Reconstruction and Representation of 3D Objects with Radial Basis Functions

Method Overview

- Input: Point cloud
- Method: Use radial basis functions (RBFs) to implicitly represent surface
  - Main task: Signed-distance function estimation
- Output: Smooth, manifold surface

Implicit Reconstruction

- Input: Set of points $P = \{p_1, p_2, \dots p_n\}$ in R^3
- Output: Manifold surface S approximating P
- Method: S is the zero set of some signed distance function f

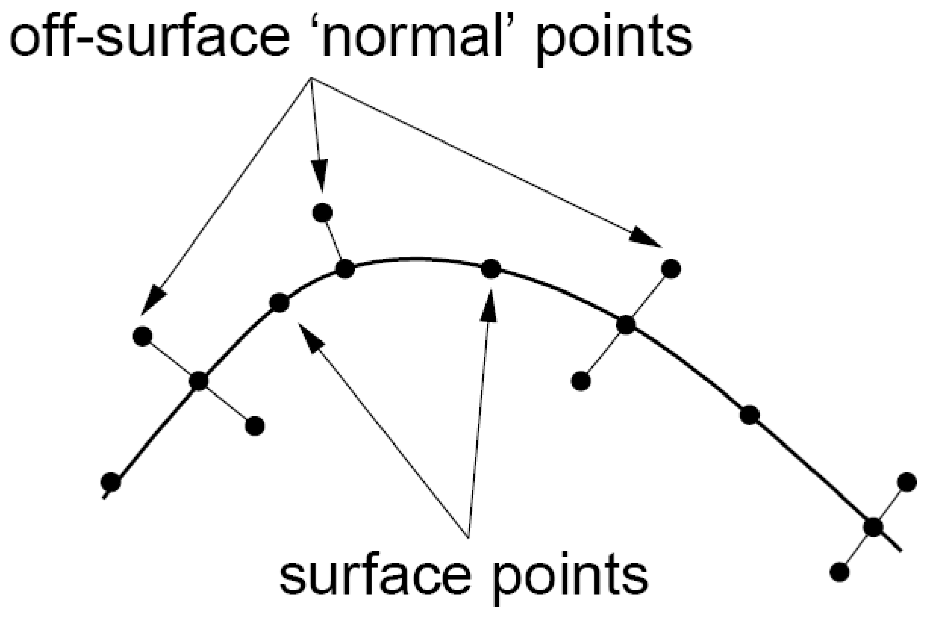
$$S = \{p_i \text{ in } R^3 \mid f(p_i) = 0\}$$

Distance Function

- Trivial Solution
  - $f(p_i) = 0$ for all p_i
- Constraints
  - On-surface Points: $f(p_i) = 0$
  - Off-surface Points: $f(p_i) = d_i$
- Solution
  - $f(x)$: signed distance function
  - d_i = distance to nearest on-surface point

Off-Surface Points

- For each point in input set
  - Add 2 off-surface points
  - One on each side of surface



(1) Generating Normals

- Input: point cloud w/o normals
- Generate normals as per [Hoppe 92]
  - Estimation from plane fitted to neighborhood
- Additionally, use consistency and/or scanner positions to resolve ambiguities
- If fails, do not define off-surface normal points

(2) Projecting Along Normals

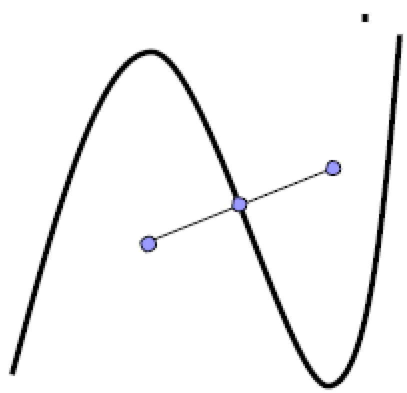
Create points → Need new constraint

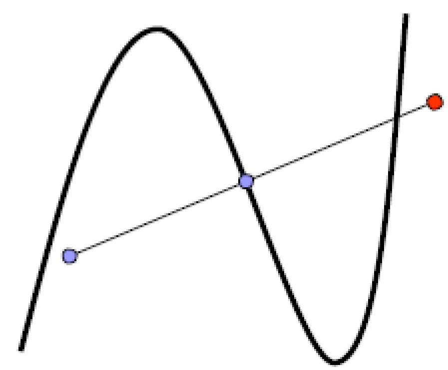
$$p_{i^{out}} = p_i + d_i$$
$$p_{i^{in}} = p_i - d_i$$

$$f(p_i) = 0$$
$$f\left(p_{i^{in}}\right) = -d_i$$
$$f(p_{i^{out}}) = +d_i \qquad i = 1 \dots n$$

