Algebraic Point Set Surface

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Introduction

- To address the instability of planar MLS
  - Low Sampling Rate
  - High Curvature
- MLS by fitting algebraic spheres (APSS)
- Contribution:
  - Improved Stability
  - Normal Estimation based on [Hoppe et al. 1992]

Geometric **vs.** Algebraic Fit

- Geometric Fit (Best-Fit) directly minimizes *the sum of the squared distances* to the given points
  - Slow (iteratively updating a parametric model)
  - Unstable for planar fit

- Algebraic Fit solves a algebraic equation F(x)=0 in least square sense
  - Not sure what we're minimizing…

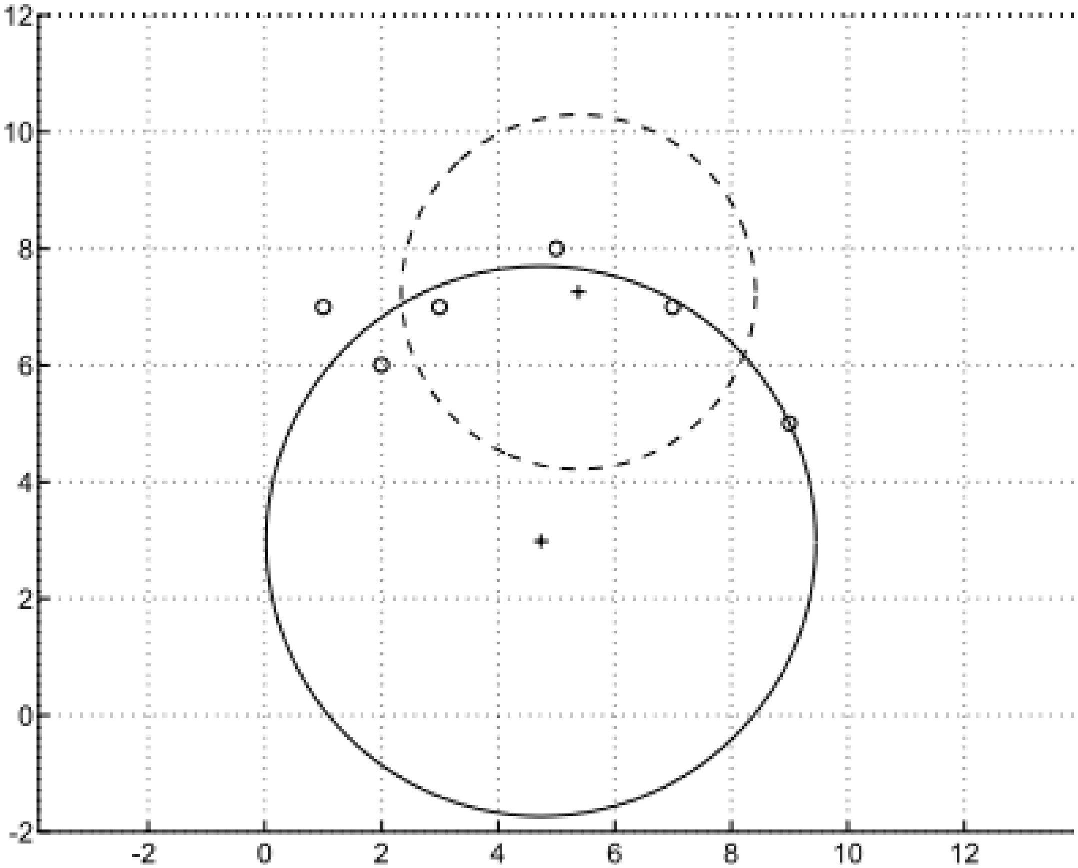


Figure 2.1: algebraic vs. best fit

Best fit
Algebraic fit

$$F(\mathbf{x}) = a\mathbf{x}^{\mathrm{T}}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} + c = 0,$$

