600.657: Mesh Processing

Chapter 4

Outline

- Review
- Diffusion Flow
- Fairing
- Signal Processing

Review

Recall:

The Laplace operator measures the divergence of the gradient of a function about a point.

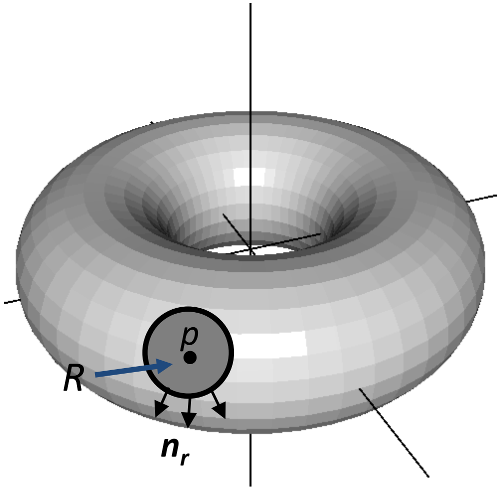
$$\Delta_S f = \mathrm{div}_S(\nabla_S f)$$

Review

<u>Recall</u>:

The divergence about a point is the measure of change across the boundary of a small region surrounding that point:

$$\mathrm{div}_S \nabla_S f(p) = \lim_{R \to p} \frac{1}{\mathrm{Area}(R)} \int_{\partial R} \langle n_r(q), \nabla_S f(q) \rangle dq$$



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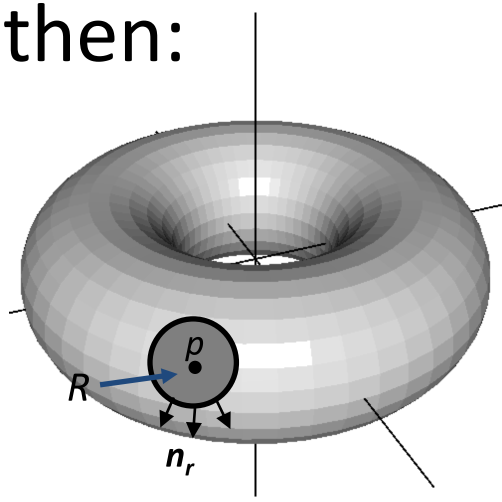
If we make *R* be a "disk with radius *r*" then:

$$\mathrm{div}_S \nabla_S f(p) \approx \lim_{R \to p} \frac{1}{\mathrm{Area}(R)} \int_{\partial R} \frac{f(q) - f(p)}{|q-p|} dq$$

$$= \lim_{R \to p} \frac{1}{\mathrm{Area}(R)} \frac{1}{r} |\partial R| \left(Avg_{\partial R}(f) - f(p) \right)$$



Review

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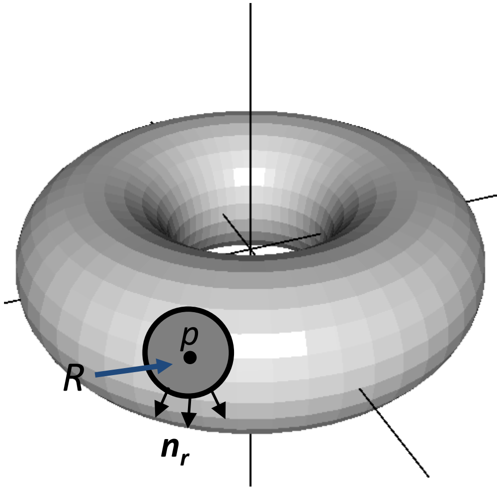
$$\mathrm{div}_S \nabla_S f(p) \approx \lim_{R \to p} \frac{1}{\mathrm{Area}(R)} \frac{1}{r} |\partial R| \left(Avg_{\partial R}(f) - f(p) \right)$$

So, the Laplacian is a measure of the difference between the value of a function at a point and the average value of its neighbors.



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Diffusion Flow

Motivation:

To smooth a function, we would like to make the value at a point be closer to the value of its neighbors.

Diffusion Flow

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To smooth a function, we would like to make the value at a point be closer to the value of its neighbors.

That is, we would like to reduce the difference between the value at a point and the average value at its neighbors (the Laplacian).