Recall:
d_i = distance to new point

Need ϵ such that
d_i < distance to any other on-surface point





Projection Constraint

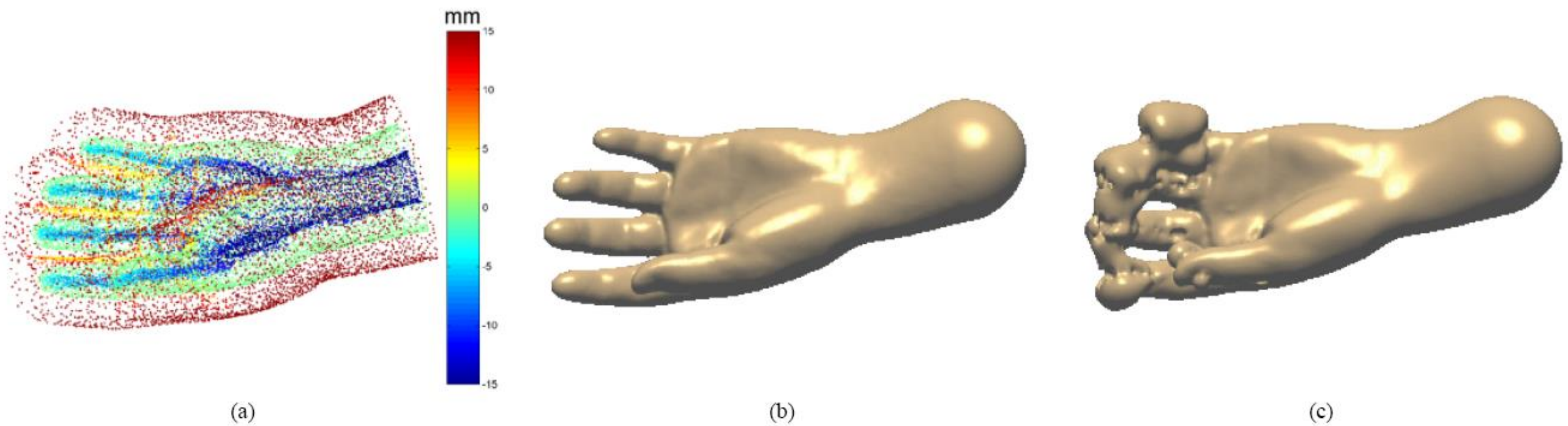


Figure 3: Reconstruction of a hand from a cloud of points with and without validation of normal lengths.

Scattered Data Interpolation Problem

- Given N points (x_i, f_i), reconstruct a function $S(x)$ such that

$$S(x_i) = f_i$$

- Constraints on $S(x)$
  - Smooth

Choosing S(x)

- S(x): $BL^{(2)}(R^3)$
  - Beppo-Levi space of distributions on R^3 with square integrable second derivatives
- Square integrable means $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$
  - Falls off quickly

Choosing S(x)

