600.657: Mesh Processing

Chapter 3

Differential Geometry

- Curves
- Surfaces

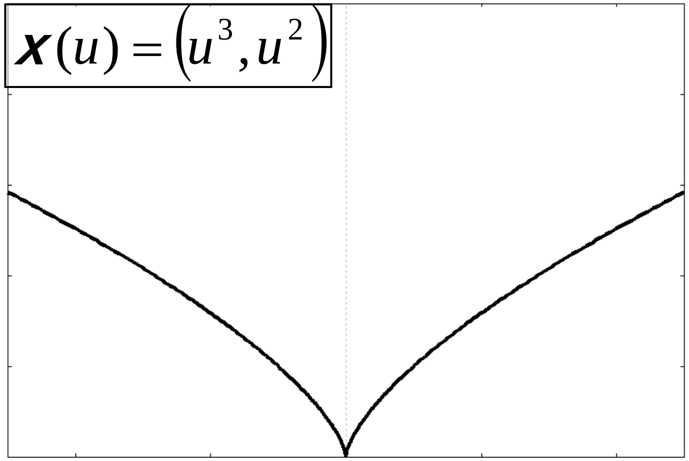
Differentiable Curves

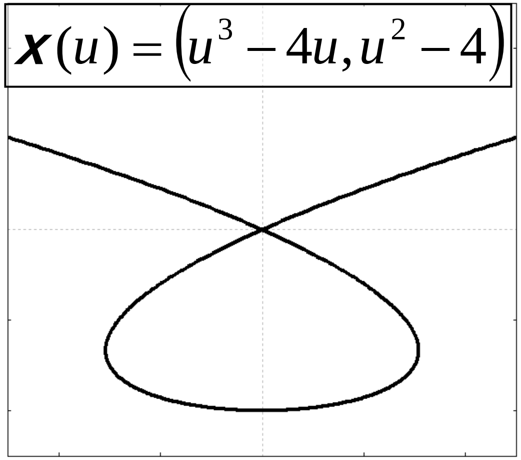
Definition:

A *parameterized differentiable curve* is a differentiable map $\boldsymbol{x}:I\to\boldsymbol{R}^2$ of an open interval $I=(a,b)$ of the real line $\boldsymbol{R}$ into $\boldsymbol{R}^2$:

$$\boldsymbol{x}(u)=(x(u),y(u))$$

where $x(u)$ and $y(u)$ are differentiable functions.



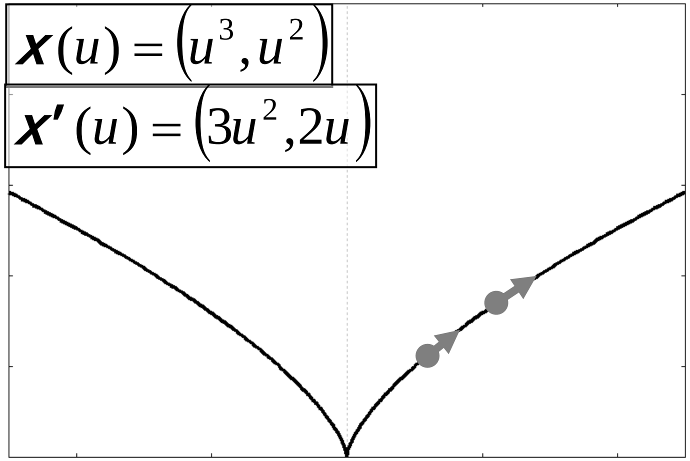


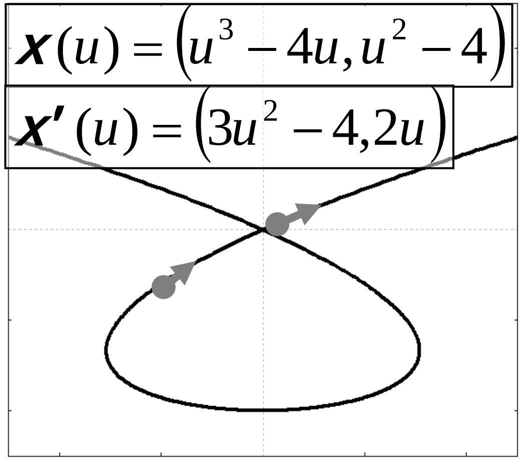
Differentiable Curves

<u>Definition</u>:

The *derivative* of the curve at $\boldsymbol{x}(u)$ is the vector, tangent to the curve, defined as:

$$\boldsymbol{x}'(u)=(x'(u),y'(u))$$





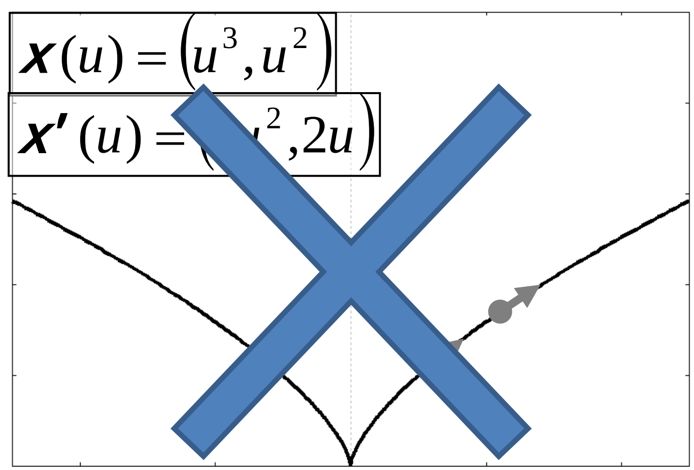
Differentiable Curves

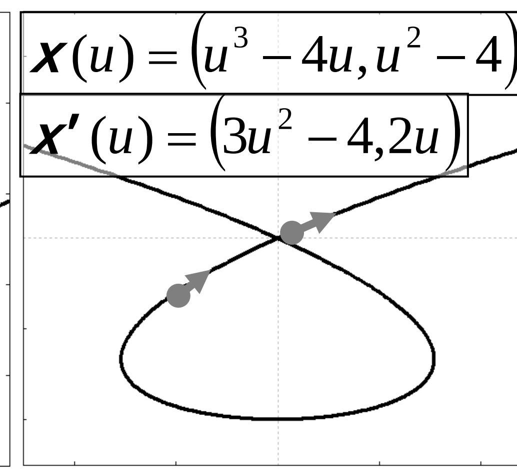
<u>Definition</u>:

The *derivative* of the curve at $\boldsymbol{x}(u)$ is the vector, tangent to the curve, defined as:

$$\boldsymbol{x}'(u)=(x'(u),y'(u))$$

The curve is said to be *regular* if $\boldsymbol{x}'(\mathrm{u})\neq 0$.





Regular Curves

Given a regular curve $\boldsymbol{x}(u)$, and given, the *arc-length* from *a* to the point *u* is:

$$s(u) = \int_a^u \left|\boldsymbol{x}'(v)\right| dv$$

Regular Curves

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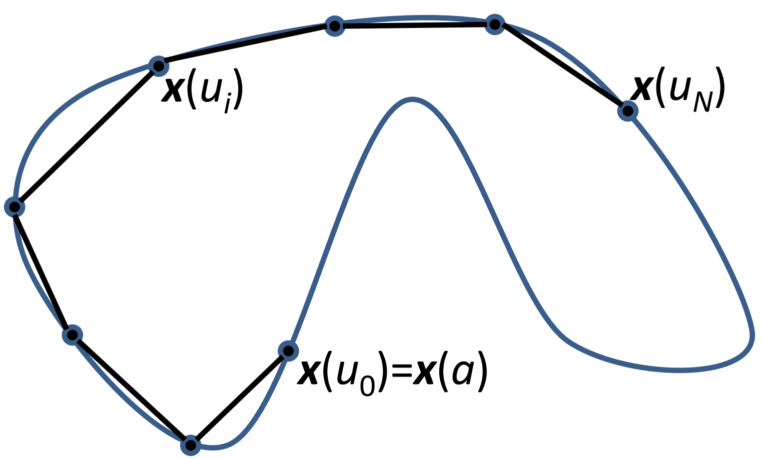
If we partition the interval $[a,u]$ into N sub-intervals, setting $\Delta u=(u-a)/N$ and $u_i=a+i\Delta u$:

$$\begin{aligned} s(u) &= \lim_{N\to\infty}\sum_{i=0}^{N-1}\left|\boldsymbol{x}(u_{i+1})-\boldsymbol{x}(u_i)\right| \\ &= \lim_{N\to\infty}\sum_{i=0}^{N-1}\frac{\left|\boldsymbol{x}(u_{i+1})-\boldsymbol{x}(u_i)\right|}{\Delta u}\Delta u \\ &= \lim_{N\to\infty}\sum_{i=0}^{N-1}\left|\boldsymbol{x}'(u_i)\right|\Delta u \\ &= \int_a^u \left|\boldsymbol{x}'(v)\right| dv \end{aligned}$$



Differentiable Curves

Definition:

We say that a regular curve is *parameterized by arc-length* if:

$$|\boldsymbol{x}'(u)| = 1$$

Differentiable Curves

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We say that a regular curve is *parameterized by arc-length* if:

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In this case:

$$s(u) = \int_a^u |\boldsymbol{x}'(v)| dv = \int_a^u dv = u - a$$

Regular Curves

Definition:

The *tangent* to the curve at $\boldsymbol{x}(u)$ is the unit vector pointing in the direction of the derivative:

$$\boldsymbol{t}(u) = \frac{\boldsymbol{x}'(u)}{\left\|\boldsymbol{x}'(u)\right\|}$$

If $\boldsymbol{x}$ is parameterized by arc-length:

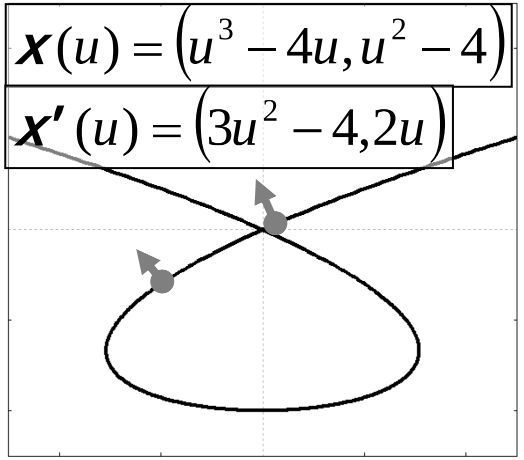
$$\boldsymbol{t}(u) = \boldsymbol{x}'(u)$$

Regular Curves

Definition:

The *normal* of the curve at $\boldsymbol{x}(u)$ is the unit vector that is perpendicular to the tangent:

$$\boldsymbol{n}(u) = \boldsymbol{t}(u)^{\perp} = \frac{\boldsymbol{x}'(u)^{\perp}}{\left\|\boldsymbol{x}'(u)\right\|} = \frac{\left(-y'(u), x'(u)\right)}{\sqrt{\left(x'(u)\right)^2 + \left(y'(u)\right)^2}}$$



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If $\boldsymbol{x}$ is parameterized by arc-length:

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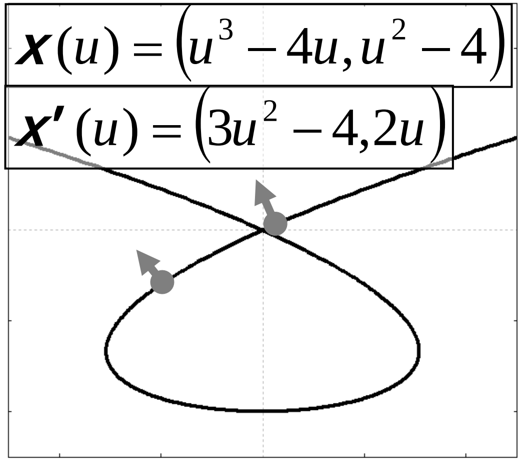
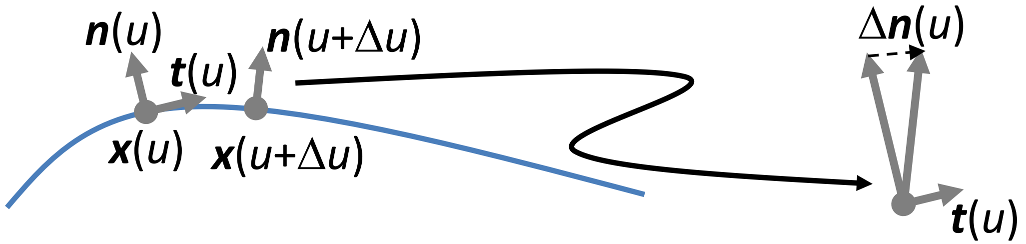


Regular Curves

<u>Claim</u>:

If we look at how the normal changes along a curve, we find that for small distances, the change is in the direction of the tangent:

$$\Delta\boldsymbol{n}(u)=\boldsymbol{n}(u+\Delta u)-\boldsymbol{n}(u)\approx \kappa(u)\boldsymbol{t}(\mathrm{u})$$



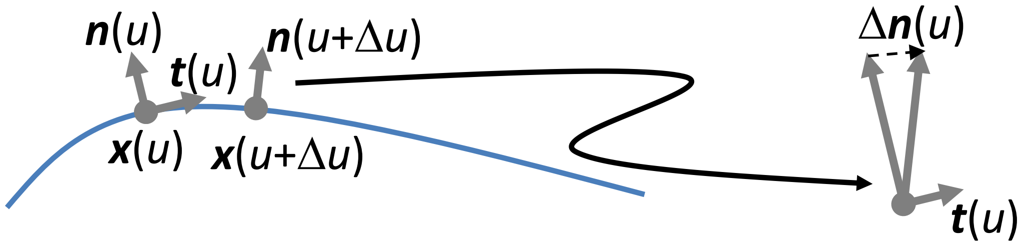
Regular Curves

$$\Delta \boldsymbol{n}(u) = \boldsymbol{n}(u+\Delta u) - \boldsymbol{n}(u) \approx \kappa(u)\boldsymbol{t}(\mathrm{u})$$