Diffusion Flow

Approach:

Given a function $f(p)$, define a family of functions $f_t(p)$ where:

1. The function defined at time t=0 is the original function.
2. The change in the function values over time is proportional to the Laplacian:

$$f_0(p) = f(p)$$

$$\frac{\partial}{\partial t} f_t(p) = \lambda \Delta_S f_t(p)$$

Diffusion Flow

$$f_0(p) = f(p)$$
$$\frac{\partial}{\partial t} f_t(p) = \lambda \Delta_S f_t(p)$$

<u>Discretizing (Explicit)</u>:

The function at time step δ(*k*+1) is set so that the difference between the function at δ(*k*+1) and δ*k* is proportional to the Laplacian at δ*k*:

$$f_{\delta(k+1)}(p) - f_{\delta k}(p) = \lambda \delta \Delta_S f_{\delta k}(p)$$

Diffusion Flow

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The function at time step $\delta(k+1)$ is set so that the difference between the function at $\delta(k+1)$ and δk is proportional to the Laplacian at δk:

$$f_{\delta(k+1)}(p) - f_{\delta k}(p) = \lambda \delta \Delta_S f_{\delta k}(p)$$

$$\Downarrow$$

$$f_{\delta(k+1)}(p) = \left(Id + \lambda \delta \Delta_S\right) f_{\delta k}(p) = \left(Id + \lambda \delta \Delta_S\right)^{k+1} f(p)$$

Diffusion Flow

$$f_0(p) = f(p)$$

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The function at time step δ(*k*+1) is set so that the difference between the function at δ(*k*+1) and δ*k* is proportional to the Laplacian at δ(*k*+1):

$$f_{\delta(k+1)}(p) - f_{\delta k}(p) = \lambda \delta \Delta_S f_{\delta(k+1)}(p)$$

$$\Downarrow$$

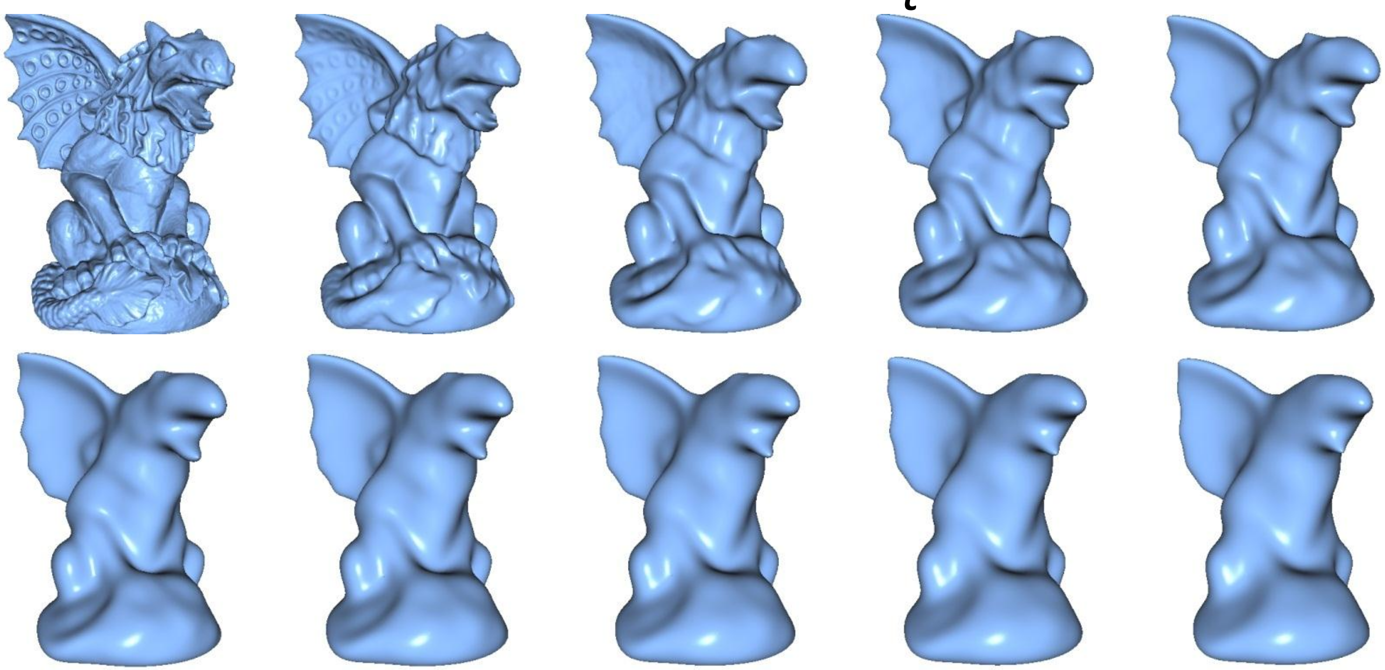
$$f_{\delta(k+1)}(p) = \left(Id - \lambda \delta \Delta_S\right)^{-1} f_{\delta k}(p) = \left(Id - \lambda \delta \Delta_S\right)^{-(k+1)} f(p)$$