- [Duchon 77] showed that the smoothest interpolant in $BL^{(2)}(R^3)$ is

$$s(x) = p(x) + \sum_{i=1}^{N} \lambda_i |x - x_i|$$

- Which is a particular example of Radial Basis Functions

$$s(x) = p(x) + \sum_{i=1}^{N} \lambda_i \phi(|x - x_i|)$$

Radial Basis Functions

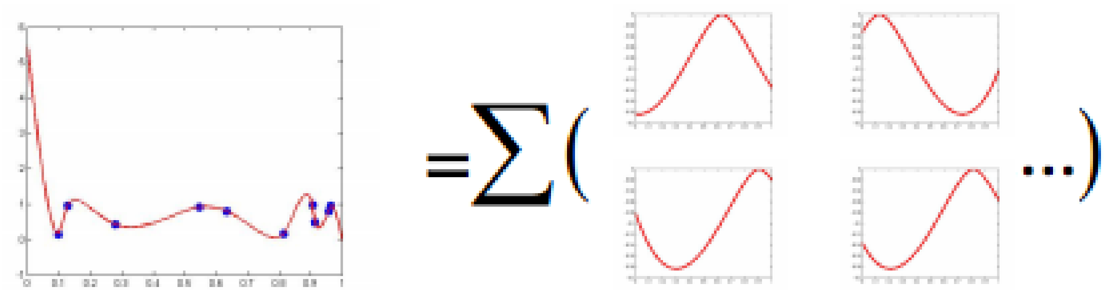
- p(x) is a low degree polynomial
- λ_i are real coefficients
- | | is the Euclidean norm

$$s(x) = p(x) + \sum_{i=1}^{N} \lambda_i \phi(|x - x_i|)$$

RBF Solution

- Assume S(x) is the weighted sum of basis functions

$$S(x) = \sum_{i=1}^{N} \lambda_i \phi(|x - x_i|)$$



RBFs Galore

Thin plate spline	$f(r) = r^2 \log(r)$	Two-variable
Multiquadric	$\varphi(r) = \sqrt{r^2 + c^2}$	Solve sparse
Gaussian	$\varphi(r) = e^{-cr^2}$	system
Polyharmonic splines		Non compact support
– Biharmonic	$f(r) = r$	
– Triharmonic	$f(r) = r^3$	

Extrapolation: Filling Holes

- Adding a polynomial p(x) to the RBF sum

$$s(x) = p(x) + \sum_{\mathrm{i}=1}^{\mathrm{N}} \lambda_{\mathrm{i}} \phi(|\mathrm{x} - \mathrm{x_i}|)$$

  - Better fitting: Can recreate polynomials exactly
  - Improves extrapolation

Radial Basis Functions

- For biharmonic spline RBF, $p(x) = \mathrm{c}_1 + \mathrm{c}_2\mathrm{x} + \mathrm{c}_3\mathrm{y} + \mathrm{c}_4\mathrm{z}$

$$\begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ c \end{pmatrix} = B \begin{pmatrix} \lambda \\ c \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

  - $A_{i,j} = \left| x_i - x_j \right|, \quad i, j = 1, \ldots, N$
  - $P_i = (1, x_i, y_i, z_i), \qquad i = 1, \ldots, N$
- B is symmetric and invertible under very mild conditions

Choosing S(x)

Side conditions for choosing λ_i

$$\sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \lambda_i x_i = \sum_{i=1}^{N} \lambda_i y_i = \sum_{i=1}^{N} \lambda_i z_i = 0.$$

$$\sum_{i=1}^{N} \lambda_i q(x_i) = 0, \text{ for all polynomials } q \text{ of degree at most } m$$

Non-Compact Support

- Biharmonic spline RBF
- Pros
  - Suitable to non-uniformly sample data
  - Handle holes
- Cons
  - A is not sparse, more computation and not scalable

Fast Methods (Magic)

- Approximated with Fast Multipole Method (FMM) [Greengard-Rokhlin 87]
  - Infinite precision not required
  - Cluster RBF centers into a hierarchy
    - Near-by clusters: Direct evaluation
    - Far-away clusters: Approximate evaluation