Solve for $B\mathbf{u} = \mathbf{0}$ with

$$B = \begin{pmatrix} x_{11}^2 + x_{12}^2 & x_{11} & x_{12} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1}^2 + x_{m2}^2 & x_{m1} & x_{m2} & 1 \end{pmatrix}$$

$$\mathbf{u} = (a, b_1, b_2, c)^{\mathrm{T}}$$

Fitting Spheres without Normals

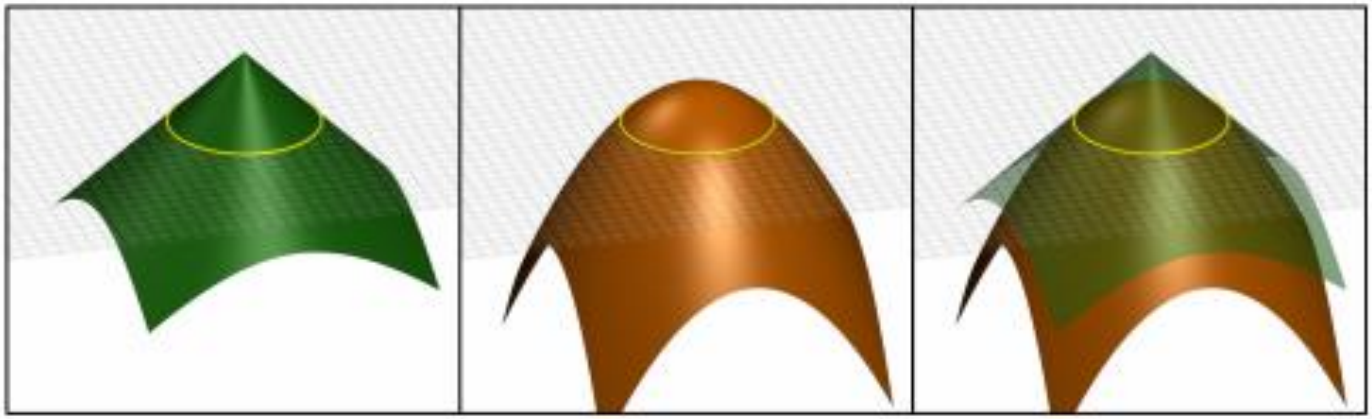
- General Setting: define the sphere as 0-set of $s_{\mathrm{u}} = [1, \mathrm{x}^T, \mathrm{x}^{\mathrm{T}}\mathrm{x}]$ with $\mathrm{u} = [\mathrm{u}_0, \dots, \mathrm{u}_{\mathrm{d+1}}] \in \mathbb{R}^{d+2}$

$$\mathbf{W}(\mathbf{x}) = \begin{bmatrix} w_0(\mathbf{x}) & & \\ & \ddots & \\ & & w_{n-1}(\mathbf{x}) \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & \mathbf{p}_0^T & \mathbf{p}_0^T\mathbf{p}_0 \\ \vdots & \vdots & \vdots \\ 1 & \mathbf{p}_{n-1}^T & \mathbf{p}_{n-1}^T\mathbf{p}_{n-1} \end{bmatrix}$$

$$\mathbf{u}(\mathbf{x}) = \underset{\mathbf{u},\, \mathbf{u}\neq\mathbf{0}}{\arg\min} \left\| \mathbf{W}^{\frac{1}{2}}(\mathbf{x})\mathbf{D}\mathbf{u} \right\|^2$$

- Additional constraint to avoid $\mathrm{u}(x) = 0$ and to make it as close to the geometric fit as possible
  - Set the norm of the gradient at the surface to 1
  - $\left||u_1, \dots, u_d|\right|^2 - 4\, u_0 u_{d+1}$

Fitting Spheres without Normals



- Lagrange's Multiplier + SVD:

$$\mathbf{D}^T\mathbf{W}(\mathbf{x})\mathbf{D}\mathbf{u}(\mathbf{x}) = \lambda\mathbf{C}\mathbf{u}(\mathbf{x}), \text{ with } \mathbf{C} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -2 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & 0 \\ -2 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Fitting Spheres *with* Normals