Mean-Curvature Flow

We can applying this approach to the function giving the position of points on the surface:

$$Id(p) = p$$

to get the smoothed surface S_t at time t.



Mean-Curvature Flow

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$$Id(p) = p$$

to get the smoothed surface S_t at time t.

Note:

Since the surface is evolving with time, we also update the Laplace operator at each time step.

$$f_0(p) = Id(p)$$
$$\frac{\partial}{\partial t} f_t(p) = \lambda \Delta_{S_t} f_t(p)$$

Mean-Curvature Flow

Using the fact that the Laplacian of this function is the mean curvature vector:

$$\Delta_S Id = -2H\boldsymbol{n}$$

this surface flow evolves points by moving them in the direction of the normal, by a value proportional to the mean-curvature:

$$\frac{\partial}{\partial t} Id_t(p) = -\lambda 2H\boldsymbol{n}(p)$$

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Alternative Derivation

Given a function f on a surface S, we can measure the Dirichlet Energy of the surface as the sum of gradient magnitudes:

$$E(f) = \int_S \left\| \nabla_S f(p) \right\|^2 dp$$

Alternative Derivation

Given a function *f* on a surface *S*, we can measure the Dirichlet Energy of the surface as the sum of gradient magnitudes:

$$E(f) = \int_S \|\nabla_S f(p)\|^2 dp$$

Given such an energy, we can try to evolve *f* so as to minimize the energy:

$$f(p) \leftarrow f(p) - \varepsilon(p)$$

where $\varepsilon(p)$ is an offset function that is zero on the boundary of *S*, corresponding to the "gradient" of the energy.

Alternative Derivation

The gradient of a function *h* at *p* is the vector *v* such that the change of *h* in any direction *w* is:

$$\lim_{t \to 0} \frac{h(p+tw)-h(p)}{t} = \langle w, v \rangle$$

Alternative Derivation

The gradient of a function *h* at *p* is the vector *v* such that the change of *h* in any direction *w* is:

$$\lim_{t\to 0}\frac{h(p+tw)-h(p)}{t}=\langle w,v\rangle$$

In our context, the “gradient” of the energy *E* at *f* is the function(al) ε such that the change of *E* in any direction η is for all η that are zero on the boundary:

$$\lim_{t\to 0}\frac{E(f+t\eta)-E(f)}{t}=\int_S \eta\cdot\varepsilon$$

Alternative Derivation

Expanding, we get:

$$E(f+t\eta) = \int_S \left\|\nabla_S f + t\nabla_S \eta\right\|^2 dp$$

$$= \int_S \left\|\nabla_S f\right\|^2 dp + t^2 \int_S \left\|\nabla_S \eta\right\|^2 dp + 2t \int_S \left\langle \nabla_S f, \nabla_S \eta \right\rangle dp$$

Alternative Derivation

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Thus, we get:

$$\lim_{t\to 0} \frac{E(f+t\eta) - E(f)}{t} = \lim_{t\to 0}\left(2\int_S \left\langle \nabla_S f, \nabla_S \eta \right\rangle dp + t\int_S \left\|\nabla_S \eta\right\|^2 dp \right)$$

$$= 2\int_S \left\langle \nabla_S f, \nabla_S \eta \right\rangle dp$$

Alternative Derivation

$$\lim_{t\to 0}\frac{E(f+t\eta)-E(f)}{t}=2\int_S \langle \nabla_S f, \nabla_S \eta\rangle dp$$