Fast Methods

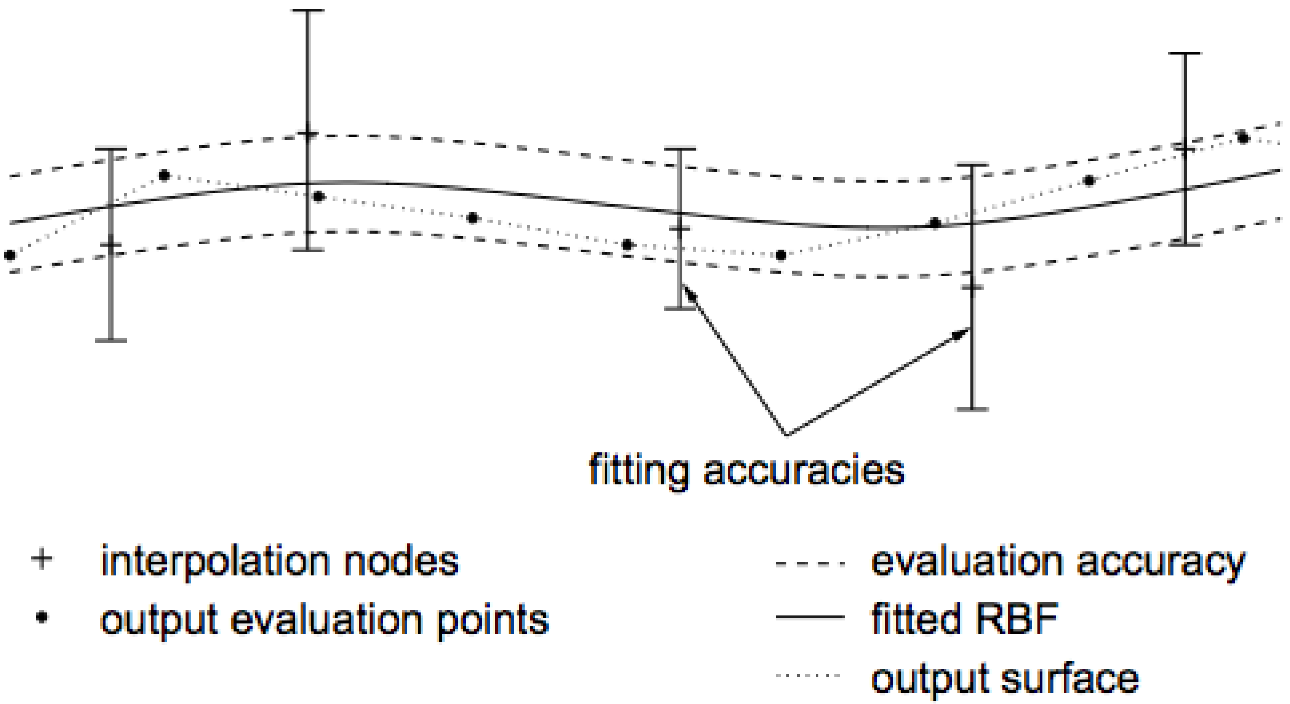


Figure 5: Illustration of fast fitting and evaluation parameters

Computational Complexity

FMM reduces both storage and computation costs

	Direct Methods	Fast Methods
Storage	$O(N^2)$	O(N)
Solving the matrix	$O(N^3)$	O(NlogN)
Evaluating a point	O(N)	O(1)*

*= after O(NlogN) setup

Reducing Number of Centers

- Use fewer centers to achieve desired accuracy
- A greedy algorithm
  - 1. Choose a subset of centers, fit an RBF to them
  - 2. Evaluate the residual, $\epsilon_i = f_i - s(p_i), i = 1 \dots n$
  - 3. If $\max\{|\epsilon_i|\}$ < *desired accuracy* then stop
  - 4. Add new centers where $|\epsilon_i|$ is large
  - 5. Re-fit RBF and go to step 2

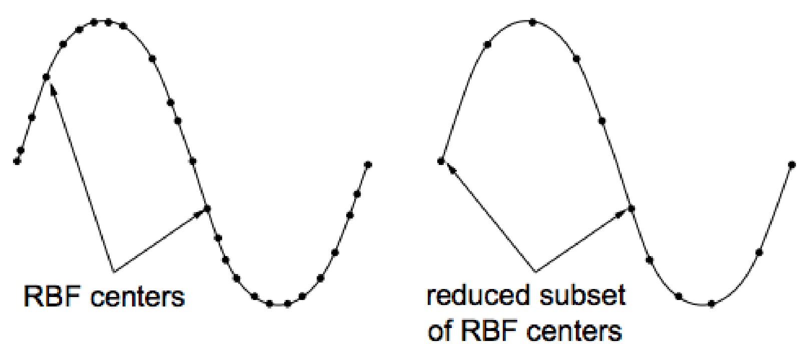


Figure 7: Illustration of center reduction.

