(a, b) \circ S(s_x, s_y) \circ T(-a, -b)$



(Use the same approach.)

Overview

2D Transformations

- Basic 2D transformations
- Matrix representation
- Matrix composition

3D Transformations

- Basic 3D transformations
- Same as 2D

3D Transformations

Same idea as 2D transformations

- Homogeneous coordinates: (x, y, z, w)
- 4×4 transformation matrices
  - » Affine

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

  - » Projective

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Basic 3D Transformations

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Identity

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Translation

Basic 3D Transformations

Pitch-Roll-Yaw Convention:

- Any rotation can be expressed as the combination of a rotation about the x-, the y-, and the z-axis.

Rotate around z axis: $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotate around y axis: $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotate around x axis: $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Basic 3D Transformations

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Rotate aroun

How do you rotate around an arbitrary axis U by angle ψ?

Rotate around x axis:
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

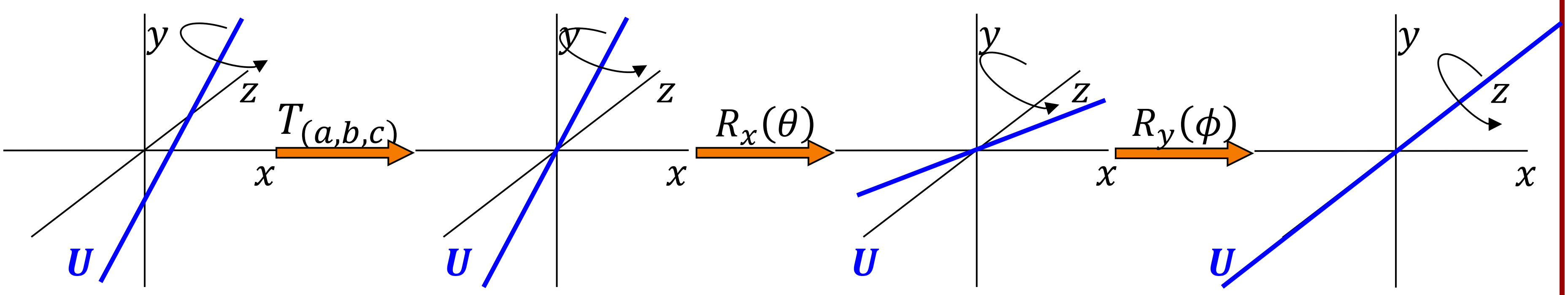
Rotation By ψ Around Arbitrary Axis U

Align U (w.l.o.g.) with the z-axis:

- $T_{(a,b,c)}$: Translate U by (a, b, c) to pass through origin
- $R_x(\theta)$: Rotate about the x-axis by θ to get U in the xz-plane
- $R_y(\phi)$: Rotate about the y-axis by ϕ to align U with the z-axis

$R_z(\psi)$: Perform rotation by ψ around the z-axis.

Do inverse of original transformation for alignment.



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- $R_y(\phi)$: Rotate about the y-axis by ϕ to align U with the z-axis

$R_z(\psi)$: Perform rotation by ψ around the z-axis.

Do inverse of original transformation for alignment.

$$p' = \underbrace{\left(R_y(\phi)\cdot R_x(\theta)\cdot T_{(a,b,c)}\right)^{-1}}\cdot R_z(\psi)\cdot \underbrace{\left(R_y(\phi)\cdot R_x(\theta)\cdot T_{(a,b,c)}\right)}p$$

Aligning Transformation