<u>Goal</u>:

Recall that we would like to express the change in energy as:

$$\lim_{t\to 0}\frac{E(f+t\eta)-E(f)}{t}=\int_S \varepsilon\cdot\eta$$

so we seek a function(al) ε such that:

$$\int_S \varepsilon\cdot\eta = 2\int_S \langle \nabla_S f, \nabla_S \eta\rangle dp$$

Alternative Derivation

$$\lim_{t\to 0}\frac{E(f+t\eta)-E(f)}{t}=2\int_S\left\langle \nabla_S f,\nabla_S\eta\right\rangle dp$$

Using the product rule, we have:

$$\mathrm{div}_S(\eta\nabla_S f)=\left\langle \nabla_S f,\nabla_S\eta\right\rangle+\eta\Delta_S f$$

Alternative Derivation

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giving:

$$\int_S \mathrm{div}_S(\eta\nabla_S f)dp=\int_S \langle \nabla_S f,\nabla_S \eta\rangle dp+\int_S \eta\Delta_S f\ dp$$

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Transforming to a boundary integral gives:

$$\int_{\partial S}\eta\nabla_S f\ ds=\int_S\left\langle \nabla_S f,\nabla_S \eta\right\rangle dp+\int_S \eta\Delta_S f\ dp$$

Alternative Derivation

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$$\int_{\partial S}\eta\nabla_S f\ ds=\int_S\langle\nabla_S f,\nabla_S\eta\rangle dp+\int_S\eta\Delta_S f\ dp$$

but since *g* is zero on the boundary, we have:

$$0=\int_S\langle\nabla_S f,\nabla_S\eta\rangle dp+\int_S\eta\Delta_S f\ dp$$

Alternative Derivation

$$\lim_{t\to 0}\frac{E(f+t\eta)-E(f)}{t}=2\int_S \langle \nabla_S f, \nabla_S \eta\rangle dp$$

$$0=\int_S \langle \nabla_S f, \nabla_S \eta\rangle dp+\int_S \eta \Delta_S f \; dp$$

Putting this together, we get:

$$\lim_{t\to 0}\frac{E(f+t\eta)-E(f)}{t}=-2\int_S \eta \Delta_S f \; dp$$

so that the "gradient" of the Dirichlet Energy at *f* is the Laplacian of *f*.

Alternative Derivation

$$\lim_{t\to 0}\frac{E(f+t\eta)-E(f)}{t} = -2\int_S \eta\Delta_S f\ dp$$

Thus, the flow that we saw before:

$$\frac{\partial}{\partial t} f_t(p) = \lambda\Delta_S f_t(p)$$

was just a gradient descent on this energy.

Surface Fairing

Instead of trying to evolve towards the minimizer of the Dirichlet Energy, we can try to solve directly for the function f that is the minimizer.

Surface Fairing

Instead of trying to evolve towards the minimizer of the Dirichlet Energy, we can try to solve directly for the function *f* that is the minimizer.

Since the gradient of the energy at *f* is (minus) the Laplacian of *f*, the function *f* is a minimizer when its Laplacian is zero:

$$\Delta_S f(p) = 0$$

Surface Fairing

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For (connected) surfaces without boundary, the minimizer is not particularly interesting, since the only functions with zero Laplacian are the constant functions.

However, for surfaces with boundary, we can prescribe the values at the boundary ∂S, and obtain non-constant functions in the interior by solving:

$$\Delta_S f(p) = 0$$
$$f(s) = \phi(s) \quad \text{for all} \quad s \in \partial S$$