Reducing Number of Centers

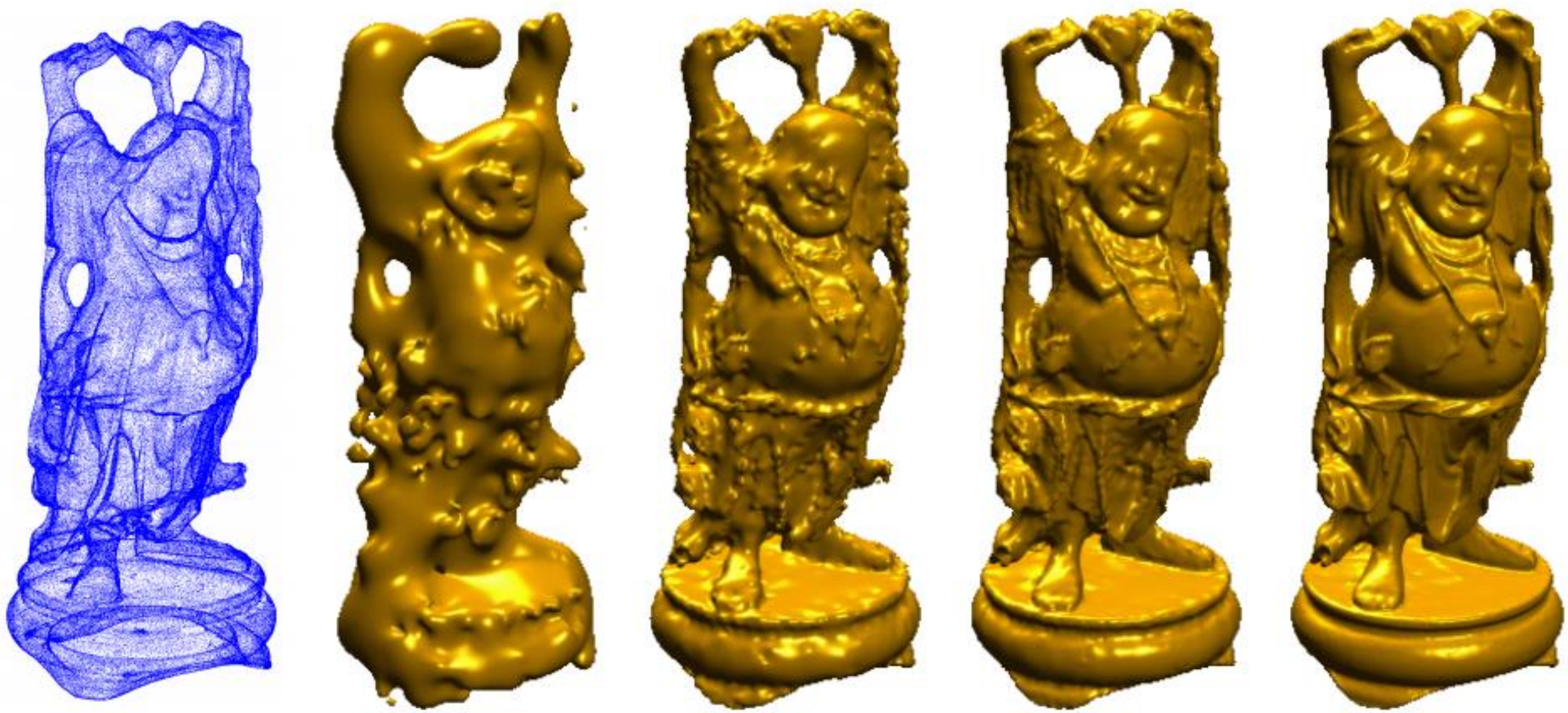


Figure 6: A greedy algorithm iteratively ts an RBF to a point cloud resulting in fewer centers in the nal function. In this case the 544,000 point cloud is represented by 80,000 centres to a relative accuracy of 5×10^{-4} in the nal frame.

Reducing Number of Centers

- Non-essential; FMM alone makes RBF feasible
- Improves storage and computation w/o reducing accuracy

(a) (b) (c)

Figure 8: RBF approximation of noisy LIDAR data. (a) 350,000 point-cloud, (b) the smooth RBF surface approximates the original point-cloud data, (c) cut-away view illustrating the RBF distance field and the preservation of the gap between the arm and the torso.