<u>Proof</u>:

Since $\boldsymbol{n}(u)$ is a unit-vector, we know that:

$$1 = \langle \boldsymbol{n}(u), \boldsymbol{n}(u) \rangle$$



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<u>Proof</u>:

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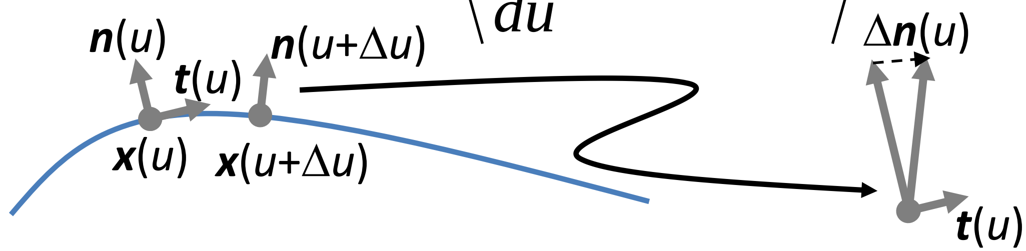
$$1=\langle \boldsymbol{n}(u),\boldsymbol{n}(u)\rangle$$

Taking derivatives of both sides, we get:

$$\begin{aligned}0&=\frac{d}{du}\langle \boldsymbol{n}(u),\boldsymbol{n}(u)\rangle\\&=2\left\langle \frac{d}{du}\boldsymbol{n}(u),\boldsymbol{n}(u)\right\rangle\end{aligned}$$



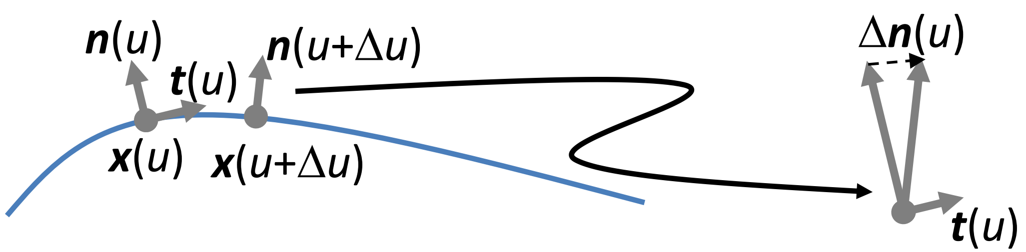
Regular Curves

$$\Delta \boldsymbol{n}(u)=\boldsymbol{n}(u+\Delta u)-\boldsymbol{n}(u)\approx \kappa(u)\boldsymbol{t}(\mathrm{u})$$

<u>Proof</u>:

$$0=\left\langle \frac{d}{du}\boldsymbol{n}(u),\boldsymbol{n}(u)\right\rangle$$

Thus, the change in the normal is perpendicular to the normal direction, so it's aligned with the tangent.



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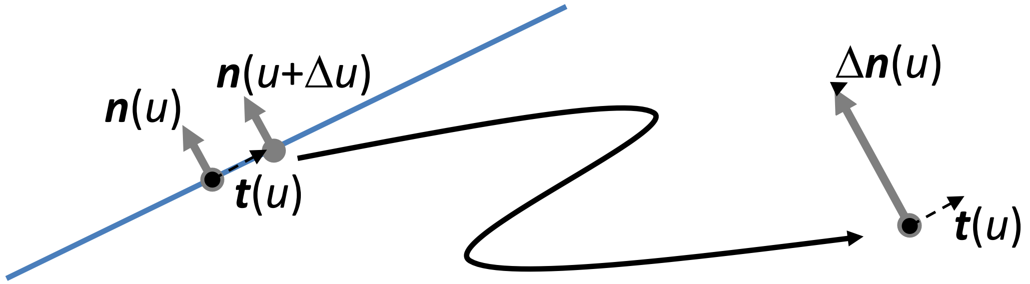
<u>Note</u>:

If we look at the value of κ we see that it's

– zero for straight curves



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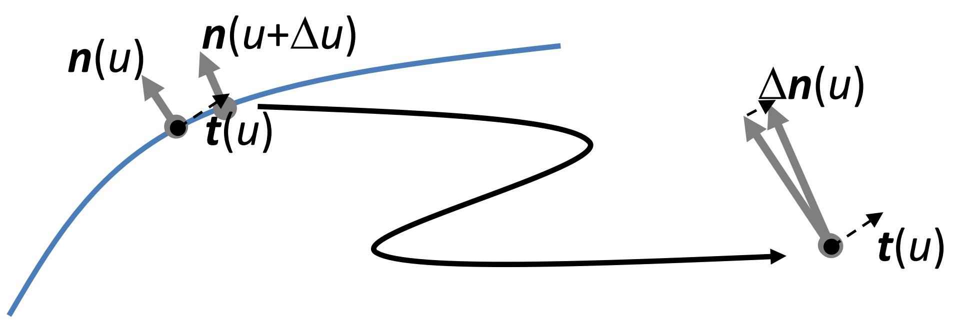
Note:

If we look at the value of κ we see that it's

- zero for straight curves
- small/positive for convex curves that turn slowly



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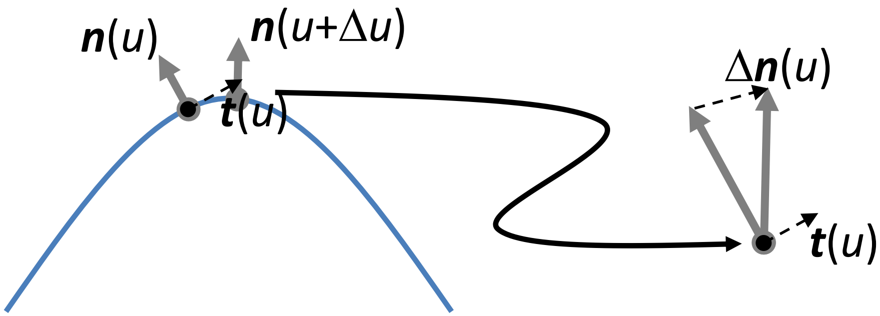
- large/positive for convex curves that turn quickly



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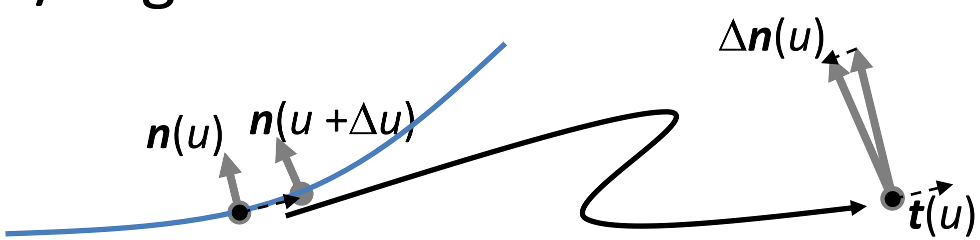
- large/positive for convex curves that turn quickly
- small/negative for concave curves that turn slowly



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- small/positive for convex curves that turn slowly
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- small/negative for concave curves that turn slowly
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Regular Curves

<u>Definition</u>:

The *curvature* at $\boldsymbol{x}(u)$ is the change in normal vector along the tangent direction relative to change in distance along the curve:

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\boldsymbol{n}(u+\Delta u) - \boldsymbol{n}(u)}{\Delta s}, \boldsymbol{t}(u) \right\rangle$$

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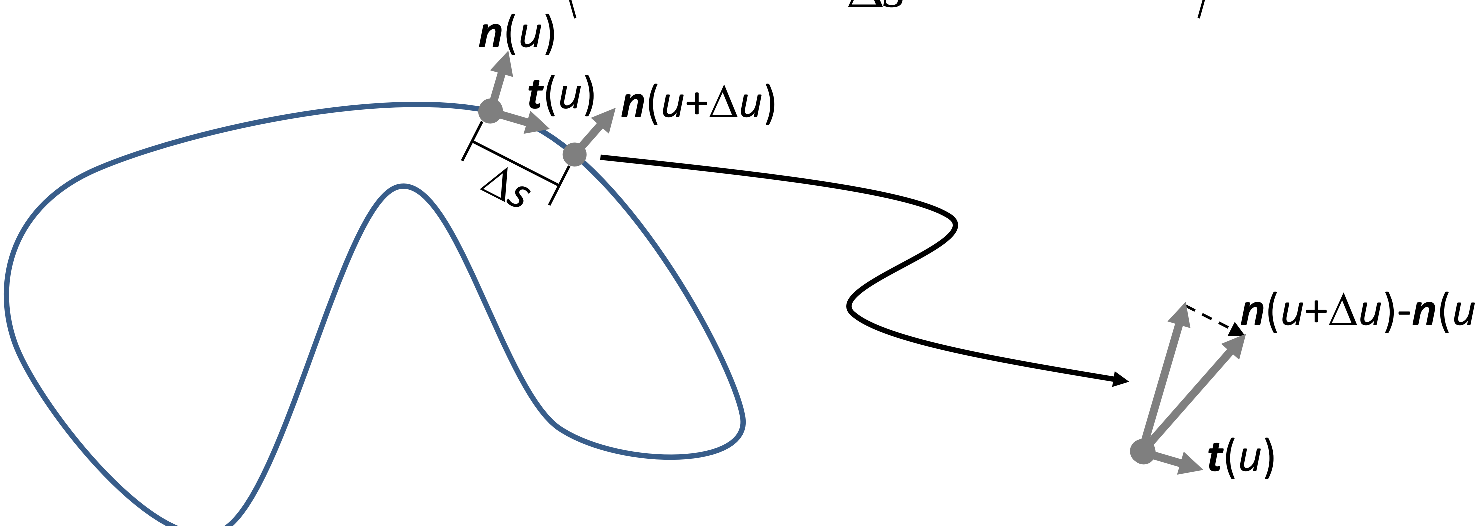

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<u>Note</u>:

If $\boldsymbol{x}$ is parameterized by arc-length, then $\Delta s=\Delta u$ so the curvature becomes:

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\boldsymbol{n}(u+\Delta u) - \boldsymbol{n}(\Delta u)}{\Delta u}, \boldsymbol{t}(u) \right\rangle = \langle \boldsymbol{n}'(u), \boldsymbol{t}(u) \rangle$$

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Otherwise, we have $\Delta s / \Delta u = |\boldsymbol{x}'(u)|$, so that:

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\boldsymbol{n}(u+\Delta u) - \boldsymbol{n}(u)}{\Delta u \cdot |\boldsymbol{x}'(u)|}, \boldsymbol{t}(u) \right\rangle = \frac{\left\langle \boldsymbol{n}'(u), \boldsymbol{t}(u) \right\rangle}{|\boldsymbol{x}'(u)|}$$

Differential Geometry

- Curves
- Surfaces

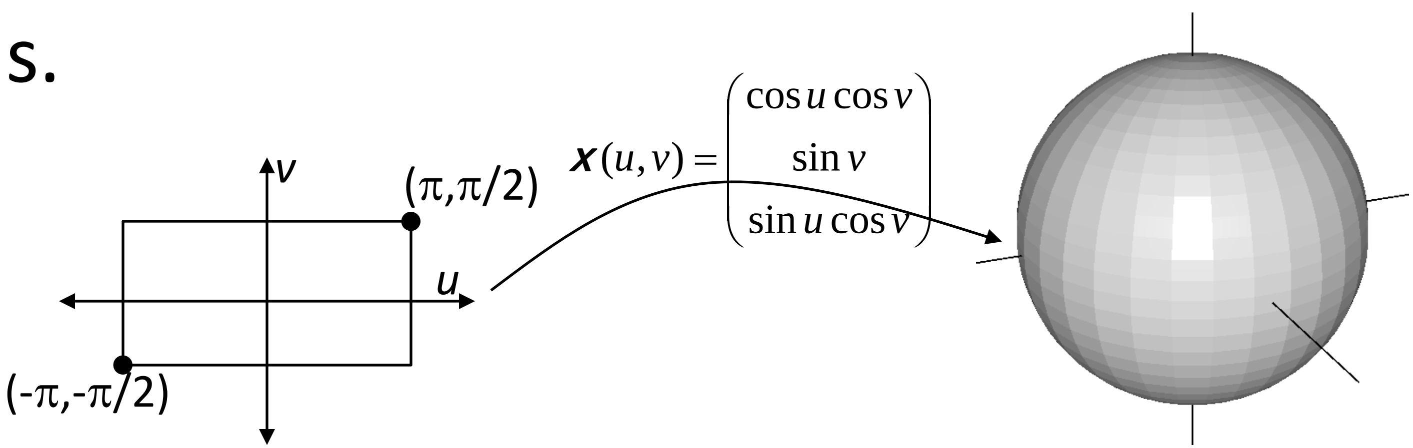
Differentiable Surfaces

<u>Definition</u>:

A *parameterized differentiable surface* is a differentiable map $\boldsymbol{x}:\Omega\rightarrow\boldsymbol{R}^3$ of an open domain $\Omega\subset\boldsymbol{R}^2$ into $\boldsymbol{R}^3$:

$$\boldsymbol{x}(u,v)=(x(u,v),y(u,v),z(u,v))$$

where $x(u,v)$, $y(u,v)$, and $z(u,v)$ are differentiable functions.