Surface Fairing

$$\Delta_S f(p) = 0$$

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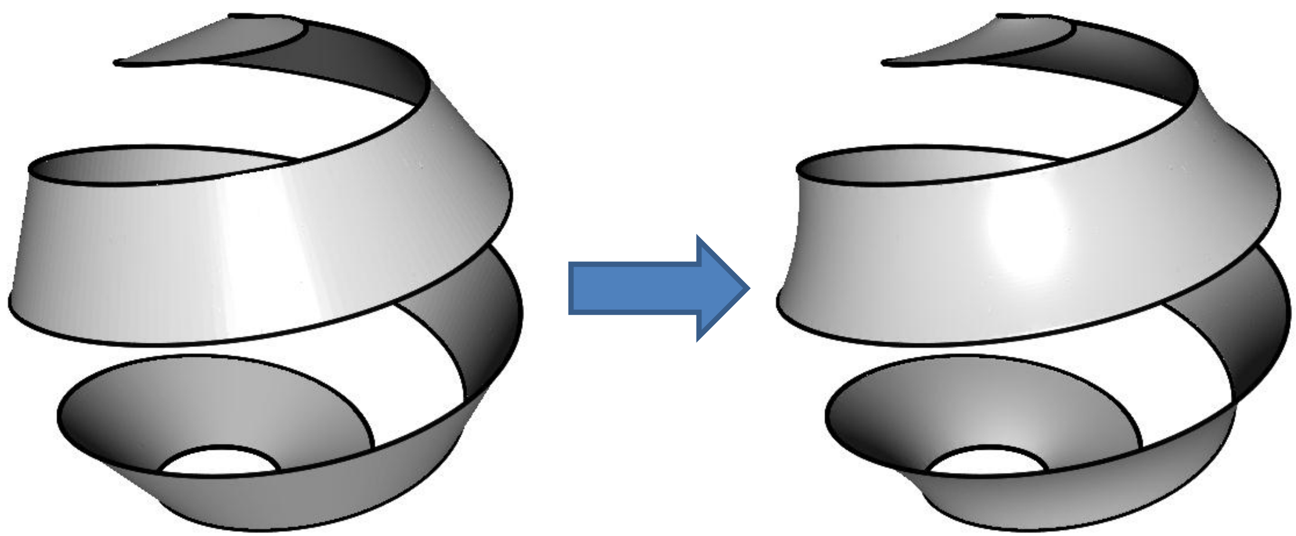
When $\phi : \partial S \rightarrow \boldsymbol{R}^3$ fixes the 3D positions of points on the boundary, we get a function mapping the old surface to an almost minimal surface with the same boundary.



Surface Fairing

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When $\phi:\partial S \to \boldsymbol{R}^3$ fixes the 3D positions of points on the boundary, we get a function mapping the old surface to an almost minimal surface with the same boundary.

The surface is not quite minimal because the mean-curvature of the new surface is only zero with respect to the old surface's Laplacian.

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The surface is not quite minimal because the mean-curvature of the new surface is only zero with respect to the old surface's Laplacian.

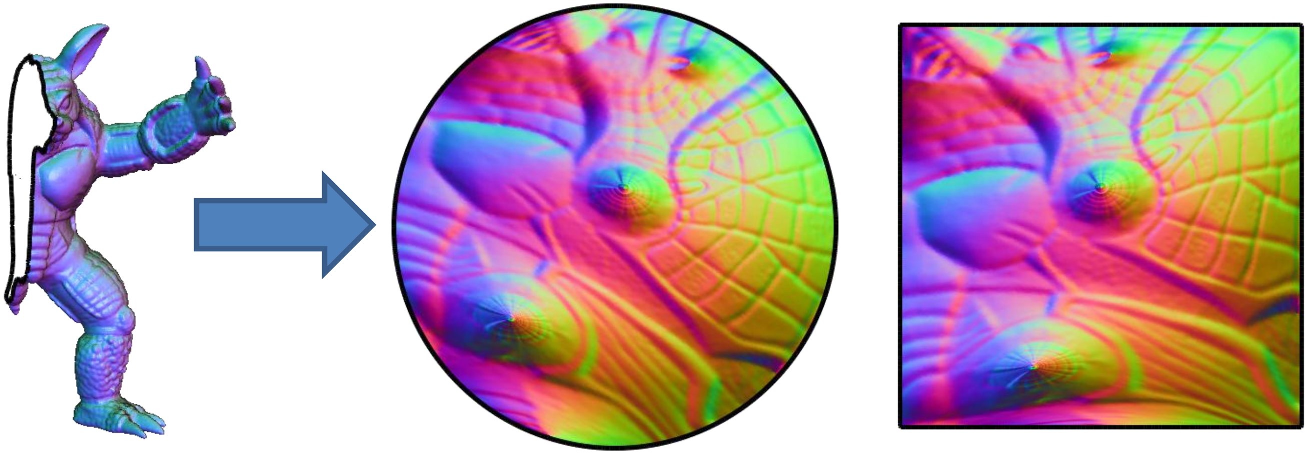
Iterating, we do get to the minimal surface.