Noisy Data

- Consider both interpolation and smoothness

$$s^* = \min_s (\rho \, ||s||^2 + \frac{1}{N} \sum_{i=1}^{N} (s(x_i) - f_i)^2)$$

  - $||s||$ measures the smoothness
  - $\rho \geq 0$ is the weight
- Linear system changes to

$$\begin{pmatrix} A - 8N\pi\rho I & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ c \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

Noisy Data

- ρ can be defined globally or specified for individual points or groups of points

(a) (b) (c)

Figure 9: (a) Exact □t, (b) medium amount of smoothing applied (the RBF approximates at data points), (c) increased smoothing.

Surface Evaluation

- Many options
  - Implicit ray tracer
  - Mesh of polygons
    - Marching cubes
- These are usually optimized for data sampled on a regular grid.

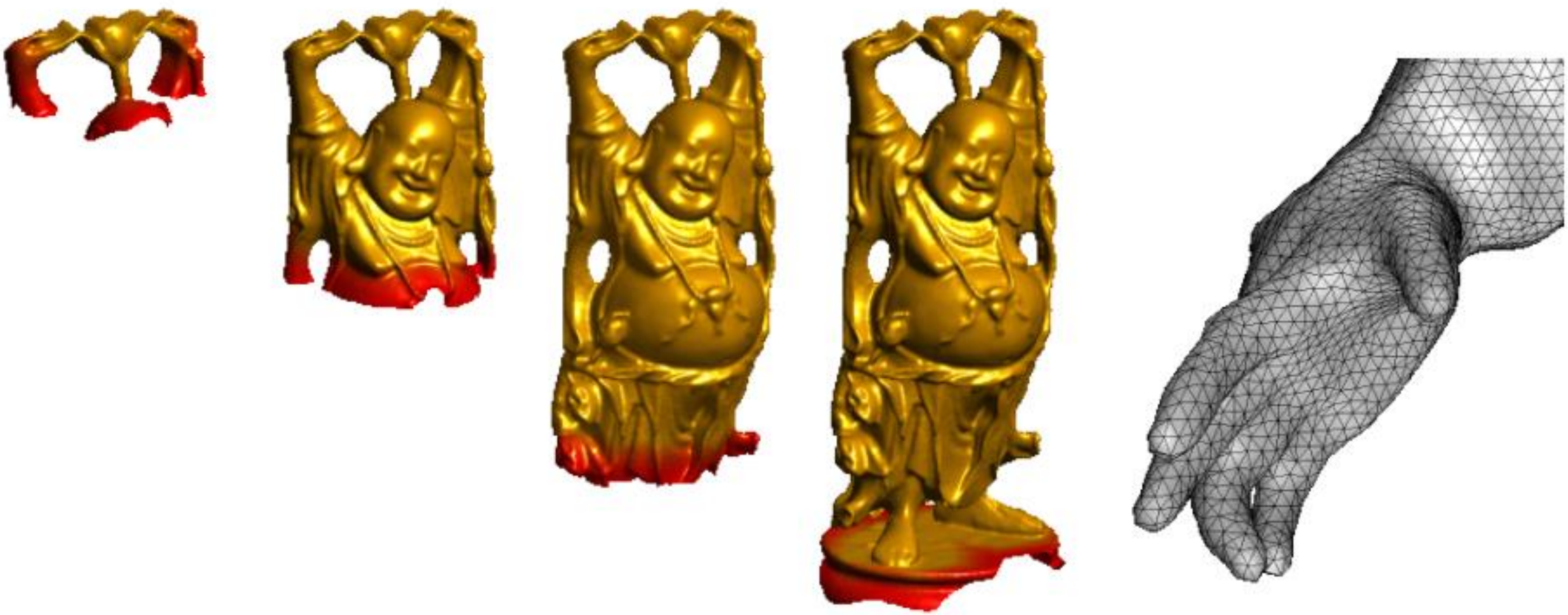
Surface Following

- A marching tetrahedra variant, optimized for surface following
  - Start from several seed points
  - Wavefronts of facets spread out across surfaces
  - Stop when intersect the bounding box
  - Makes use of simple gradient definition near surface

Surface Following

- Advantages
  - Outputs mesh w/ fewer thin triangles
  - Evaluate RBFs at fewer points
  - Only need to reference vertices along advancing wavefronts during computation

Surface Following



(a) (b)

Figure 10: Iso-surfacing an RBF. (a) Surface-following from a single seed, (b) example of an optimised mesh.

