600.657: Mesh Processing

Chapter 5

Outline

- Tutte Embeddings
- Harmonic Maps
- Conformal Maps
- Stretch Minimization

Parameterization

<u>Definition</u>:

A *parameterization* of a surface *S* is a bijective (diffeomorophic) map to/from a region $\Omega \subset \boldsymbol{R}^2$ from/to the surface *S*:

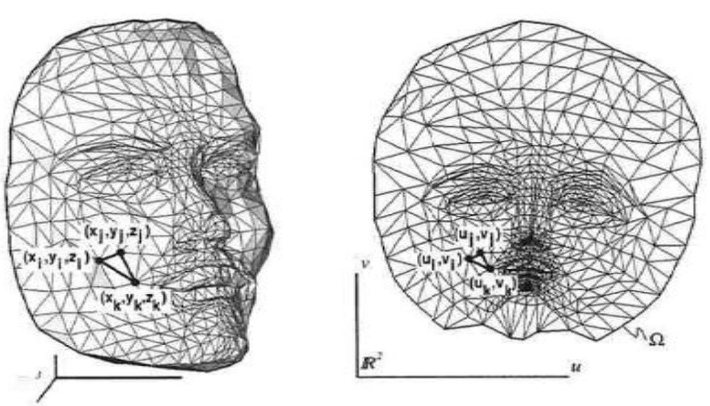
$$\varphi : S \leftrightarrow \Omega$$

Parameterization

Definition:

For discrete surfaces, with vertices $\{x_1,\ldots,x_n\}$, the parameterization can be characterized by where it maps the vertices:

$$\varphi : x_i \rightarrow w_i = (u_i, v_i)$$



[Botsch et al., Polygon Mesh Processing]

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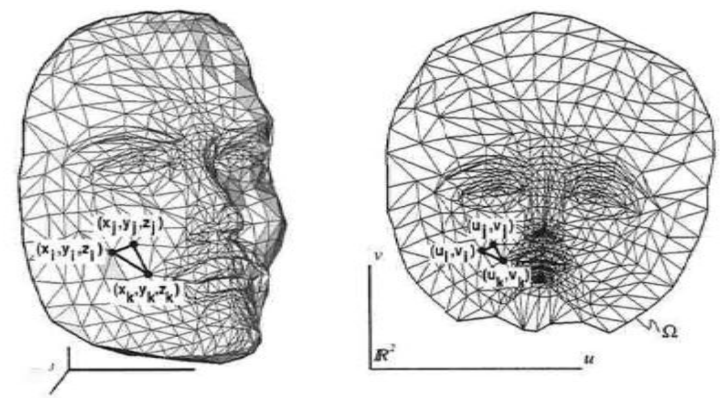
Using barycentric coordinates the, map can be extended to the interior of the triangles:

$$\varphi(\alpha_i x_i + \alpha_j x_j + \alpha_k x_k) \mapsto \alpha_i \varphi(x_i) + \alpha_j \varphi(x_j) + \alpha_k \varphi(x_k)$$



[Botsch et al., Polygon Mesh Processing]

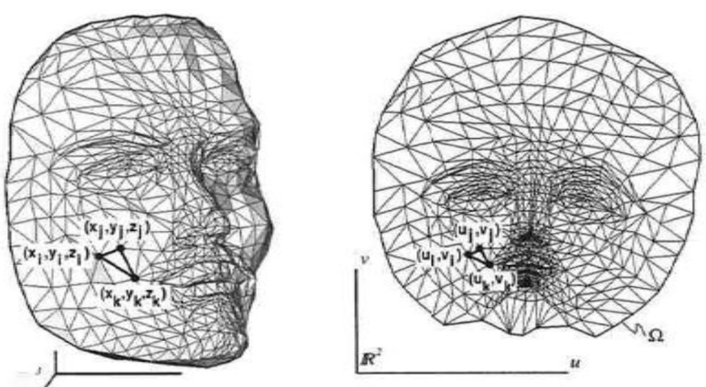
Parameterization

Goals:

- The parameterization cannot self-intersect:

$$\varphi(x) \neq \varphi(y) \quad \forall\, x, y \in S \mid x \neq y$$

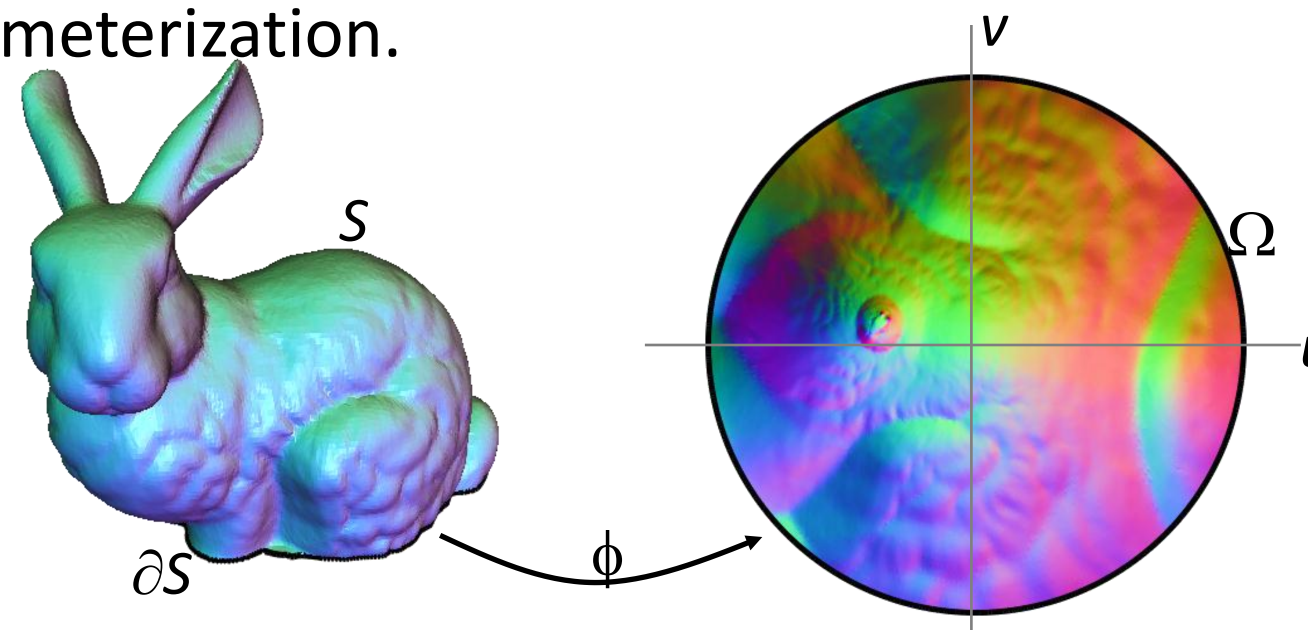
- The parameterization should minimally distort the surface in mapping it from 3D to 2D.



[Botsch et al., Polygon Mesh Processing]

Tutte's Theorem

Given a triangulated surface homeomorphic to a disk, if the (*u*,*v*) coordinates at the boundary vertices lie on a convex polygon, and if the coordinates of the internal vertices are a convex combination of their neighbors, then the (*u*,*v*) coordinates form a valid parameterization.



Tutte's Theorem (Implication)

Given a surface *S* and:

1. Constraints $\phi(x_i)=c_i$ for all boundary vertices $x_i \in \partial S$ to lie on a convex polygon *C*,
2. Weights α_{ij} associated with every pair of neighboring vertices $x_i \in S^\circ$ and $x_j \in S$ such that:

$$\alpha_{ij} \geq 0 \quad \text{and} \quad \sum_j \alpha_{ij} = 1 \ \forall i$$

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Then the solution to the constrained system:

$$w_i = \sum_{j \neq i} \alpha_{ij} w_j, \qquad w_i = c_i \ \ \forall x_i \in \partial S$$

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<u>Note</u>: This does not mean that this gives a **good** parameterization.

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gives a valid parameterization.