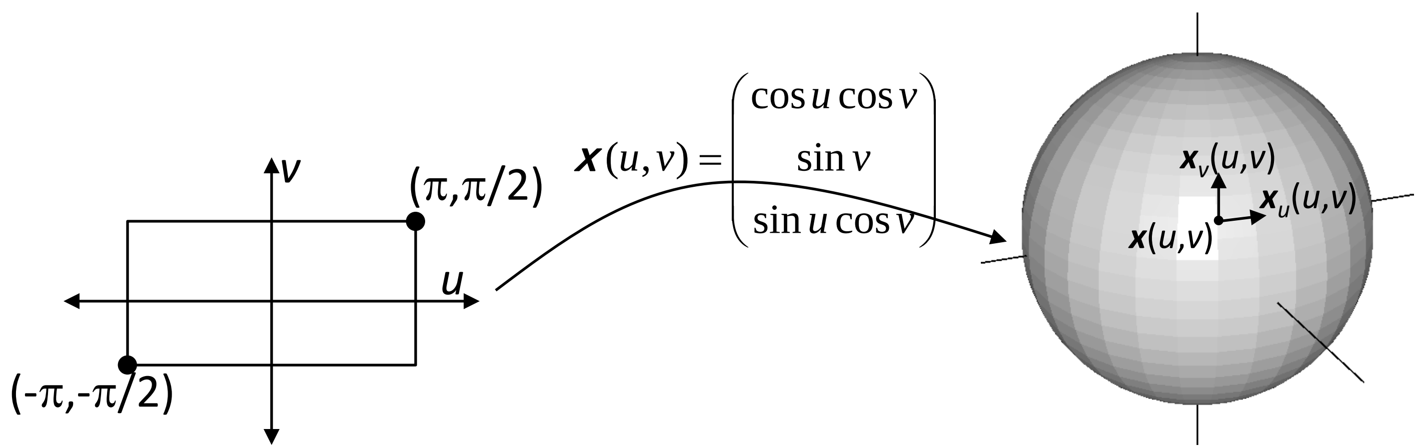


Differentiable Surfaces

<u>Definition</u>:

The *derivatives* of the surface at $\boldsymbol{x}(u,v)$ are the vectors:

$$\boldsymbol{x}_u(u,v) = \frac{\partial \boldsymbol{x}(u,v)}{\partial u} = \begin{pmatrix} \partial x / \partial u \\ \partial y / \partial u \\ \partial z / \partial u \end{pmatrix} \qquad \boldsymbol{x}_v(u,v) = \frac{\partial \boldsymbol{x}(u,v)}{\partial v} = \begin{pmatrix} \partial x / \partial v \\ \partial y / \partial v \\ \partial z / \partial v \end{pmatrix}$$



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Definition:

The surface is said to be *regular* if at each point (u,v) the derivatives/tangents $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are linearly independent.

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This is equivalent to the statement:

$$\boldsymbol{x}_u \times \boldsymbol{x}_v \neq 0$$

i.e. that a normal (line) can be defined everywhere.

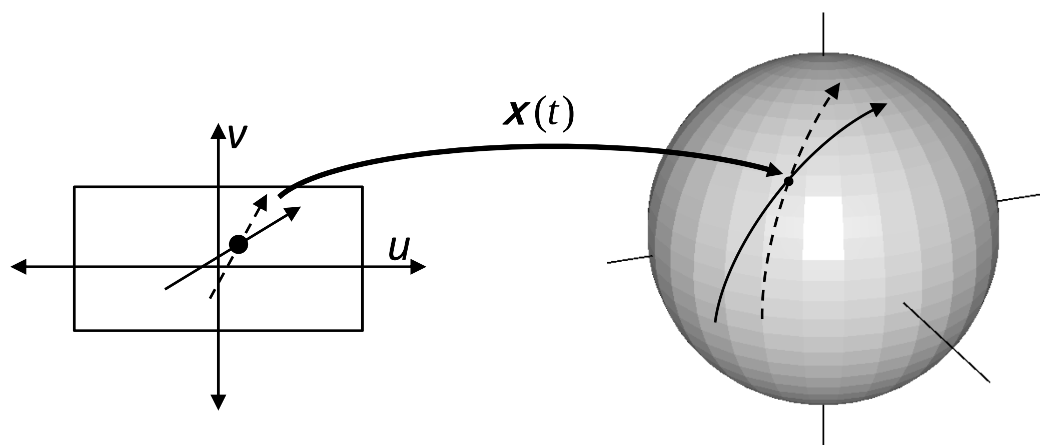
Differentiable Surfaces

$$\boldsymbol{x}_u(u,v)=\frac{\partial \boldsymbol{x}(u,v)}{\partial u} \qquad \boldsymbol{x}_v(u,v)=\frac{\partial \boldsymbol{x}(u,v)}{\partial v}$$

<u>Definition</u>:

Given a point $p_0=(u_0,v_0)\in\Omega$ and given a direction $w=(u_w,v_w)$ in the parameter space, we can define the (3D) curve:

$$\boldsymbol{x}(t)=\boldsymbol{x}\left(p_0+tw\right)$$



Differentiable Surfaces

$$\boldsymbol{x}_u(u,v)=\frac{\partial \boldsymbol{x}(u,v)}{\partial u} \qquad \boldsymbol{x}_v(u,v)=\frac{\partial \boldsymbol{x}(u,v)}{\partial v}$$

<u>Definition</u>:

$$\boldsymbol{x}(t)=\boldsymbol{x}\left(p_0+tw\right)$$

Taking the derivative at *t*=0, we get:

$$\boldsymbol{x}'(0)=w_u\boldsymbol{x}_u+w_v\boldsymbol{x}_v=\boldsymbol{J}(w)$$

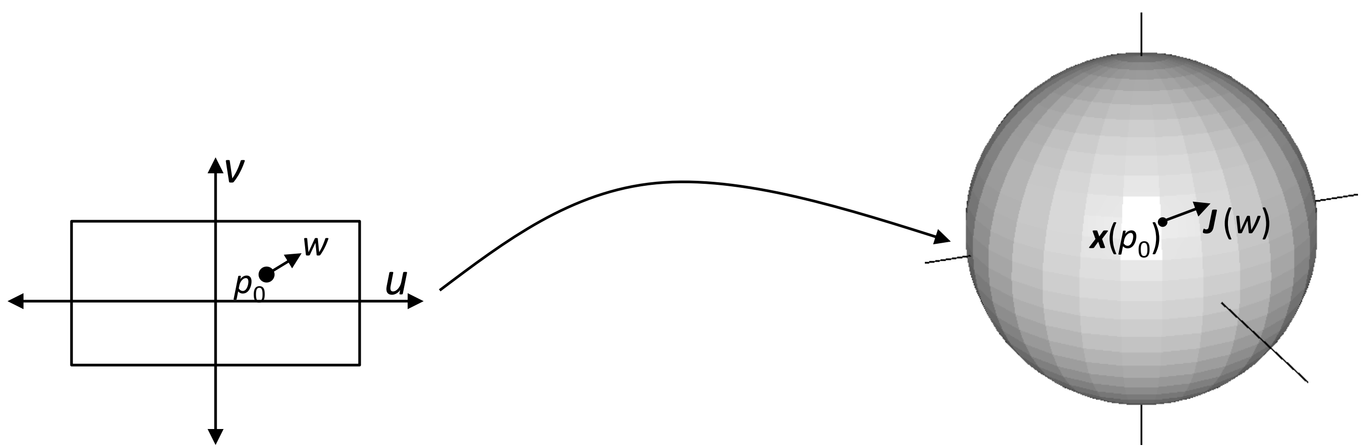
where ***J*** is the Jacobian matrix taking directions in Ω to tangent vectors on the surface:

$$\boldsymbol{J}=\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

Metric Properties

Thus, given a point $p_0=(u_0,v_0)\in\Omega$ and given a direction $w=(u_w,v_w)$, we can use the Jacobian to compute the length of the corresponding tangent vector over $\boldsymbol{x}(p_0)$:

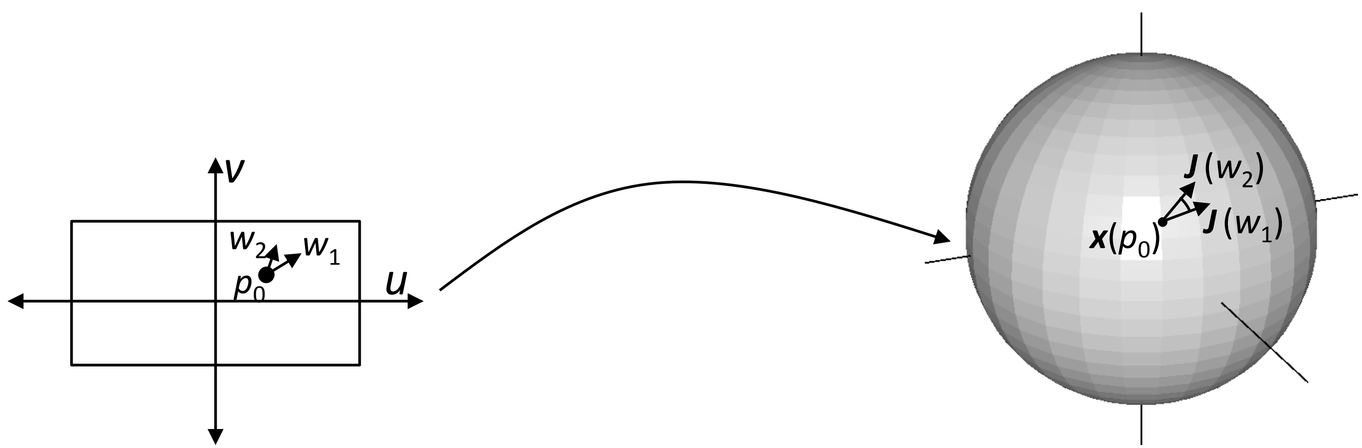
$$length^2=\left\|\boldsymbol{J}w\right\|^2=w^t\boldsymbol{J}^t\boldsymbol{J}w$$



Metric Properties

Similarly, given a point $p_0=(u_0,v_0)\in\Omega$ and given directions $w_1=(u_1,v_1)$ and $w_2=(u_2,v_2)$ we can use the Jacobian to compute the angle of the corresponding tangent vectors over $\boldsymbol{x}(p_0)$:

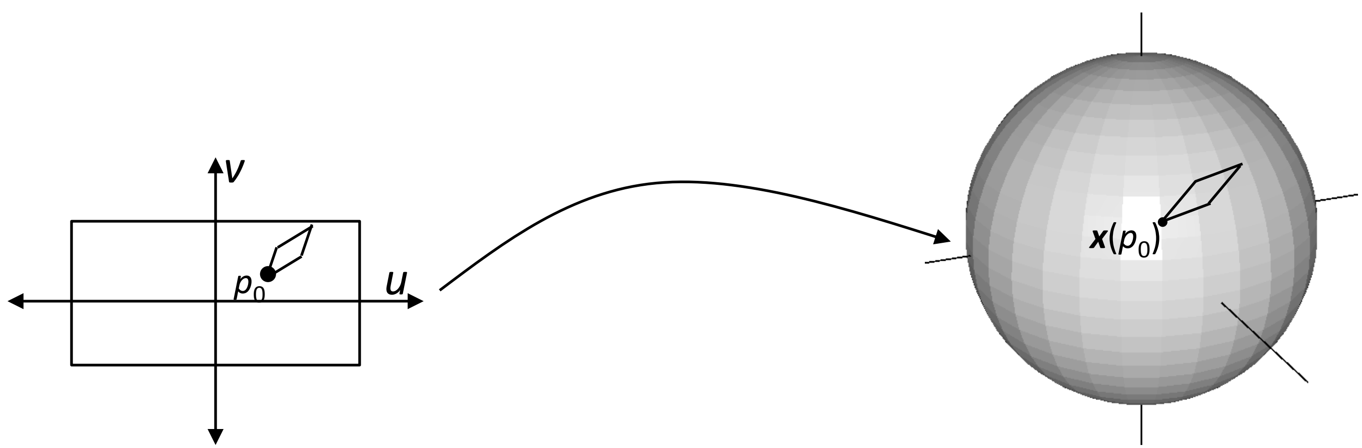
$$\cos(angle)=\frac{\left\langle \boldsymbol{J}w_1,\boldsymbol{J}w_2\right\rangle}{\left\|\boldsymbol{J}w_1\right\|\left\|\boldsymbol{J}w_2\right\|}=\frac{w_1^t\boldsymbol{J}^t\boldsymbol{J}w_2}{\sqrt{w_1^t\boldsymbol{J}^t\boldsymbol{J}w_1}\sqrt{w_2^t\boldsymbol{J}^t\boldsymbol{J}w_2}}$$



Metric Properties

Finally, given a point $p_0=(u_0,v_0)\in\Omega$ and given directions $w_1=(u_1,v_1)$ and $w_2=(u_2,v_2)$ we can use the Jacobian to compute the area of the corresponding parallelogram in the tangent space:

$$area = length_1 \cdot length_2 \cdot \sin(angle)$$



Metric Properties

$$length^2 = \|\boldsymbol{J}w\|^2 = w^t \boldsymbol{J}^t \boldsymbol{J}w$$

$$\cos(angle) = \frac{\langle \boldsymbol{J}w_1, \boldsymbol{J}w_2 \rangle}{\|\boldsymbol{J}w_1\| \|\boldsymbol{J}w_2\|} = \frac{w_1^t \boldsymbol{J}^t \boldsymbol{J}w_2}{\sqrt{w_1^t \boldsymbol{J}^t \boldsymbol{J}w_1} \sqrt{w_2^t \boldsymbol{J}^t \boldsymbol{J}w_2}}$$

$$area = length_1 \cdot length_2 \cdot \sin(angle)$$

Definition:

The matrix ***I*=*J*^{t}*J*** is called the *first fundamental form* and provides all the information needed to measure distances, angles, and areas on the surface.