Results: Mesh Repair

Figure 11: An RBF has automatically filled small holes and extrapolated across occluded regions in the scan data (left), to produce a closed, water-tight model (right). The complex topology of the statue has been preserved.

Results: Large, Complicated Datasets

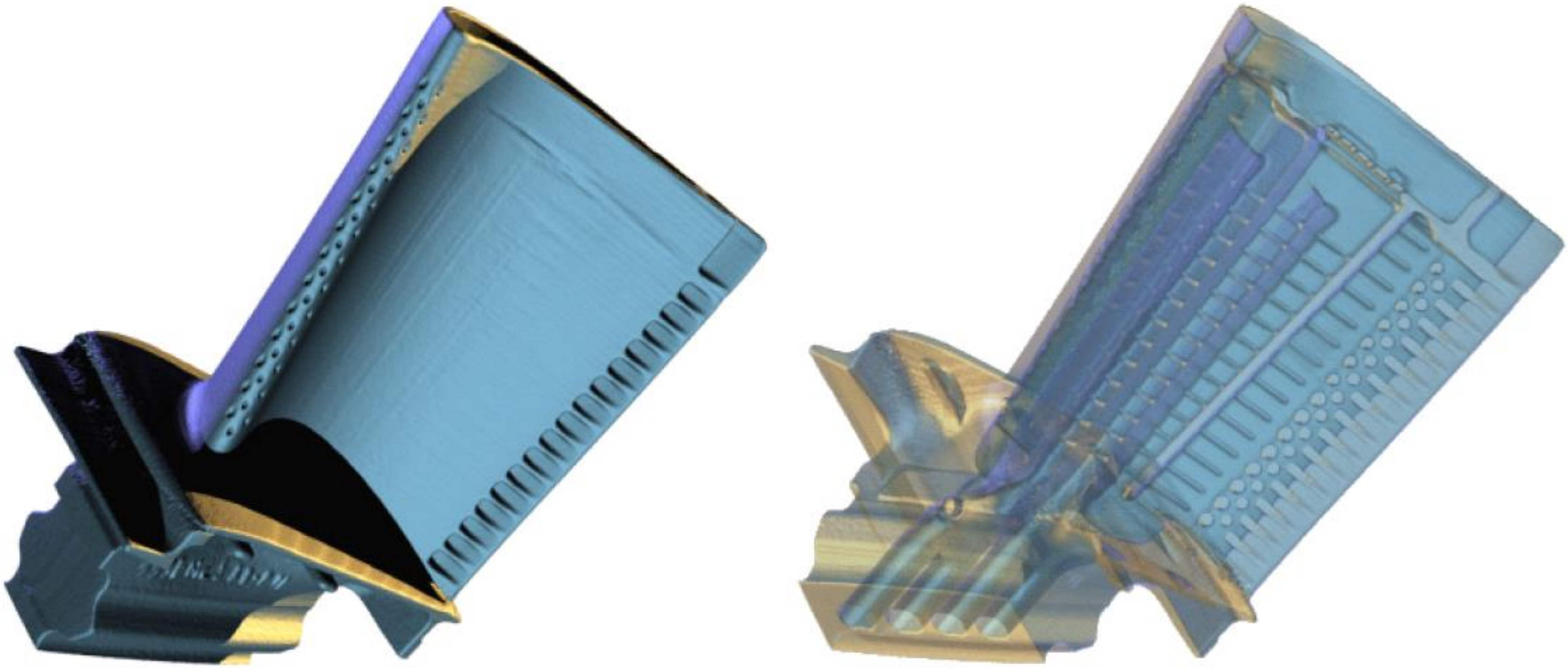


Figure 14: Solid and semi-transparent renderings of an RBF model of a turbine blade containing intricate internal structure. The RBF has 594,000 centers.