Harmonic Maps

Recall:

The Dirichlet Energy is a measure of the "smoothness" of a map:

$$E(\phi) = \int_S \left\| \nabla_S \phi(p) \right\|^2 dp$$

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Since the "gradient" of the energy at ϕ is $\Delta_S \phi$, the minimizer can be found by solving:

$$\Delta_S \phi = 0$$

$$\phi(x) = c(x) \quad \forall x \in \partial S$$

Harmonic Maps

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In the discrete setting, we have the cotangent Laplacian:

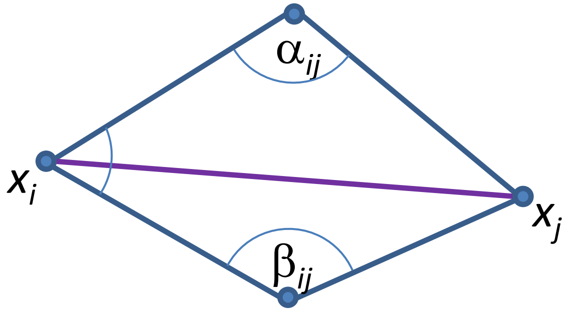
$$L_{ij} = \frac{1}{2|R_i|}\left(\cot \alpha_{ij} + \cot \beta_{ij}\right)$$

$$L_{ii} = -\sum_j L_{ij}$$

which gives the linear system:

$$\sum_{j \neq i} L_{ij} w_j - \sum_{j \neq i} L_{ij} w_i = 0 \quad \forall x_i \in S^\circ$$

$$w_i = c_i \quad \forall x_i \in \partial S$$



Harmonic Maps

$$\sum_{j\neq i} L_{ij} w_j - \sum_{j\neq i} L_{ij} w_i = 0 \quad \forall x_i \in S^\circ$$

Re-writing this, we get:

$$\frac{\sum_{j\neq i} L_{ij} w_j}{\sum_{j\neq i} L_{ij}} = w_i \quad \forall x_i \in S^\circ$$

which expresses the position of each $w_i=\phi(x_i)$ as the average of the positions of its neighbors.

Harmonic Maps

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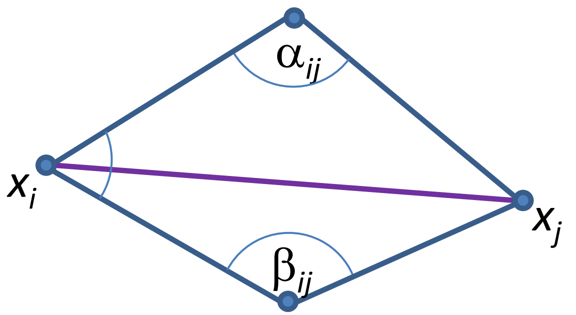
Note: Since the values of L_{ij} are not guaranteed to be positive, Tutte's theorem does not guarantee that the parameterization is valid.

Harmonic Maps

In particular, if the triangulation is not Delaunay then the associated entry in the cotangent Laplacian will be negative:

$$\alpha_{ij} + \beta_{ij} > \pi \iff L_{ij} < 0$$

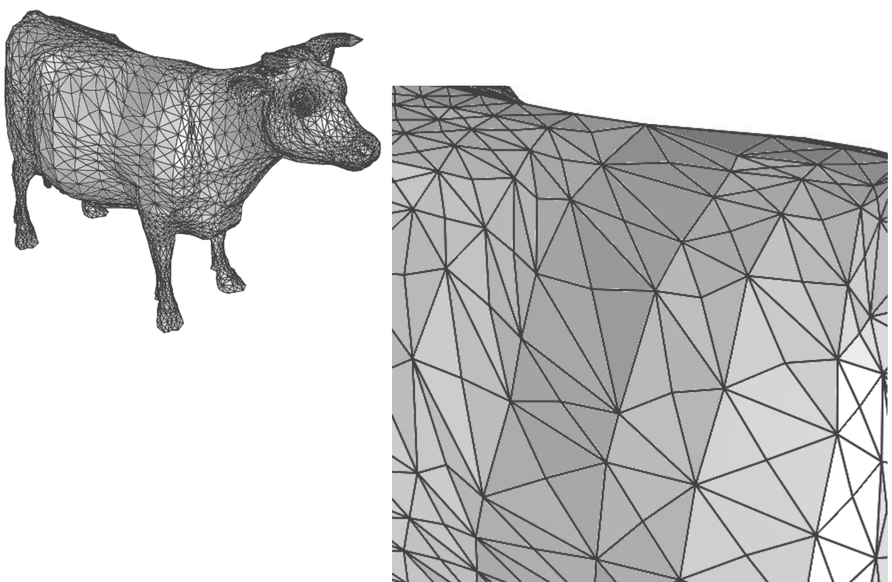


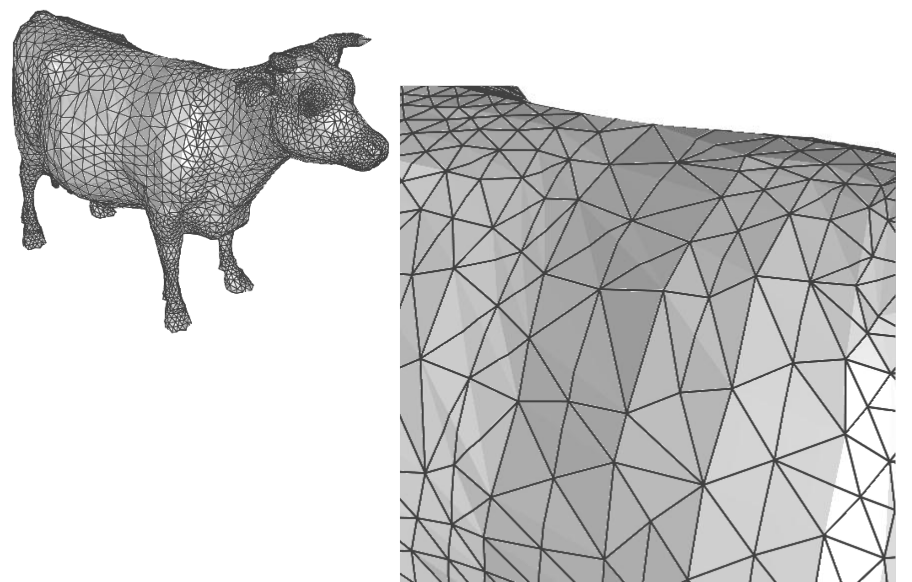
Harmonic Maps

$$\alpha_{ij} + \beta_{ij} > \pi \;\Leftrightarrow\; L_{ij} < 0$$

<u>Possible Fixes</u>:

– Re-triangulate the mesh (using edge-flips) to ensures that the triangulation is Delaunay [Fisher *et al*., 2007]





Harmonic Maps

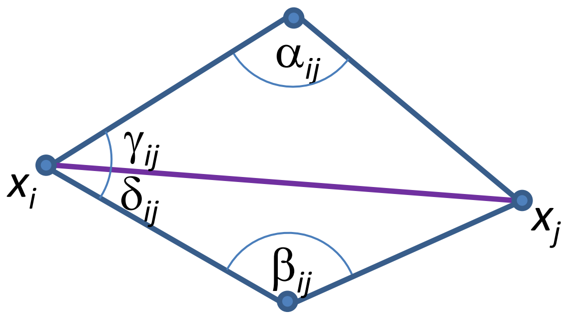
$$\alpha_{ij} + \beta_{ij} > \pi \iff L_{ij} < 0$$

<u>Possible Fixes</u>:

– Re-triangulate the mesh (using edge-flips) to ensures that the triangulation is Delaunay [Fisher *et al*., 2007]

– Use weights that are guaranteed to be non-negative, e.g. mean-value weights of Floater:

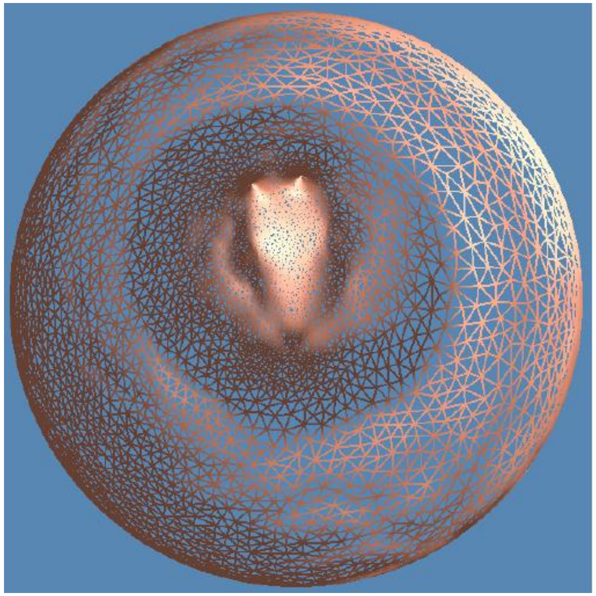
$$L_{ij} = \frac{1}{\|x_i - x_j\|}\left(\tan\frac{\gamma_{ij}}{2} + \tan\frac{\delta_{ij}}{2}\right)$$

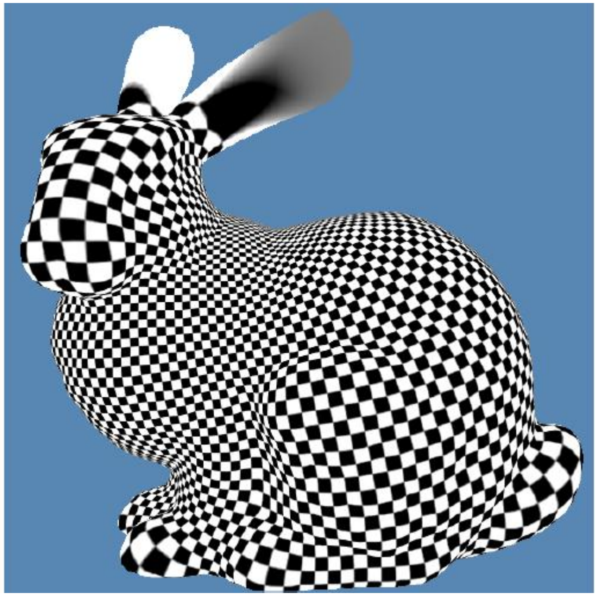


Conformal Maps

<u>Goal</u>:

To come up with a parameterization of the surface *S* that preserves the angles.



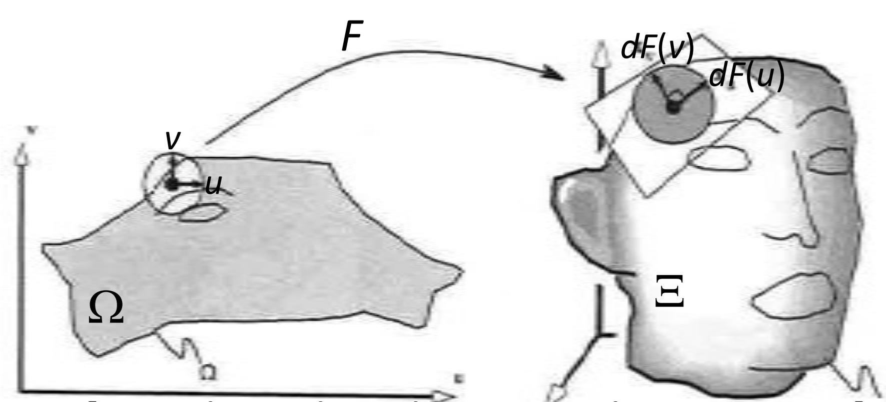


[*Global Conformal Surface Parameterization*, Gu and Yau]

Conformal Maps

A (differentiable) mapping $F{:}\Omega \rightarrow \Xi$ from one 2D domain to another 2D domain is angle-preserving if:

$$\frac{\langle dF(u), dF(v)\rangle}{\|dF(u)\|\|dF(v)\|} = \frac{\langle u, v\rangle}{\|u\|\|v\|}$$



[Botsch et al., Polygon Mesh Processing]