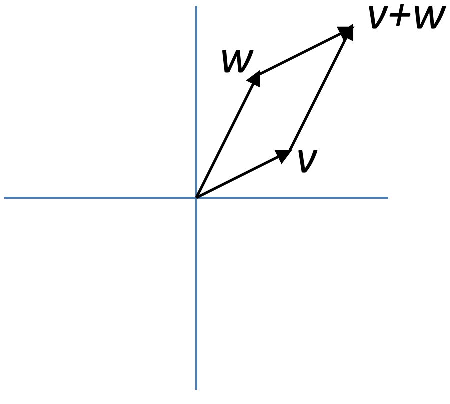
$$\boldsymbol{I} = \boldsymbol{J}^t \boldsymbol{J} = \begin{pmatrix} \boldsymbol{X}_u^t \\ \\ \boldsymbol{X}_v^t \end{pmatrix} \begin{pmatrix} \boldsymbol{X}_u & \boldsymbol{X}_v \end{pmatrix} = \begin{pmatrix} \langle \boldsymbol{X}_u, \boldsymbol{X}_u \rangle & \langle \boldsymbol{X}_u, \boldsymbol{X}_v \rangle \\ \\ \langle \boldsymbol{X}_v, \boldsymbol{X}_u \rangle & \langle \boldsymbol{X}_v, \boldsymbol{X}_v \rangle \end{pmatrix}$$

Metric Properties

<u>Note</u>:

Given vectors *v* and *w* in $\boldsymbol{R}^n$, the area of the parallelogram spanned by *v* and *w* is:

$$
\begin{aligned}
Area(v,w) &= |v|\cdot|w|\cdot\sin\big(Angle(v,w)\big) \\
&= |v|\cdot|w|\cdot\sqrt{1-\cos^2 Angle(v,w)} \\
&= |v|\cdot|w|\cdot\sqrt{1-\frac{\langle v,w\rangle^2}{|v|^2|w|^2}} \\
&= \sqrt{|v|^2|w|^2-\langle v,w\rangle^2}
\end{aligned}
$$



Metric Properties

$$Area(v,w)=\sqrt{|v|^2|w|^2-\langle v,w\rangle}$$

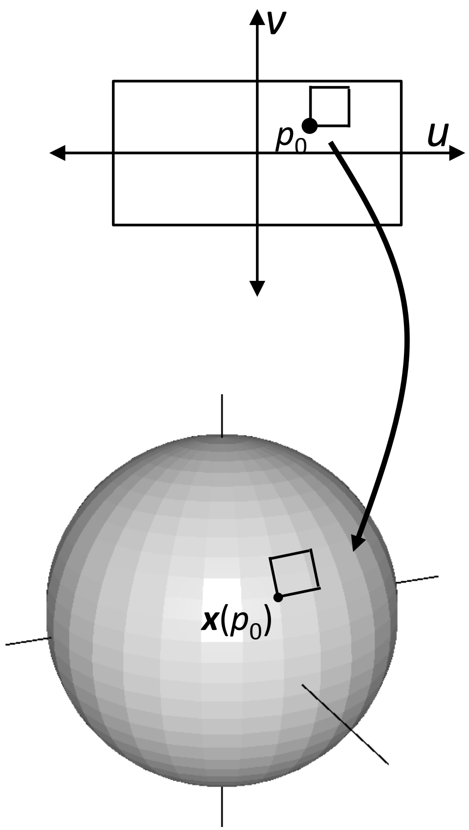
<u>Note</u>:

Since the first fundamental form is defined as:

$$\boldsymbol{I}=\boldsymbol{J}^t\boldsymbol{J}=\begin{pmatrix}\langle \boldsymbol{x}_u,\boldsymbol{x}_u\rangle & \langle \boldsymbol{x}_u,\boldsymbol{x}_v\rangle \\ \langle \boldsymbol{x}_v,\boldsymbol{x}_u\rangle & \langle \boldsymbol{x}_v,\boldsymbol{x}_v\rangle\end{pmatrix}$$

in mapping from Ω to the surface, the area of a tiny patch of surface gets scaled by:

$$\sqrt{\|\boldsymbol{x}_u\|^2\|\boldsymbol{x}_v\|^2-\langle \boldsymbol{x}_u,\boldsymbol{x}_v\rangle^2}=\sqrt{\det \boldsymbol{I}}$$

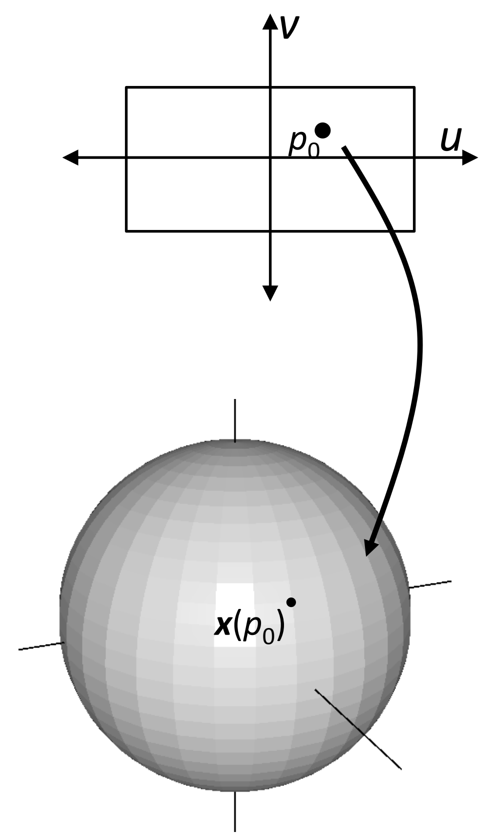


Metric Properties

Example (Sphere):

We can parameterize the sphere (almost everywhere) by the map:

$$\boldsymbol{x}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix}^t$$



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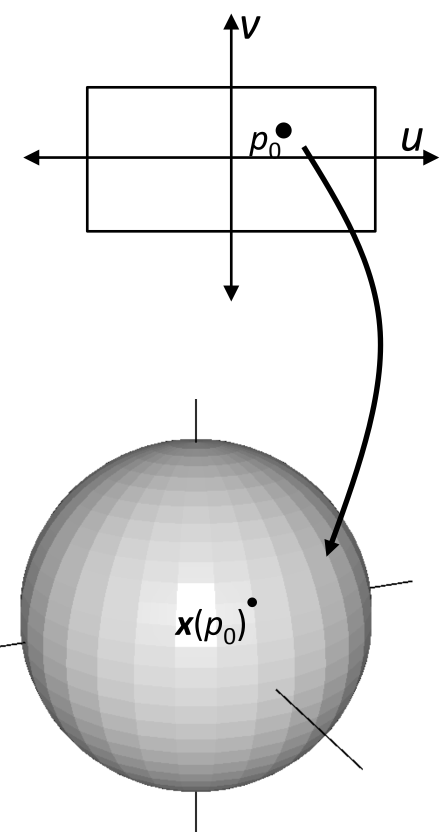
Then the partial derivatives are:

$$\boldsymbol{x}_u(u,v) = \begin{pmatrix} -\sin u \cos v & 0 & \cos u \cos v \end{pmatrix}$$
$$\boldsymbol{x}_v(u,v) = \begin{pmatrix} -\cos u \sin v & \cos v & -\sin u \sin v \end{pmatrix}$$



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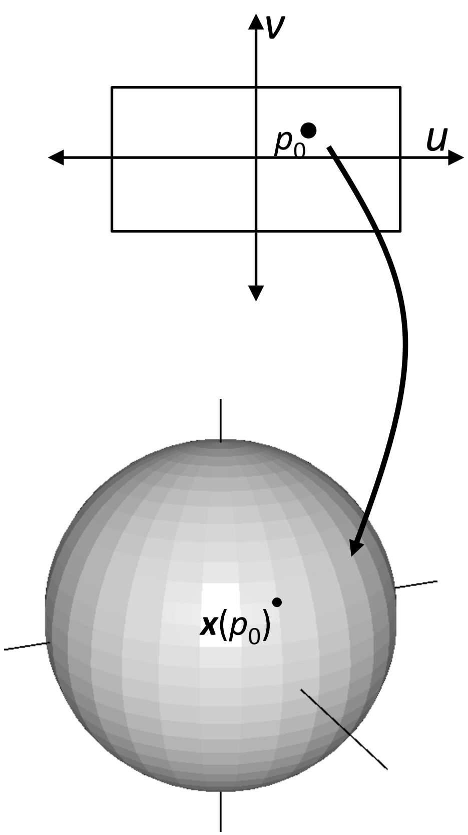
We can parameterize the sphere (almost everywhere) by the map:

$$\boldsymbol{x}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix}^t$$



Then the partial derivatives are:

$$\boldsymbol{x}_u(u,v) = \begin{pmatrix}-\sin u \cos v & 0 & \cos u \cos v\end{pmatrix}$$
$$\boldsymbol{x}_v(u,v) = \begin{pmatrix}-\cos u \sin v & \cos v & -\sin u \sin v\end{pmatrix}$$

Which gives:

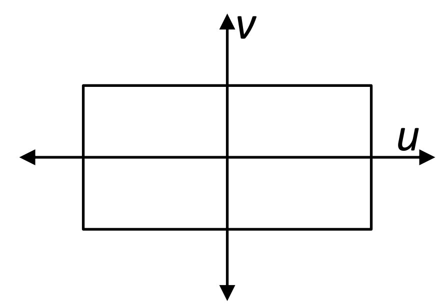
$$\boldsymbol{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

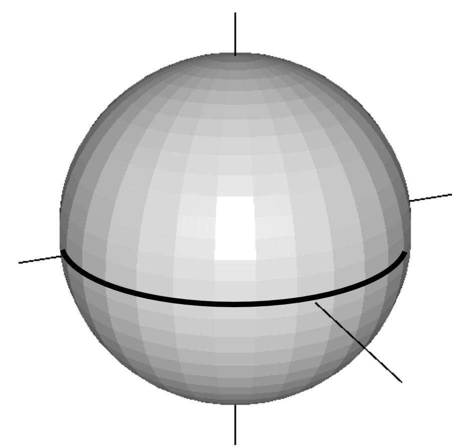
Metric Properties

$$\boldsymbol{x}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix}^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the length of the equator?





Metric Properties

$$\boldsymbol{x}(u,v)=\begin{pmatrix}\cos u\cos v & \sin v & \sin u\cos v\end{pmatrix}^t \qquad \boldsymbol{I}(u,v)=\begin{pmatrix}\cos^2 v & 0\\ 0 & 1\end{pmatrix}$$

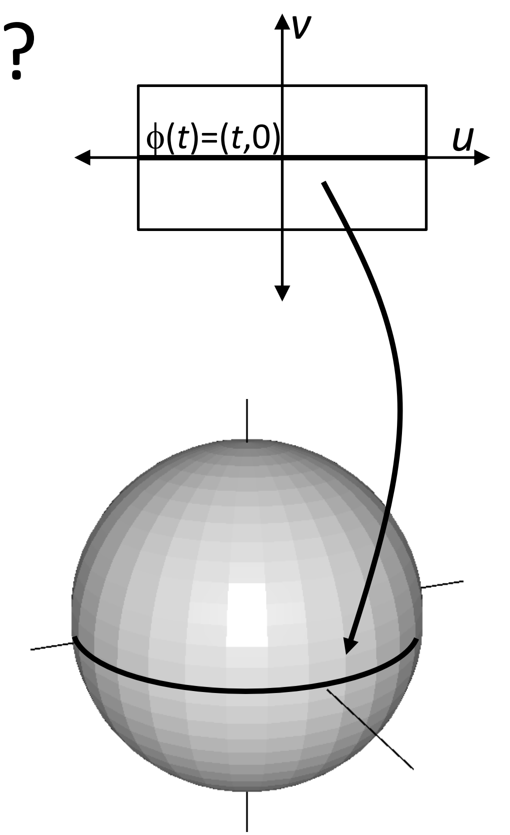
<u>Example (Sphere)</u>:

- What is the length of the equator?

The equator is the image of:

$\phi(t)=(t,0)$ with $t\in[-\pi,\pi]$

under the parameterization.

