Moving Least Squares

[McLain, 1974]

Preliminaries

Gradients:

Given a function $F: \mathbb{R}^3 \to \mathbb{R}$, the *gradient* of *F* is the vector valued function $\nabla F: \mathbb{R}^3 \to \mathbb{R}^3$, with:

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right).$$

Preliminaries

Extrema:

Given a function $F: \mathbb{R}^3 \rightarrow \mathbb{R}$, a point $p \in \mathbb{R}^3$ is an *extremum* of F only if the gradient of F is zero at p:

$$\nabla F|_p = 0.$$

Preliminaries

Dot Products:

If $F: \mathbb{R}^3 \to \mathbb{R}$ has the form:

$$F(p) = \langle p, q \rangle$$

for some fixed $q \in \mathbb{R}^3$, then

$$\nabla F|_p = q^t.$$

Preliminaries

<u>Dot Products</u>:

If $F: \mathbb{R}^3 \to \mathbb{R}$ has the form:

$$F(p) = \langle p, q \rangle^n$$

for some fixed $q \in \mathbb{R}^3$, then

$$\nabla F|_p = n \langle p, q \rangle^{n-1} q^t.$$

Preliminaries

Dot Products:

If $F\colon \mathbb{R}^3 \to \mathbb{R}$ has the form:

$$F(p) = \langle p, p \rangle,$$

then

$$\nabla F|_p = 2p^t.$$

Preliminaries

Dot Products:

If $F: \mathbb{R}^3 \to \mathbb{R}$ has the form:

$$F(p) = \langle p, Mp \rangle,$$

then

$$\nabla F|_p = p^t M^t + p^t M.$$

Preliminaries

<u>Quadratic Polynomials</u>:

If $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a quadratic polynomial:

$$F(x,y) = c_{00} + c_{10}x + c_{01}y + c_{11}xy + c_{20}x^2 + c_{02}y^2,$$

then we can write out $F(x,y)$ as a dot-product

$$F(x,y) = \langle v, Q(x,y) \rangle$$

with:

$$v = (c_{00}, c_{10}, c_{01}, c_{11}, c_{20}, c_{02})^t$$

and:

$$Q(x,y) = (1, x, y, xy, x^2, y^2)^t.$$

Preliminaries

Lagrangians:

The extrema of F, subject to the constraint $G(p) = c$, can be found by solving:

$$\nabla F\Big|_p = \lambda \nabla G\Big|_p$$

(That is, at p, the function F should only be changing in a direction that is perpendicular to the constraint.)

Preliminaries

Lagrangians and Symmetric Matrices:

If M is a symmetric matrix then, p is an extremum of:

$$F(p) = \langle p, Mp \rangle = p^t M p,$$

subject to the constraint $\|p\|^2 = 1$ if and only if p is an eigenvector of M:

$$\lambda 2p^t = \nabla F|_p = p^t M^t + p^t M = 2p^t M.$$

Or equivalently:

$$\lambda p = Mp.$$

Challenge

Given a set of 2D points $\{p_i\} \subset \mathbb{R}^2$, with associated real values $\phi_i \in \mathbb{R}$, define a function $\Phi: \mathbb{R}^2 \to \mathbb{R}$ fits/approximates the sample data.

Weighted Averaging

Given a set of 2D points $\{p_i\} \subset \mathbb{R}^2$, with associated real values $\phi_i \in \mathbb{R}$, define:

$$\Phi(p) = \frac{\sum_i \Theta(\|p_i - p\|)\phi_i}{\sum_i \Theta(\|p_i - p\|)}$$

Properties of the weight function $\Theta(p)$:

- It should drop off with distance
- Drop off too slow $\rightarrow$ blurring
- Drop off too slow $\rightarrow$ numerical instability
- Interpolation $\Rightarrow \Theta(0) = \infty$

[McLain 1974]

Key Idea:

Weighted averaging can be viewed as a form of function fitting:

Let $\mathcal{L}$ be a space of functions.

At each point p, find $\phi_p \in \mathcal{L}$ minimizing:

$$E(\phi_p) = \sum_i \Theta(\|p_i - p\|)\big(\phi_p(p_i) - \phi_i\big)^2$$

and set:

$$\Phi(p) = \phi_p(p).$$

[McLain 1974]

Constant Functions:

When $\mathcal{L}$ is the space of constant functions, this reduces to solving for $\phi_p \in \mathbb{R}$ minimizing:

$$E(\phi_p) = \sum_i \Theta(\|p_i - p\|)(\phi_p - \phi_i)^2.$$

Taking the gradient and setting to zero gives:

$$\phi_p = \frac{\sum_i \Theta(\|p_i - p\|)\phi_i}{\sum_i \Theta(\|p_i - p\|)} = \Phi(p),$$

which is just the weighted-average from above.

[McLain 1974]

Quadratic Polynomials:

When $\mathcal{L}$ is the space of quadratic polynomials, this reduces to solving for $\phi_p \in \mathbb{R}^6$ minimizing :

$$E(\phi_p) = \sum_i \Theta(\|p_i - p\|)\big(\langle \phi_p, Q(p_i)\rangle - \phi_i\big)^2.$$

Taking the gradient and setting to zero gives:

$$0 = \sum_i \Theta(\|p_i - p\|)\big(\langle \phi_p, Q(p_i)\rangle - \phi_i\big)Q(p_i).$$

Equivalently, ϕ_p is the solution to the 6×6 system

$$\phi_p = \big(\sum_i \Theta(\|p_i - p\|) \cdot Q(p_i) \cdot Q^t(p_i)\big)^{-1}\big(\sum_i \Theta(\|p_i - p\|) \cdot \phi_i \cdot Q(p_i)\big).$$

This approach can be generalized to:

- Polynomials of arbitrary degree
- Functions in arbitrary dimensions

MLS vs. Splines

MLS

- A continuous set of functions "glued" together with the weight function.
- Continuity defined by the continuity of the weight function.
- No structure on the distribution of sample points.

Splines

- A discrete set of functions designed to glue together at the end points.
- Continuity defined by the order of the polynomial at the joints.
- Sample points are distributed with a regular grid topology.