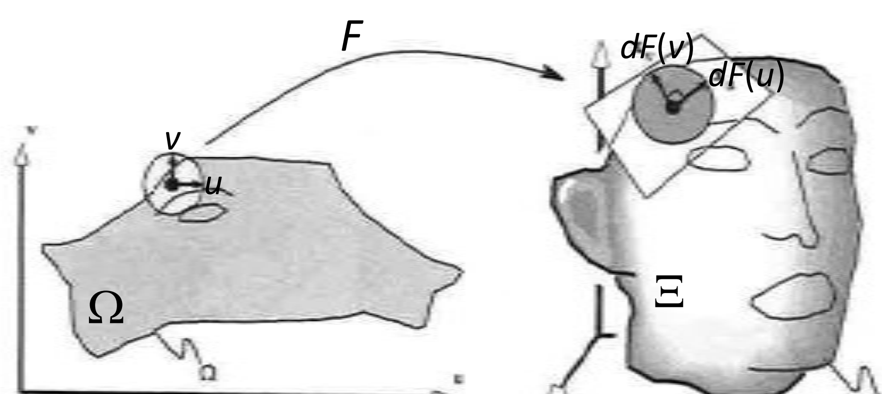
Conformal Maps

$$\frac{\langle dF(u), dF(v)\rangle}{\|dF(u)\|\|dF(v)\|} = \frac{\langle u, v\rangle}{\|u\|\|v\|}$$

This is equivalent* to the statement that:

$$dF = \lambda\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

i.e. that locally, the map is a combination of a rotation and a uniform scale.



[*Assuming orientation preserving]

[Botsch et al., Polygon Mesh Processing]

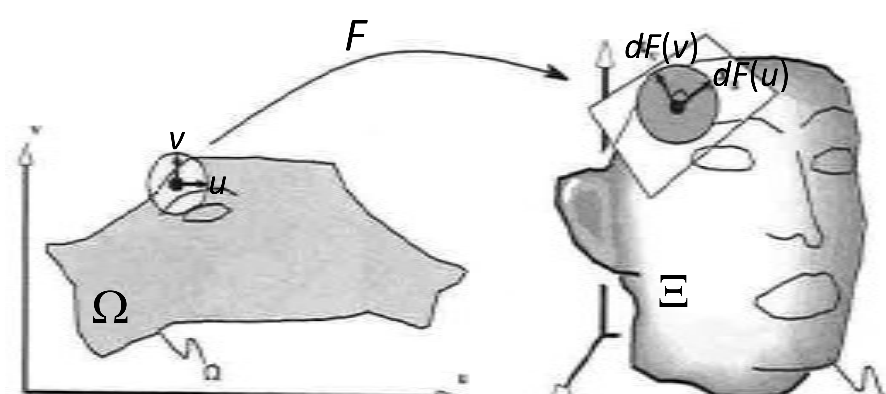
Conformal Maps

$$dF = \lambda \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

This is can also be expressed as the condition that equivalent to the statement that:

$$\frac{\partial F}{\partial y} = \left(\frac{\partial F}{\partial x} \right)^{\perp}$$

where *x* and *y* are an orthonormal frame for Ω.



[Botsch et al., Polygon Mesh Processing]

Conformal Maps

Goal:

Given a triangulated surface S and given a parameterization sending vertex x_i to w_i, we would like a measure of the extent to which the map is angle preserving.

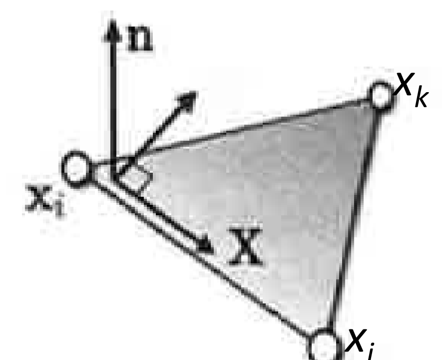
Conformal Maps

Given a triangulated surface S and given a triangle T=(x_i,x_j,x_k)$\in S$ with normal n, we can define an orthonormal frame over T by setting X to be the unit-vector:

$$X = \frac{x_j - x_i}{\left\| x_j - x_i \right\|}$$

and setting Y to be the orthogonal vector lying in the plane of T:

$$Y = n \times X$$

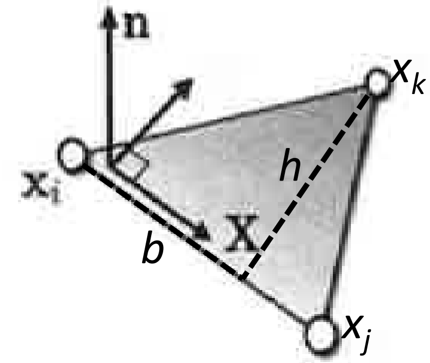


[Botsch et al., Polygon Mesh Processing]