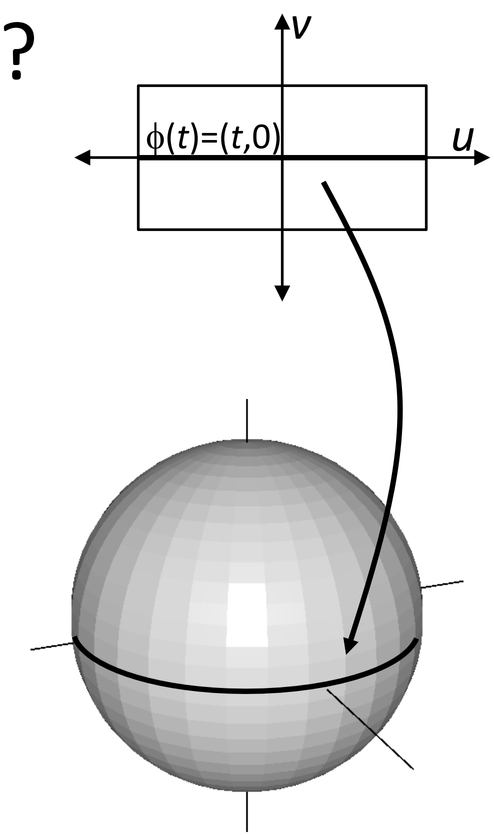


Metric Properties

$$\boldsymbol{X}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix}^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the length of the equator?



$$length(\boldsymbol{X} \circ \phi) = \int_{-\pi}^{\pi} \sqrt{\phi'(t)^t \boldsymbol{I} \phi'(t)}\, dt$$

$$= \int_{-\pi}^{\pi} \sqrt{(1,0)^t \begin{pmatrix}\cos^2(0) & 0 \\ 0 & 1\end{pmatrix} (1,0)}\, dt$$

$$= \int_{-\pi}^{\pi} dt$$

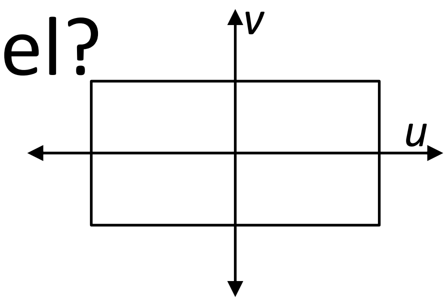
$$= 2\pi$$

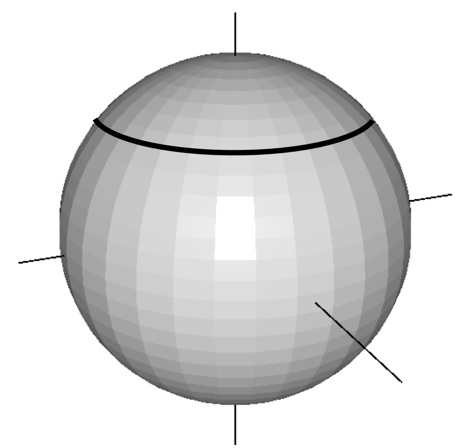
Metric Properties

$$\boldsymbol{x}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix}^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

Example (Sphere):

- What is the length of the w^{th} parallel?



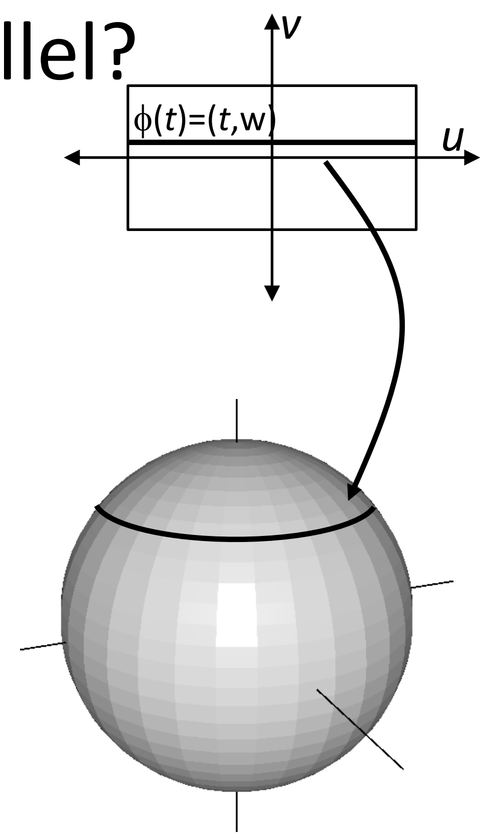


Metric Properties

$$\boldsymbol{x}(u,v) = \begin{pmatrix} \cos u \cos v & \sin v & \sin u \cos v \end{pmatrix}^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the length of the w^{th} parallel?



The w^{th} parallel is the image of:

$$\phi(t)=(t,w) \quad \text{with } t\in[-\pi, \pi]$$

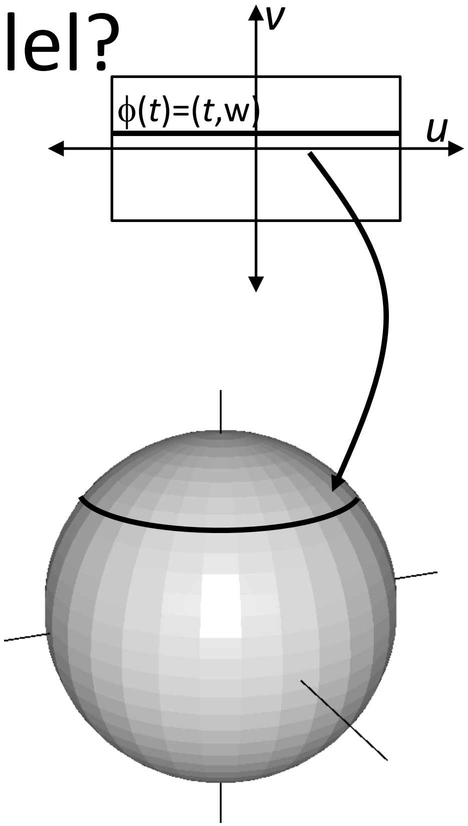
under the parameterization.

Metric Properties

$$\boldsymbol{X}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix}^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the length of the w^{th} parallel?



$$length(\boldsymbol{X}\circ\phi) = \int_{-\pi}^{\pi} \sqrt{\phi'(t)^t \boldsymbol{I} \phi'(t)}\,dt$$

$$= \int_{-\pi}^{\pi} \sqrt{(1,0)^t \begin{pmatrix}\cos^2 w & 0 \\ 0 & 1\end{pmatrix}(1,0)}\,dt$$

$$= \int_{-\pi}^{\pi} \cos w \, dt$$

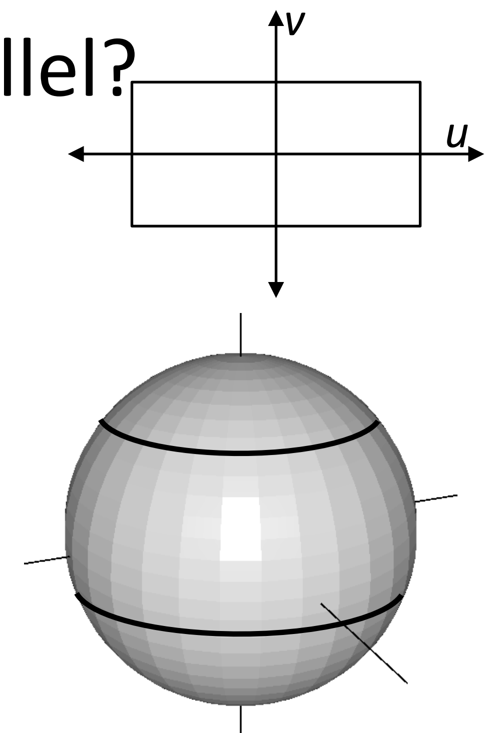
$$= 2\pi \cos w$$

Metric Properties

$$\boldsymbol{x}(u,v) = \begin{pmatrix} \cos u \cos v & \sin v & \sin u \cos v \end{pmatrix}^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the area of the band between the w_1th parallel and the w_2th parallel?

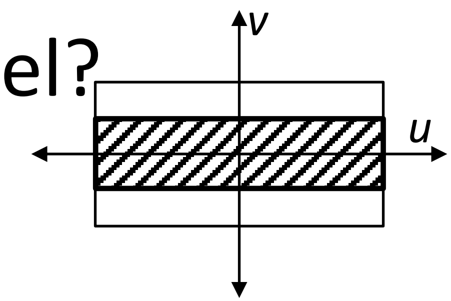


Metric Properties

$$\boldsymbol{x}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix}^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

<u>Example (Sphere)</u>:

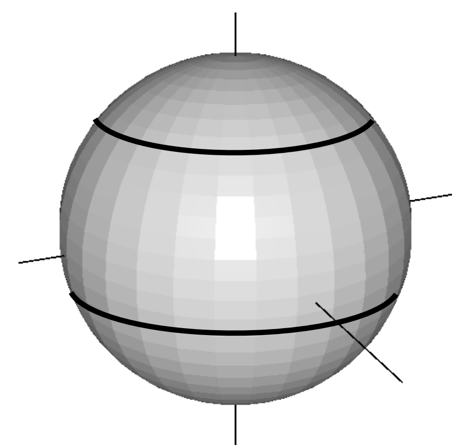
- What is the area of the band between the w_1th parallel and the w_2th parallel?



The band is the image of:

$\phi(s,t)=(s,t)$ with $s\in[-\pi,\pi]$, $t\in[w_1,w_2]$

under the parameterization.

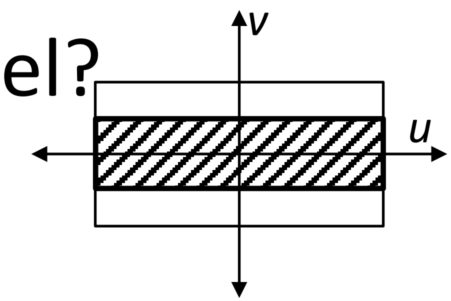


Metric Properties

$$\boldsymbol{X}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix}^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the area of the band between the w_1th parallel and the w_2th parallel?

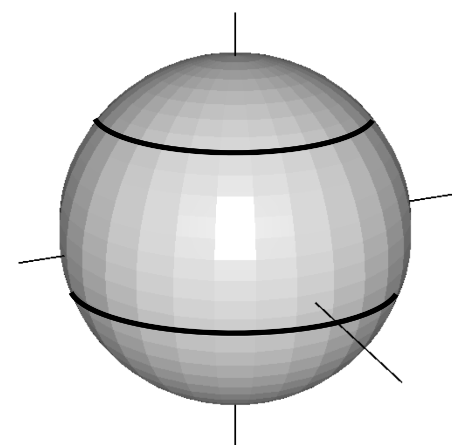


$$area(\boldsymbol{X} \circ \phi) = \int_{w_1}^{w_2} \int_{-\pi}^{\pi} \sqrt{\det \boldsymbol{I}}\, ds\, dt$$

$$= \int_{w_1}^{w_2} \int_{-\pi}^{\pi} \cos t\, ds\, dt$$

$$= 2\pi \int_{w_1}^{w_2} \cos t\, dt$$

$$= 2\pi\left(\sin w_2 - \sin w_1\right)$$

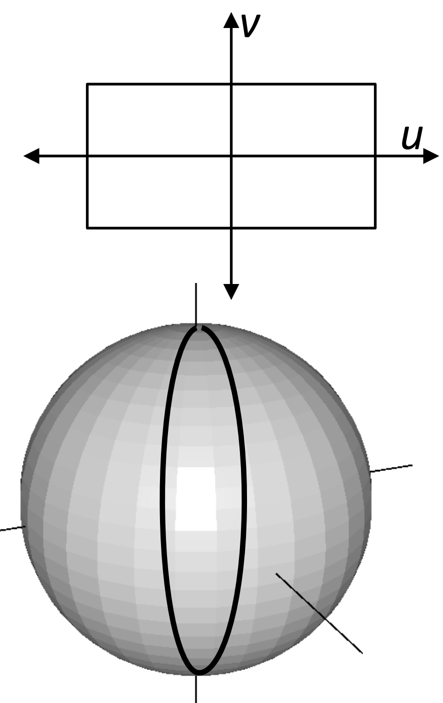


Metric Properties

$$\boldsymbol{x}(u,v)=\begin{pmatrix}\cos u\cos v & \sin v & \sin u\cos v\end{pmatrix}^t \qquad \boldsymbol{I}(u,v)=\begin{pmatrix}\cos^2 v & 0\\ 0 & 1\end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the area of the band between the w_1th and the w_2th meridians?



Metric Properties

$$\boldsymbol{x}(u,v) = \left(\cos u \cos v \quad \sin v \quad \sin u \cos v\right)^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

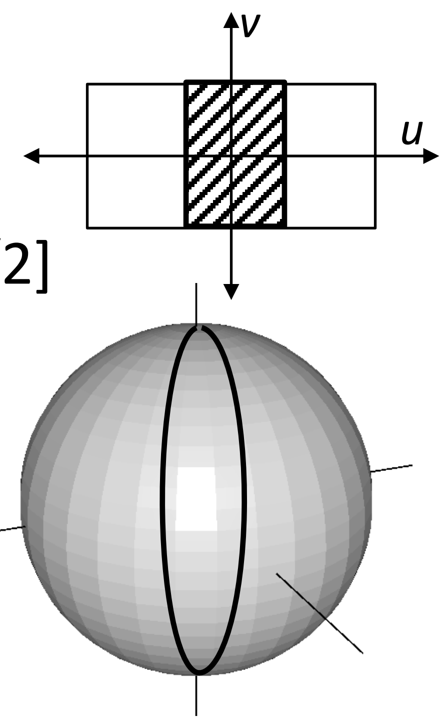
Example (Sphere):

- What is the area of the band between the w_1^{th} and the w_2^{th} meridians?

The band is the image of:

$$\phi(s,t)=(s,t) \quad \text{with } s \in [w_1, w_2],\ t \in [-\pi/2, \pi/2]$$

under the parameterization.

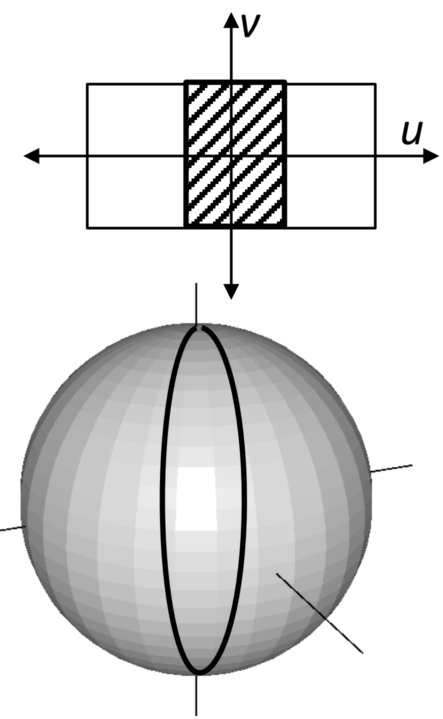


Metric Properties

$$\boldsymbol{x}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix}^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the area of the band between the w_1th and the w_2th meridians?