- Constraining by $\nabla s_{\mathrm{u}}(p_i) = n_i$, resulting in (with β to control the importance of normal constraints

$$\mathbf{W}^{\frac{1}{2}}(\mathbf{x})\mathbf{D}\mathbf{u} = \mathbf{W}^{\frac{1}{2}}(\mathbf{x})\,\mathbf{b} \tag{7}$$

where

$$\mathbf{W}(\mathbf{x}) = \begin{bmatrix} \ddots & & & & \\ & w_i(\mathbf{x}) & & & \\ & & \beta w_i(\mathbf{x}) & & \\ & & & \ddots & \\ & & & & \beta w_i(\mathbf{x}) \\ & & & & & \ddots \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \vdots & \vdots & \vdots \\ 1 & \mathbf{p}_i^T & \mathbf{p}_i^T\mathbf{p}_i \\ 0 & \mathbf{e}_0^T & 2\mathbf{e}_0^T\mathbf{p}_i \\ \vdots & \vdots & \vdots \\ 0 & \mathbf{e}_{d-1}^T & 2\mathbf{e}_{d-1}^T\mathbf{p}_i \\ \vdots & \vdots & \vdots \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \vdots \\ 0 \\ \mathbf{e}_0^T\mathbf{n}_i \\ \vdots \\ \mathbf{e}_{d-1}^T\mathbf{n}_i \\ \vdots \end{bmatrix}. \tag{8}$$

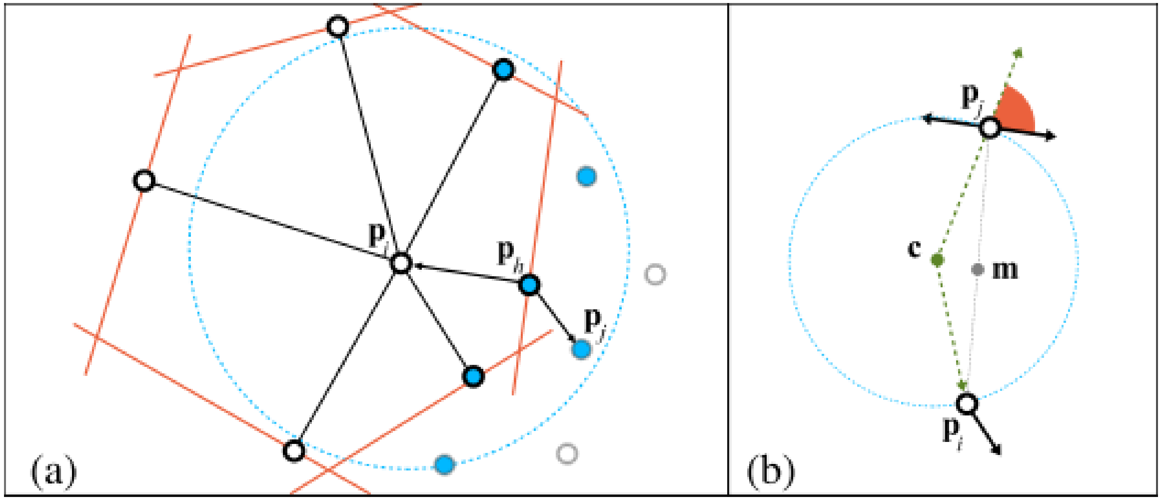
Normal Estimation

$$\mathbf{n}_i \approx \nabla s_{\mathbf{u}(\mathbf{p}_i)}(\mathbf{p}_i) = \begin{bmatrix} 0 & \mathbf{e}_0^T & 2\mathbf{e}_0^T\mathbf{p}_i \\ \vdots & \vdots & \vdots \\ 0 & \mathbf{e}_{d-1}^T & 2\mathbf{e}_{d-1}^T\mathbf{p}_i \end{bmatrix} \mathbf{u}(\mathbf{p}_i).$$

- Precompute the “confidence” at each sample by the relative magnitude of the smallest eigenvalue
- Propagate the direction based on the idea of *minimum spanning tree* [Hoppe et. al 1992]

Normal Estimation

- Propagate the direction based on the idea of *minimum spanning tree* [Hoppe et. al 1992]
  - Different k-NN selection
  - Different edge weight function



  - Different propagation: fitting a sphere at $m = \frac{(p_i + p_j)}{2}$ and check if $\nabla s_{\mathbf{u}(\mathbf{m})}(\mathbf{p}_i)^T \mathbf{n}_i \cdot \nabla s_{\mathbf{u}(\mathbf{m})}(\mathbf{p}_j)^T \mathbf{n}_j < 0.$