Surface Fairing

$$\Delta_S f(p) = 0$$

$$f(s) = \phi(s) \quad \text{for all} \quad s \in \partial S$$

When $\phi: \partial S \rightarrow \boldsymbol{R}^2$ prescribes the desired 2D position of points on the boundary, we get a mapping from the surface to the 2D plane.



Surface Fairing

Generalizations:

Although we derived mean-curvature flow by minimizing the gradient norms:

$$E(f)=\int_S \left\|\nabla_S f(p)\right\|^2 dp$$

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Although we derived mean-curvature flow by minimizing the gradient norms:

$$E(f) = \int_S \left\| \nabla_S f(p) \right\|^2 dp$$

We can also try to minimize higher order derivatives, like the change in gradients:

$$E(f) = \int_S \left\| \nabla_S \nabla_S f(p) \right\|^2 dp$$

Surface Fairing

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or the change of the change in gradients…

Surface Fairing

Generalizations:

When minimizing the gradient norms, we obtained the flow:

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Surface Fairing

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When minimizing the change in gradients we get:

$$\frac{\partial}{\partial t} f_t(p) = \lambda \Delta_S \Delta_S f_t(p)$$

Surface Fairing

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When minimizing the change in gradients we get:

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and so on.

Surface Fairing

Generalizations:

If we specify the boundary values, we obtain the minimizer of the Dirichlet Energy by solving:

$$\Delta_S f(p) = 0$$

$$f(s) = \phi(s) \quad \text{for all} \quad s \in \partial S$$

Surface Fairing

Generalizations:

If we specify the boundary values, we obtain the minimizer of the Dirichlet Energy by solving:

$$\Delta_S f(p) = 0$$

$$f(s) = \phi(s) \quad \text{for all} \quad s \in \partial S$$

When minimizing the change in gradients, we obtain the minimizer by solving:

$$\Delta_S \Delta_S f(p) = 0$$

$$f(s) = \phi(s) \qquad \text{for all} \quad s \in \partial S$$

$$\nabla_S f(s) = \vec{V}(s) \quad \text{for all} \quad s \in \partial S$$

Surface Fairing

Generalizations:
Applying this to the function giving the 3D positions of points on the surface gives a way to smoothly fill in holes:

[Botsch et al., Polygon Mesh Processing]

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Discrete Dirichlet Energy

Recall:

To compute the Dirichlet Energy of a function *f* on *S*, we integrate the square norms of the gradient of *f*:

$$E(f) = \int_S \left\| \nabla_S f(p) \right\|^2 dp$$

Discrete Dirichlet Energy

$$E(f) = \int_S \left\| \nabla_S f(p) \right\|^2 dp$$

If *S* has no boundary, or we restrict ourselves to looking at functions that vanish on the boundary, we get:

$$E(f) = \int_S \left\| \nabla_S f(p) \right\|^2 dp = -\int_S f \cdot \Delta_S f \, dp$$

Discrete Dirichlet Energy

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If *S* has no boundary, or we restrict ourselves to looking at functions that vanish on the boundary, we get:

$$E(f) = \int_S \left\| \nabla_S f(p) \right\|^2 dp = -\int_S f \cdot \Delta_S f \; dp$$

Thus, if we represent the function $f=\{f_1,\ldots,f_n\}$ by its values on the vertices, and we set *L* to be the discrete (cotangent) Laplacian, we get:

$$E(f) = -f^t L f$$

Discrete Dirichlet Energy

$$E(f) = -f^t L f$$

Q: What is the most "energetic" function?

Discrete Dirichlet Energy

$$E(f) = -f^t L f$$

Q: What is the most "energetic", unit-norm function?

$$\arg\max_f = \frac{-f^t L f}{f^t f}$$

Discrete Dirichlet Energy

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Q: What is the most "energetic", unit-norm function?