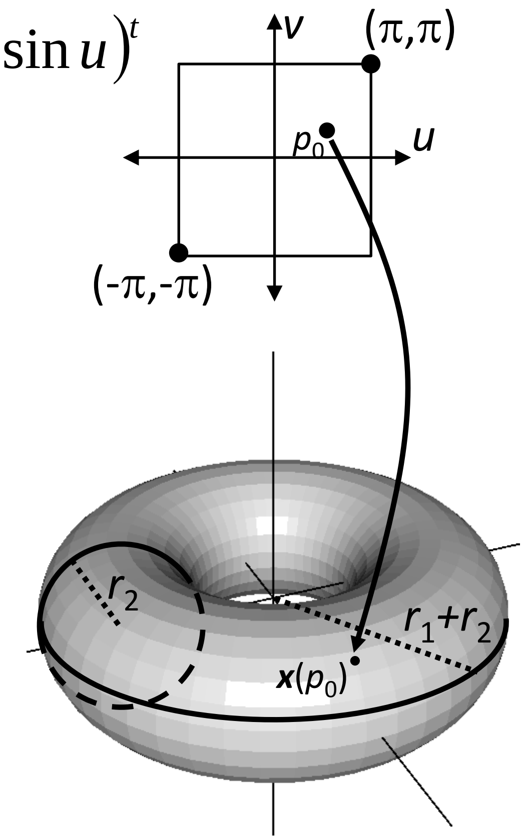
$$\begin{aligned} area(\boldsymbol{x} \circ \phi) &= \int_{-\pi/2}^{\pi/2} \int_{w_1}^{w_2} \sqrt{\det \boldsymbol{I}}\, ds\, dt \\ &= \int_{-\pi/2}^{\pi/2} \int_{w_1}^{w_2} \cos t \, ds\, dt \\ &= (w_2 - w_1) \int_{-\pi/2}^{\pi/2} \cos t \; dt \\ &= (w_2 - w_1)\big(\sin(\pi/2) - \sin(-\pi/2)\big) \\ &= 2(w_2 - w_1) \end{aligned}$$

Metric Properties

Example (Torus):

We can parameterize the torus by the map:

$$\boldsymbol{x}(u,v) = \big((r_1 + r_2\sin v)\cos u \quad r_2\cos v \quad (r_1 + r_2\sin v)\sin u\big)^t$$



Metric Properties

<u>Example (Torus)</u>:

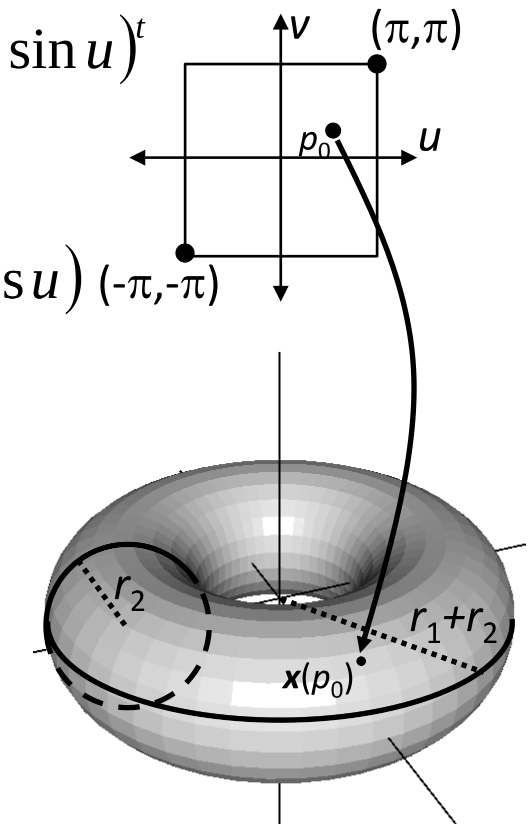
We can parameterize the torus by the map:

$$\boldsymbol{x}(u,v) = \big((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\big)^t$$

Then the partial derivatives are:

$$\boldsymbol{x}_u(u,v) = \big(-(r_1 + r_2 \sin v)\sin u \quad 0 \quad (r_1 + r_2 \sin v)\cos u\big)$$

$$\boldsymbol{x}_v(u,v) = \big(r_2 \cos v \cos u \quad -r_2 \sin v \quad r_2 \cos v \sin u\big)$$



Metric Properties

<u>Example (Torus)</u>:

We can parameterize the torus by the map:

$$\boldsymbol{x}(u,v) = \left((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\right)^t$$

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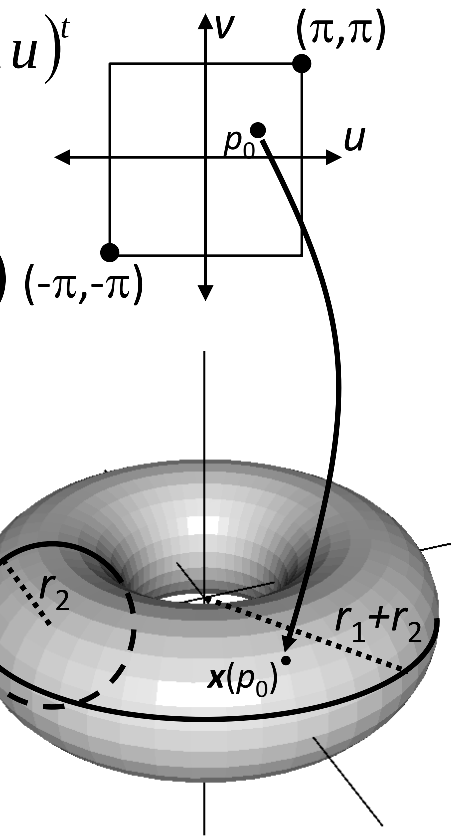
$$\boldsymbol{x}_u(u,v) = \left(-(r_1 + r_2 \sin v)\sin u \quad 0 \quad (r_1 + r_2 \sin v)\cos u\right)$$

$$\boldsymbol{x}_v(u,v) = \left(r_2 \cos v \cos u \quad -r_2 \sin v \quad r_2 \cos v \sin u\right)$$



Which gives:

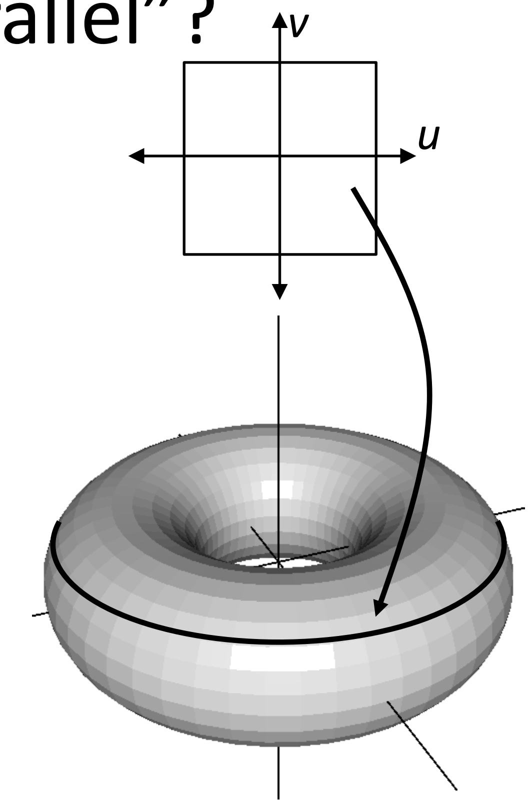
$$\boldsymbol{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

Metric Properties

$$\boldsymbol{x}(u,v) = \left((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\right)^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

<u>Example (Torus)</u>:

- What is the length of the w^{th} "parallel"?

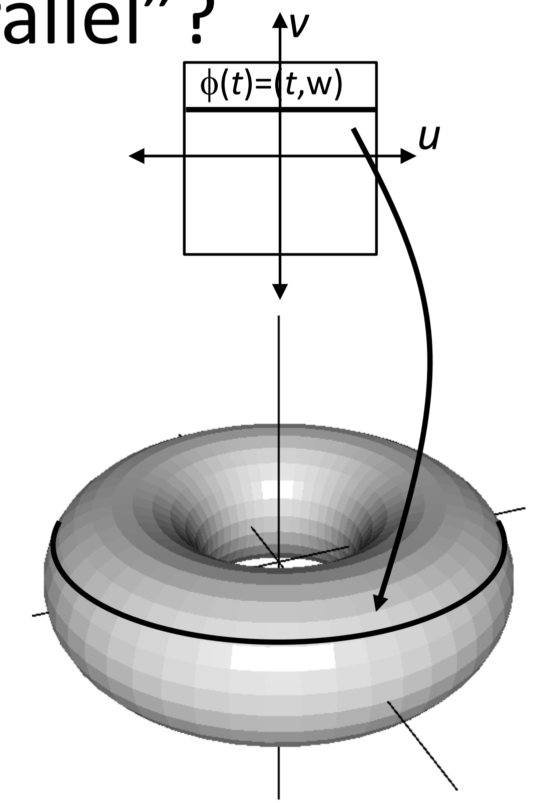


Metric Properties

$$\boldsymbol{x}(u,v) = \left((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\right)^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

<u>Example (Torus)</u>:

- What is the length of the w^{th} "parallel"?



The w^{th} parallel is the image of:

$$\phi(t) = (t, w) \quad \text{with } t \in [-\pi, \pi]$$

under the parameterization.

Metric Properties

$$\boldsymbol{x}(u,v) = \left((r_1 + r_2\sin v)\cos u \quad r_2\cos v \quad (r_1 + r_2\sin v)\sin u\right)^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix} (r_1 + r_2\sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

<u>Example (Torus)</u>:

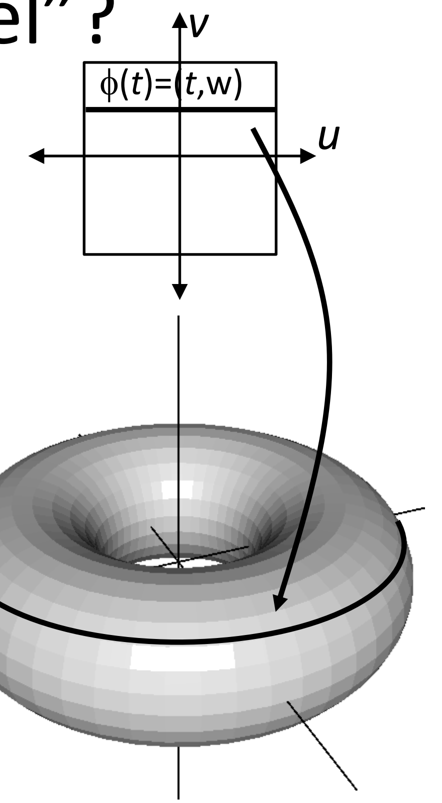
- What is the length of the w^{th} "parallel"?

$$length(\boldsymbol{x}\circ\phi) = \int_{-\pi}^{\pi} \sqrt{\phi'(t)^t \boldsymbol{I} \phi'(t)}\, dt$$

$$= \int_{-\pi}^{\pi} \sqrt{(1,0)^t \begin{pmatrix} (r_1 + r_2\sin w)^2 & 0 \\ 0 & r_2^2 \end{pmatrix} (1,0)}\, dt$$

$$= \int_{-\pi}^{\pi} r_1 + r_2 \sin w\, dt$$

$$= 2\pi(r_1 + r_2 \sin w)$$

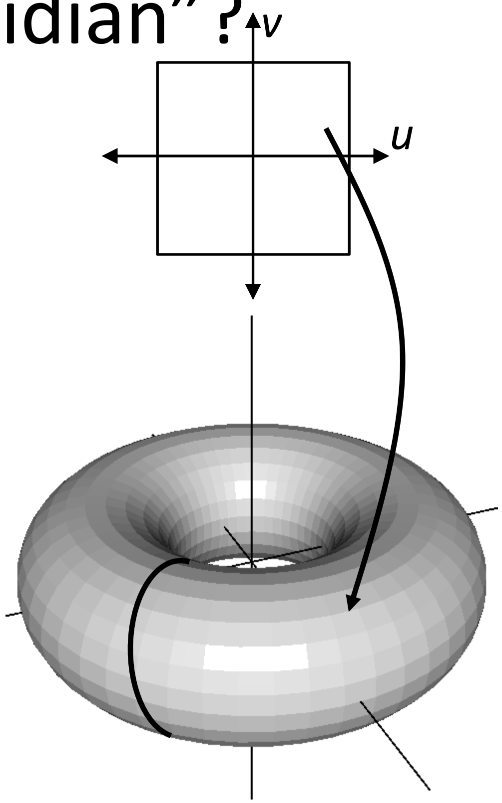


Metric Properties

$$\boldsymbol{x}(u,v)=\big((r_1+r_2\sin v)\cos u \quad r_2\cos v \quad (r_1+r_2\sin v)\sin u\big)^t \qquad \boldsymbol{I}(u,v)=\begin{pmatrix}(r_1+r_2\sin v)^2 & 0\\ 0 & r_2^2\end{pmatrix}$$

<u>Example (Torus)</u>:

- What is the length of the w^{th} "meridian"?