Conformal Maps

Within the triangle *T*, the mapping can be expressed with respect to the orthonormal basis (*X*,*Y*) as:

$$\phi\left(x_i + uX + vY\right) = w_i + \frac{1}{2|T|}u\left[-hw_i + hw_j\right] + \frac{1}{2|T|}v\left[-\left(\|x_j - x_i\| - b\right)w_i - bw_j + \|x_j - x_i\|w_k\right]$$



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Conformal Maps

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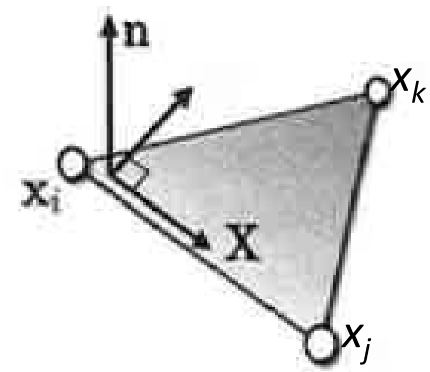
Taking the derivative, we get:

$$d\phi\big|_T = \frac{1}{2|T|}\left(-hw_i + hw_j \;\middle|\; \left(b - \|x_j - x_i\|\right)w_i - bw_j + \|x_j - x_i\| w_k\right)$$



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Thus, we can measure the lack of conformality by looking at the size of the difference:

$$E(\phi) = \sum_T \frac{1}{|T|}\left\|\frac{\partial\phi|_T}{\phi X} - \left(\frac{\partial\phi|_T}{\phi Y}\right)^{\perp}\right\|^2$$

$$= \sum_T \frac{1}{|T|}\left\|\left(-h|_T w_i + h|_T w_j\right) - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left(\left(b|_T - \left\|x_j - x_i\right\|\right)w_i - b|_T w_j + \left\|x_j - x_i\right\|w_k\right)\right\|^2$$

Conformal Maps

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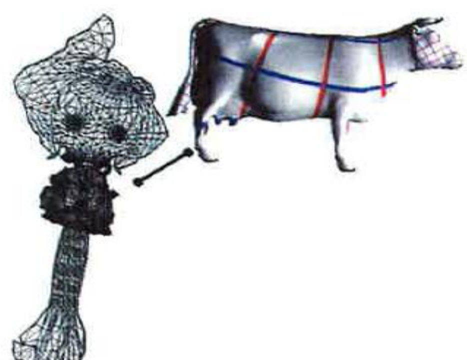
Since the parameterization minimizing this energy is invariant to translations and rotations, an optimal solution by constraining the position of (at least) two points on the mesh.



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Conformal Maps

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To avoid (arbitrarily) fixing positions, other methods formulate the problem in a way that supports “free” boundaries.

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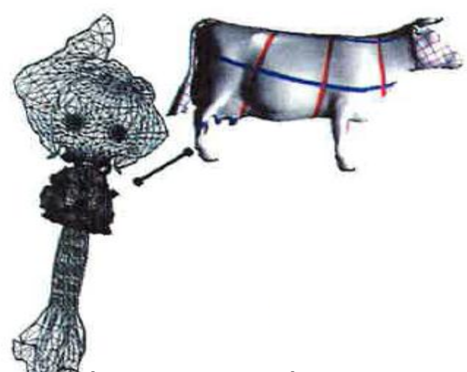

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Conformal Maps

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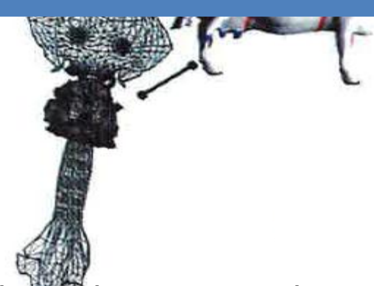
<u>Question</u>: Doesn't the Mobius transformation allow for three fixed points?



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