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A: The maximizer of the energy is the eigenvector* of *L* with the largest (negative) eigenvalue.

*generalized eigenvector

Discrete Dirichlet Energy

$$E(f) = -f^t L f$$

Q: What is the most “energetic”, unit-norm function?

$$\arg\max_f = \frac{-f^t L f}{f^t f}$$

A: The next maximizer of the energy is the eigenvector* of *L* with the next largest (negative) eigenvalue.

*generalized eigenvector

Discrete Dirichlet Energy

$$E(f) = -f^t L f$$

Computing all the eigenvectors* of the Laplace operator, we get a set of functions $\{f^1,\ldots,f^n\}$ with associated eigenvalues $\{-\lambda_1 \leq \ldots \leq -\lambda_n\}$ such that:

$$L f^i = \lambda_i f^i$$

[Vallet et al. 2008]

*generalized eigenvector

Discrete Dirichlet Energy

$$E(f) = -f^t L f$$

Computing all the eigenvectors* of the Laplace operator, we get a set of functions $\{b^1,\ldots,b^n\}$ with associated eigenvalues $\{-\lambda_1 \leq \ldots \leq -\lambda_n\}$ such that:

$$Lb^i = \lambda_i b^i$$

Since the Laplace operator is symmetric and (negative) semi-definite, the eigenvectors are all orthogonal, and the eigenvalues are all negative.

*generalized eigenvector

Signal Processing

$$Lb^i = \lambda_i b^i$$

<u>Definition</u>:

The values λ_i are called the *natural frequencies* of the surface and the functions b^i are called the *manifold/mesh-harmonics*.

Signal Processing

$$Lb^i = \lambda_i b^i$$

<u>Alternate Smoothing</u>:

Given a function *f* defined on the mesh, we can express *f* as a linear combination of the manifold harmonics:

$$f = \sum_i f_i b^i$$

Signal Processing

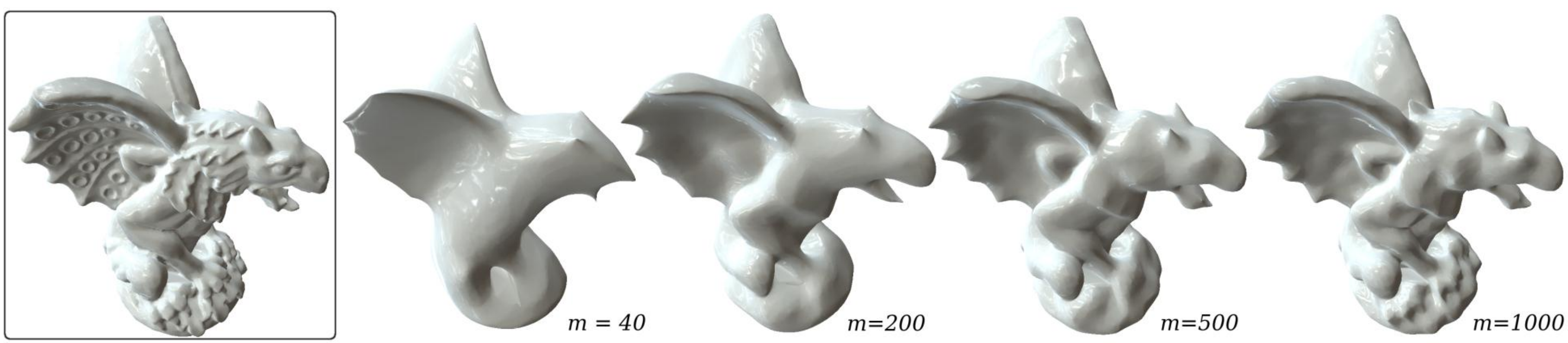
$$f = \sum_i f_i b^i$$

Alternate Smoothing:

So smoothing *f* corresponds to dampening the more energetic frequencies:

$$\text{Smooth}(f) = \sum_i w_i f_i b^i$$

where w_i is a set of weights that drops off with frequency (e.g. w_i=1/0 for small/large *i*.)