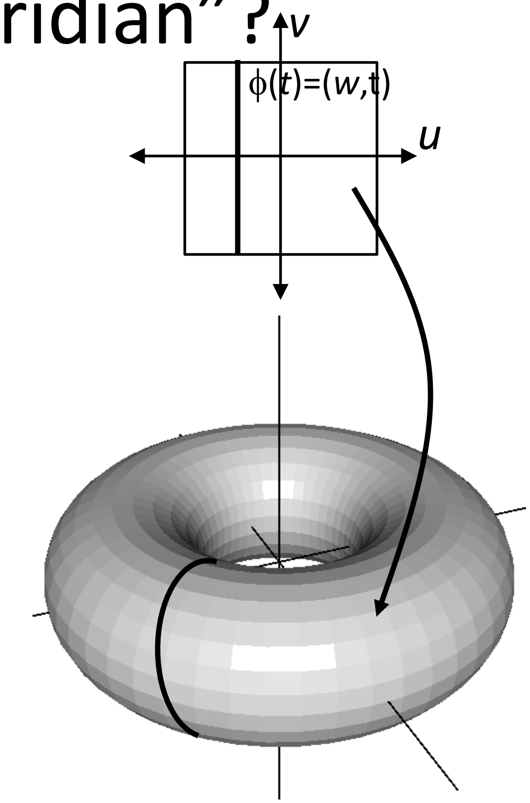


Metric Properties

$$\boldsymbol{x}(u,v) = \big((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\big)^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

<u>Example (Torus)</u>:

- What is the length of the w^{th} "meridian"?



The w^{th} meridian is the image of:

$$\phi(t)=(w,t) \quad \text{with } t\in[-\pi, \pi]$$

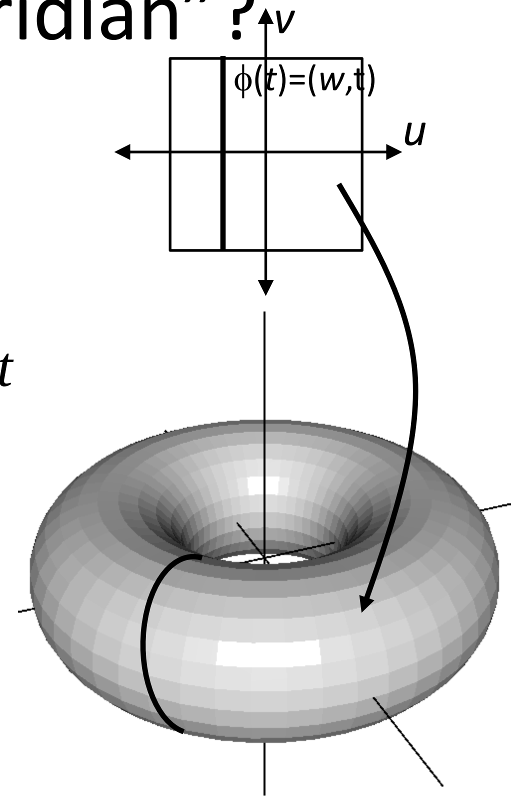
under the parameterization.

Metric Properties

$$\mathbf{x}(u,v) = \left((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\right)^t \quad \mathbf{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

<u>Example (Torus)</u>:

- What is the length of the w^{th} "meridian"?



$$length(\mathbf{x} \circ \phi) = \int_{-\pi}^{\pi} \sqrt{\phi'(t)^t \mathbf{I} \phi'(t)}\, dt$$

$$= \int_{-\pi}^{\pi} \sqrt{(0,1)^t \begin{pmatrix} (r_1 + r_2 \sin w)^2 & 0 \\ 0 & r_2^2 \end{pmatrix} (0,1)}\, dt$$

$$= \int_{-\pi}^{\pi} r_2 \, dt$$

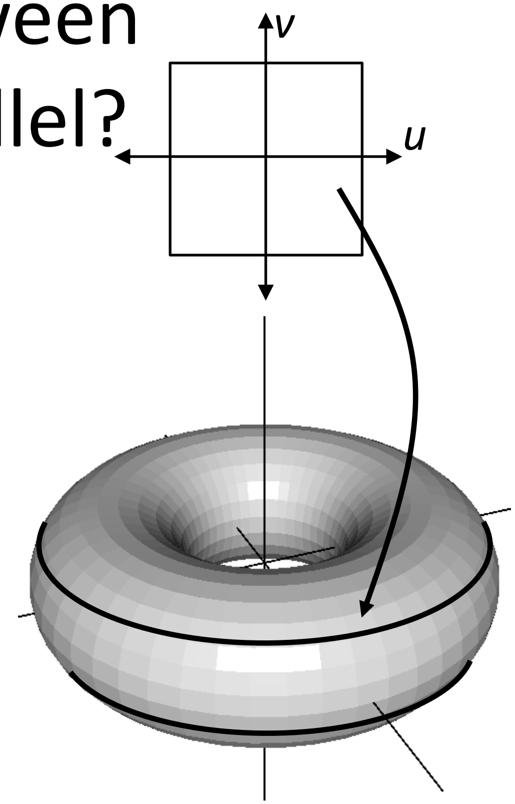
$$= 2\pi r_2$$

Metric Properties

$$\boldsymbol{x}(u,v) = \big((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\big)^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

<u>Example (Torus)</u>:

- What is the area of the band between the w_1th parallel and the w_2th parallel?



Metric Properties

$$\boldsymbol{x}(u,v) = \big((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\big)^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

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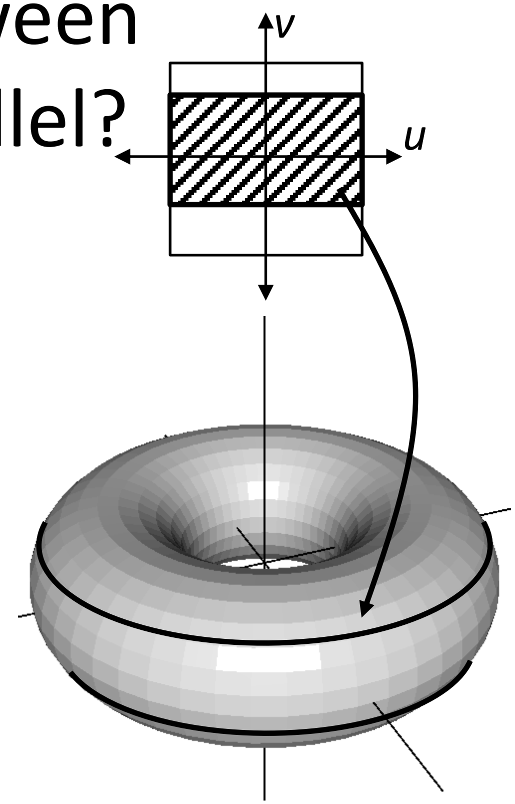
- What is the area of the band between the w_1th parallel and the w_2th parallel?



The band is the image of:

$\phi(s,t)=(s,t)$ with $s \in [-\pi,\pi]$, $t \in [w_1,w_2]$

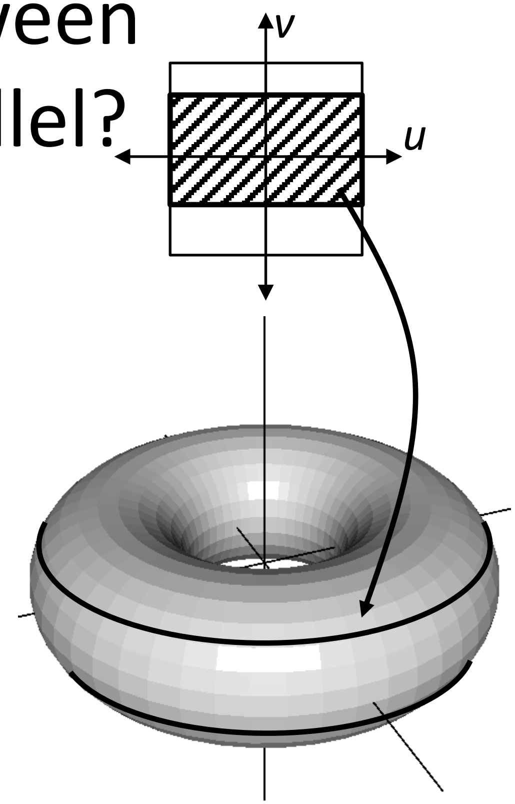
under the parameterization.

Metric Properties

$$\boldsymbol{x}(u,v) = \big((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\big)^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

Example (Torus):

- What is the area of the band between the w_1th parallel and the w_2th parallel?



$$area(\boldsymbol{x} \circ \phi) = \int_{w_1}^{w_2} \int_{-\pi}^{\pi} \sqrt{\det \boldsymbol{I}}\, ds\, dt$$

$$= \int_{w_1}^{w_2} \int_{-\pi}^{\pi} (r_1 + r_2 \sin t) r_2 \; ds\, dt$$

$$= 2\pi \int_{w_1}^{w_2} (r_1 + r_2 \sin t) r_2 \; dt$$

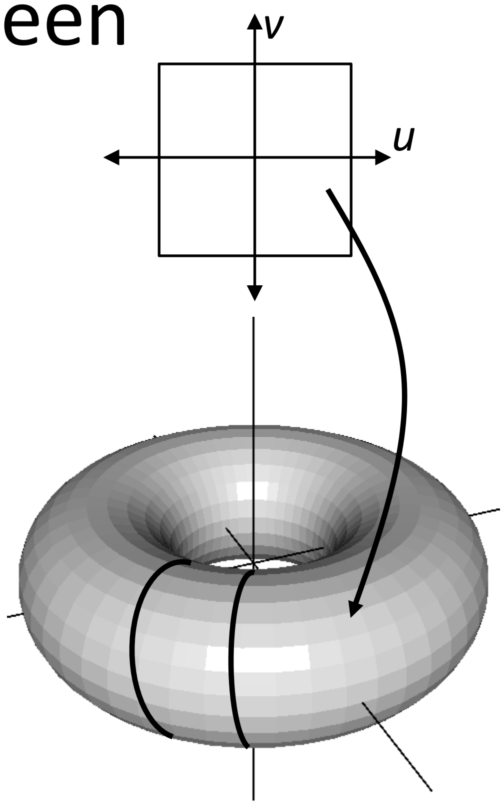
$$= 2\pi\big(r_1(w_2 - w_1) + r_2(\cos w_1 - \cos w_2)\big) r_2$$

Metric Properties

$$\boldsymbol{x}(u,v) = \left((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\right)^t \quad \boldsymbol{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

<u>Example (Torus)</u>:

- What is the area of the band between the w_1^th and the w_2^th meridian?



Metric Properties

$$\boldsymbol{x}(u,v) = \big((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\big)^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

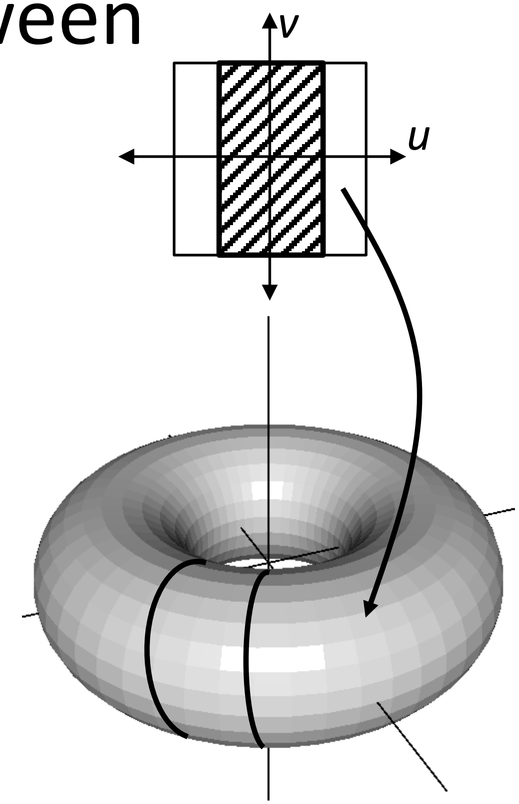
<u>Example (Torus)</u>:

- What is the area of the band between the w_1th and the w_2th meridian?

The band is the image of:

$\phi(s,t)=(s,t)$ with $s \in [w_1,w_2]$, $t \in [-\pi,\pi]$

under the parameterization.

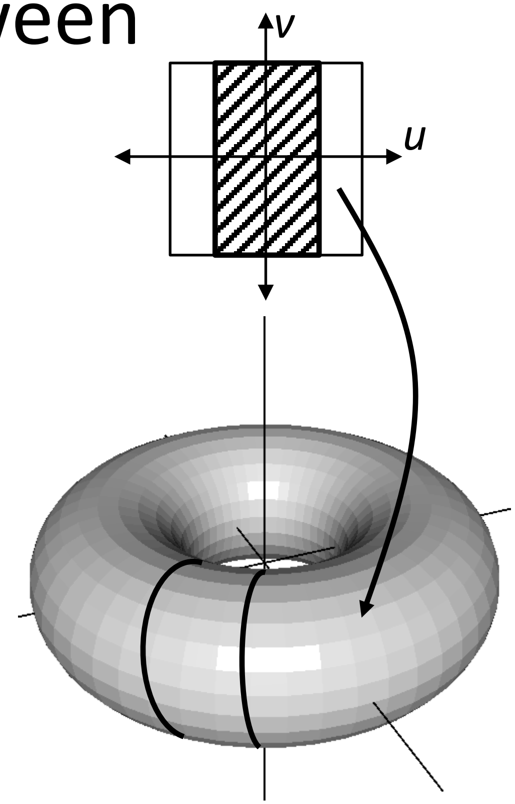


Metric Properties

$$\boldsymbol{x}(u,v)=\big((r_1+r_2\sin v)\cos u \quad r_2\cos v \quad (r_1+r_2\sin v)\sin u\big)^t \qquad \boldsymbol{I}(u,v)=\begin{pmatrix}(r_1+r_2\sin v)^2 & 0\\ 0 & r_2^2\end{pmatrix}$$

<u>Example (Torus)</u>:

- What is the area of the band between the w_1th and the w_2th meridian?