Results: Non-uniform Sampling

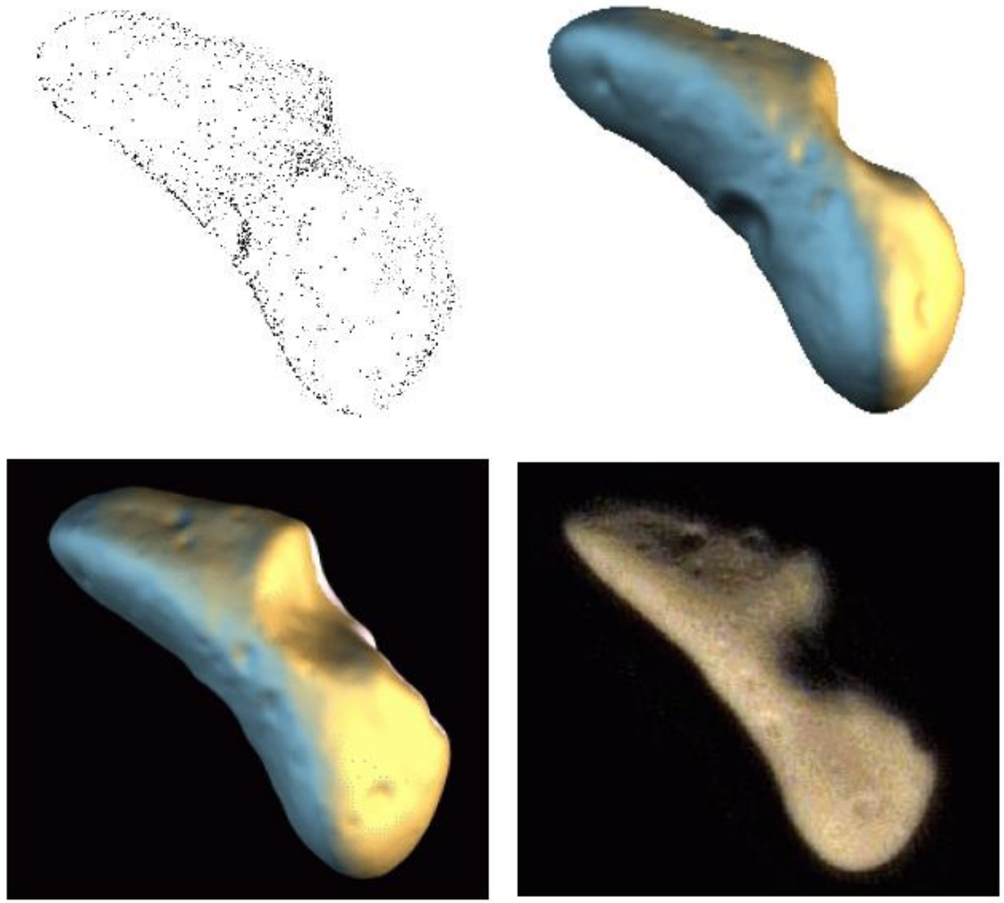


Figure 13: RBF reconstruction of the asteroid Eros from non-uniformly distributed range data (top). Photograph and model from a similar view (bottom).

Conclusions

- FMM makes it feasible to model complicated objects with RBF
  - Model complex scanned objects via RBF with Constructive Solid Geometry framework
  - Visualize data obtained on irregular grids
  - Repair existing meshes

Future Work

- Improve center reduction algorithm
  - Decompose global RBF description into implicit surface patches
  - Allows local manipulation and ray tracing
- Improve fitting and evaluation speeds
  - Parallel processing
  - Align data to a grid (may be incompletely sampled)

References

- Implicit reconstruction overview: http://graphics.stanford.edu/courses/cs468-10-fall/LectureSlides/04_Surface_Reconstruction.pdf
- Tutorial on RBFs: http://www.cs.technion.ac.il/~cs236329/tutorials/RBF.pdf
- Vladimir Savchenko's Shaping Modeling Lecture 10: http://cis.k.hosei.ac.jp/~vsavchen/SML/
- Carr et al. Reconstruction and Representation of 3D Objects with Radial Basis Functions. 2001.