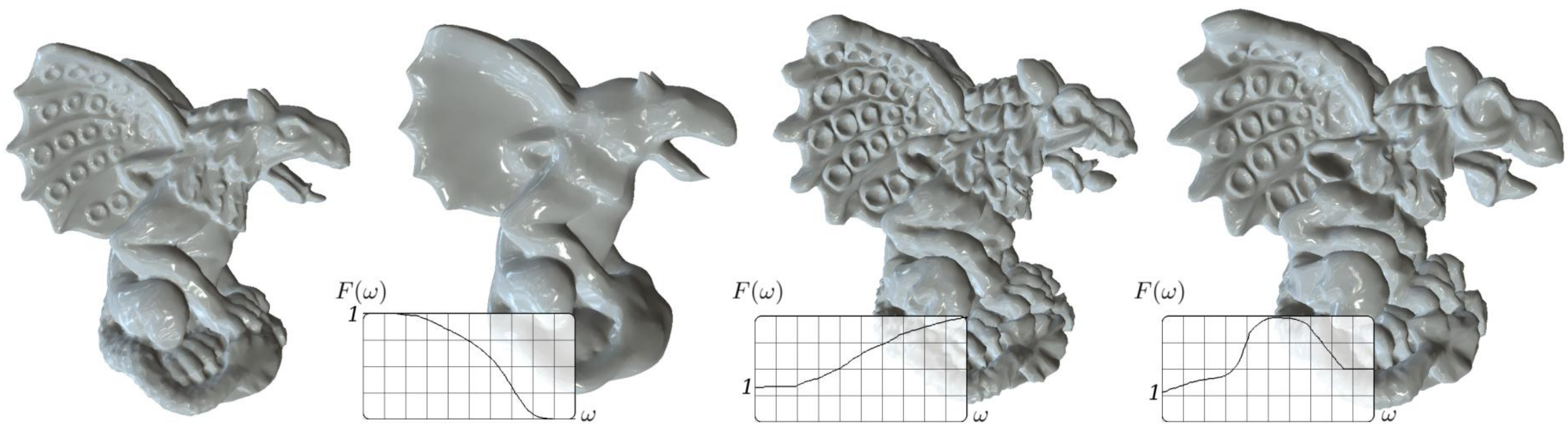
[Vallet et al. 2008]

Signal Processing

$$f = \sum_i f_i b^i$$

Alternate Smoothing:

Of course, we can also generalize the approach to a broader class of surface edits by selectively amplifying/dampening different frequencies:



[Vallet et al. 2008]

Signal Processing

$$f = \sum_i f_i b^i$$

<u>Recall</u>:

In performing diffusion flow with step size δ, we considered two different integration schemes:

– Explicit: $f(p) \leftarrow \left(Id + \delta\Delta_S\right) f(p)$

– Semi-Implicit: $f(p) \leftarrow \left(Id - \delta\Delta_S\right)^{-1} f(p)$

Signal Processing

$$f = \sum_i f_i b^i$$

<u>Recall</u>:

If *f* is an eigenvector of the Laplacian, with eigenvalue -λ, this gives:

– Explicit: $$f(p) \leftarrow (1 - \delta\lambda) f(p)$$

– Semi-Implicit: $$f(p) \leftarrow \frac{1}{1 + \delta\lambda} f(p)$$

Signal Processing

– Explicit: $f(p) \leftarrow (1-\delta\lambda) f(p)$

– Semi-Implicit: $f(p) \leftarrow \dfrac{1}{1+\delta\lambda} f(p)$

That is:

– Explicit: scales the $-\lambda$-th frequency by $(1-\delta\lambda)^k$.

– Implicit: scales the $-\lambda$-th frequency by $1/(1+\delta\lambda)^k$.

Signal Processing

- Explicit: $$f(p) \leftarrow (1-\delta\lambda) f(p)$$
- Semi-Implicit: $$f(p) \leftarrow \frac{1}{1+\delta\lambda} f(p)$$

That is, after *k* iterations:

- Explicit: scales the -λ-th frequency by $(1-\delta\lambda)^k$.
- Implicit: scales the -λ-th frequency by $1/(1+\delta\lambda)^k$.

<u>Explicit</u>: Smooth/converges for small time steps ($\delta\lambda<2$)
<u>Implicit</u>: Smooth/converges for all time steps.