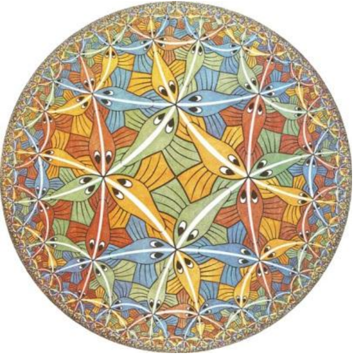


$$area(\boldsymbol{x}\circ\phi)=\int_{-\pi}^{\pi}\int_{w_1}^{w_2}\sqrt{\det \boldsymbol{I}}\,ds\,dt$$

$$=\int_{-\pi}^{\pi}\int_{w_1}^{w_2}\big(r_1+r_2\sin t\big)r_2\;ds\,dt$$

$$=(w_2-w_1)\int_{-\pi}^{\pi}\big(r_1+r_2\sin t\big)r_2\;dt$$

$$=(w_2-w_1)2\pi r_1 r_2$$



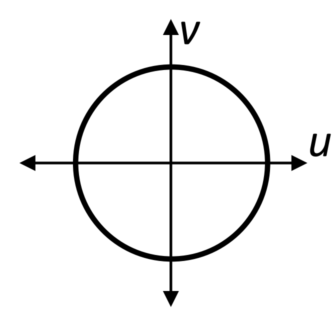
Metric Properties

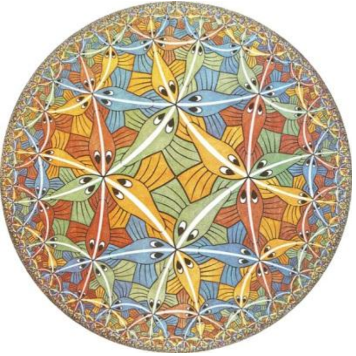
<u>Example (Hyperbolic Plane)</u>:

If we are given the first fundamental form, we can ignore the embedding of the surface in 3D, and integrate directly.

Consider the domain $\Omega=\{u,v \mid (u^2+v^2<1)\}$, with the first fundamental form:

$$\mathbf{I}(u,v)=\begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$



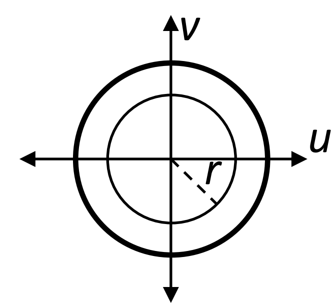


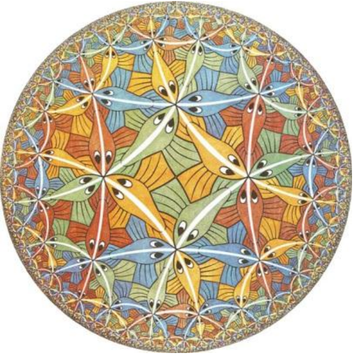
Metric Properties

$$\Omega = \left\{(u,v) \mid u^2 + v^2 < 1\right\} \qquad \mathbf{I}(u,v) = \begin{pmatrix} \dfrac{1}{1-u^2-v^2} & 0 \\ 0 & \dfrac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

- What is the length of the circle with radius *r*?





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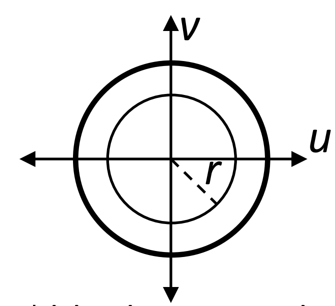
$$\Omega = \left\{(u,v) \mid u^2 + v^2 < 1\right\} \qquad \mathbf{I}(u,v) = \begin{pmatrix} \dfrac{1}{1-u^2-v^2} & 0 \\ 0 & \dfrac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

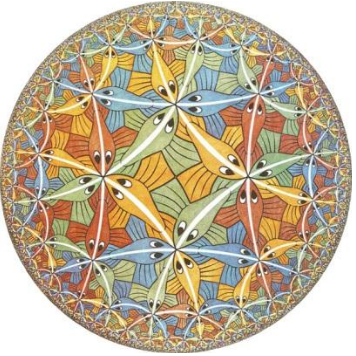
- What is the length of the circle with radius r?

The circle is described by:

$$\phi(s)=r(\cos s, \sin s) \quad \text{with } s\in[0,2\pi].$$



$\phi(t)=r(\cos t,\sin t)$



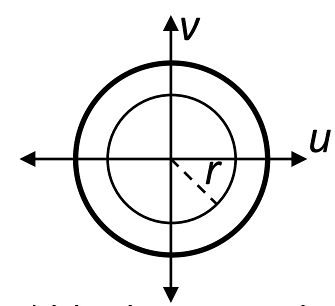
Metric Properties

$$\Omega = \{(u,v) \mid u^2 + v^2 < 1\} \qquad \mathbf{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

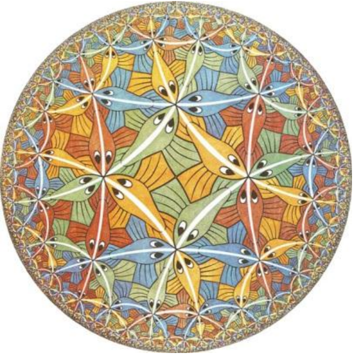
<u>Example (Hyperbolic Plane)</u>:

- What is the length of the circle with radius *r*?

$$\begin{aligned} length(\phi) &= \int_0^{2\pi} \sqrt{\phi'(t)^t \mathbf{I} \phi'(t)}\, dt \\ &= \int_0^{2\pi} \sqrt{r(-\sin t, \cos t)^t \begin{pmatrix} \frac{1}{1-r^2} & 0 \\ 0 & \frac{1}{1-r^2} \end{pmatrix} r(-\sin t, \cos t)}\, dt \\ &= \int_0^{2\pi} \sqrt{\frac{r^2}{1-r^2}}\, dt \\ &= 2\pi r \sqrt{\frac{1}{1-r^2}} \end{aligned}$$



$\phi(t)=r(\cos t, \sin t)$

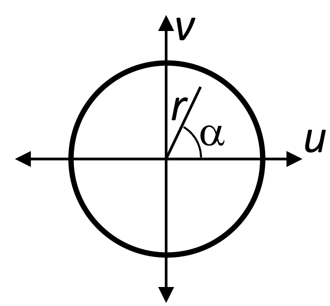


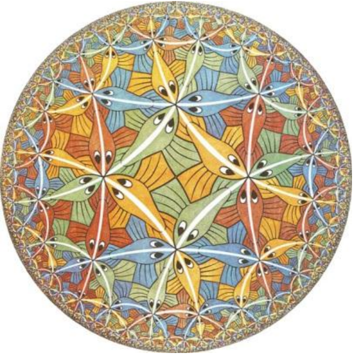
Metric Properties

$$\Omega = \left\{(u,v) \mid u^2 + v^2 < 1\right\} \qquad \mathbf{I}(u,v) = \begin{pmatrix} \dfrac{1}{1-u^2-v^2} & 0 \\ 0 & \dfrac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

- What is the length of the segment with angle α and radius r?





Metric Properties

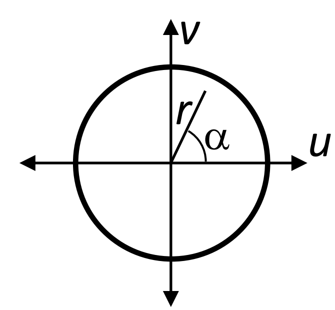
$$\Omega = \left\{(u,v) \mid u^2 + v^2 < 1\right\} \qquad I(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

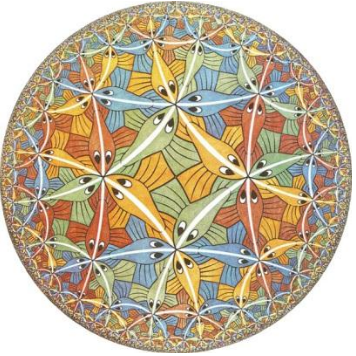
- What is the length of the segment with angle α and radius r?

The segment is described by:

$\phi(s)=s(\cos\alpha, \sin\alpha)$ with $s\in[0,r]$.



$\phi(s)=s(\cos\alpha,\sin\alpha)$



Metric Properties

$$\Omega = \{(u,v) \mid u^2 + v^2 < 1\} \qquad \mathbf{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

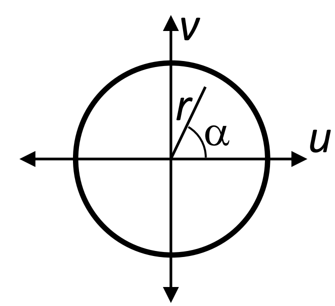
<u>Example (Hyperbolic Plane)</u>:

- What is the length of the segment with angle α and radius r?

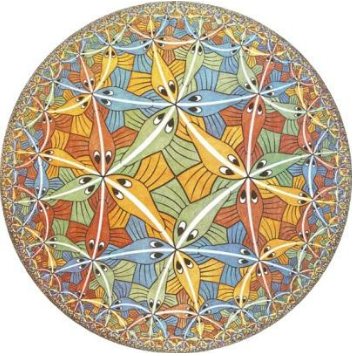
$$length(\phi) = \int_0^r \sqrt{\phi'(s)^t \mathbf{I} \phi'(s)}\, ds$$

$$= \int_0^r \sqrt{(\cos\alpha, \sin\alpha)^t \begin{pmatrix} \frac{1}{1-s^2} & 0 \\ 0 & \frac{1}{1-s^2} \end{pmatrix} (\cos\alpha, \sin\alpha)}\, ds$$

$$= \int_0^r \frac{1}{1-s^2}\, ds = \frac{1}{2} \log \frac{1+r}{1-r}$$



$\phi(s) = s(\cos\alpha, \sin\alpha)$

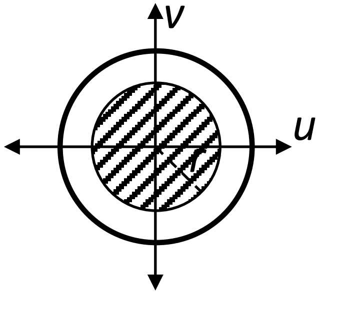


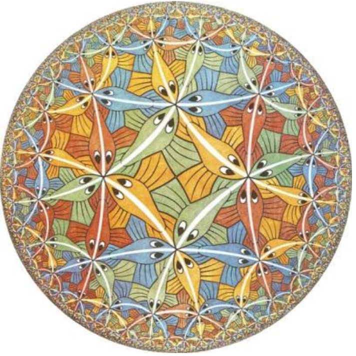
Metric Properties

$$\Omega = \left\{(u,v) \mid u^2 + v^2 < 1\right\} \qquad I(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

- What is the area of the region with radius less than *r*?





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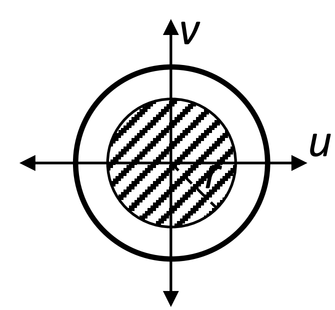
$$\Omega = \left\{ (u,v) \mid u^2 + v^2 < 1 \right\} \qquad \mathbf{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

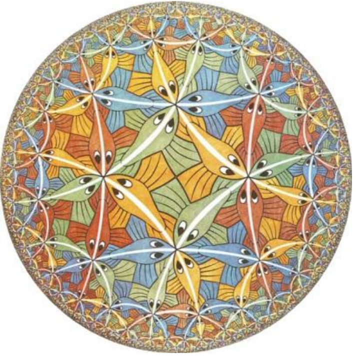
- What is the area of the region with radius less than *r*?

The region is the image of:

$$\phi(s,\mathrm{t}) = s(\cos t, \sin t) \quad \text{with } s \in [0,r],\ t \in [-\pi,\pi].$$



$\phi(s,\mathrm{t})=s(\cos t,\sin t)$



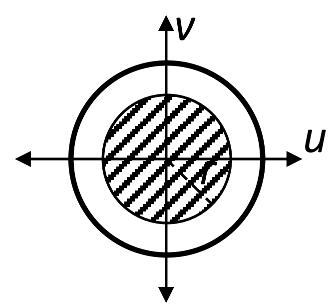
Metric Properties

$$\Omega = \left\{(u,v) \mid u^2 + v^2 < 1\right\} \qquad \mathbf{I}(u,v) = \begin{pmatrix} \dfrac{1}{1-u^2-v^2} & 0 \\ 0 & \dfrac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

- What is the area of the region with radius less than *r*?

$$\begin{aligned} area(\phi) &= \int_{-\pi}^{\pi}\int_{0}^{r} \sqrt{\det \mathbf{I}}\, s\, ds\, dt \\ &= \int_{-\pi}^{\pi}\int_{0}^{r} \frac{s}{1-s^2}\, ds\, dt \\ &= 2\pi \int_{0}^{r} \frac{s}{1-s^2}\, ds \\ &= -\pi \ln\left(1-r^2\right) \end{aligned}$$



$\phi(s,t)=s(\cos t,\sin t)$