FFTs in Graphics and Vision

Surface Reconstruction

Shape Spectrum

There are many ways to represent a shape:

- Point Set
- Polygon Soup
- Polygonal Mesh
- Solid Model

Equivalence of Representations

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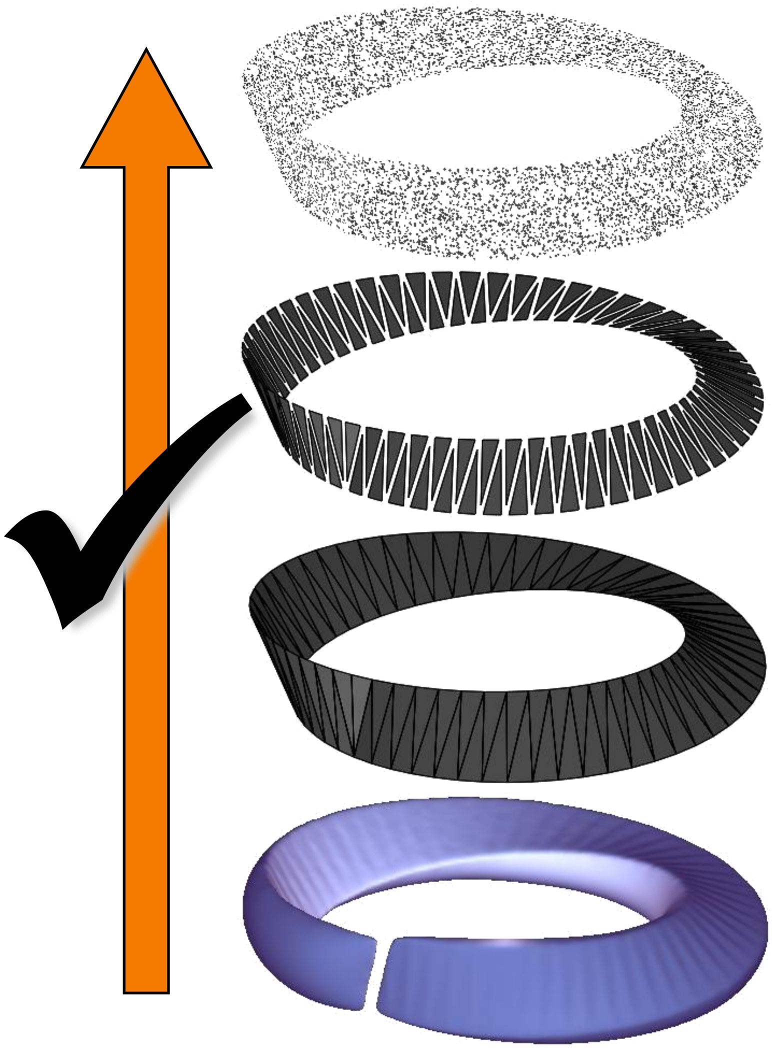
In one direction, the transition between representations is straight-forward



Equivalence of Representations

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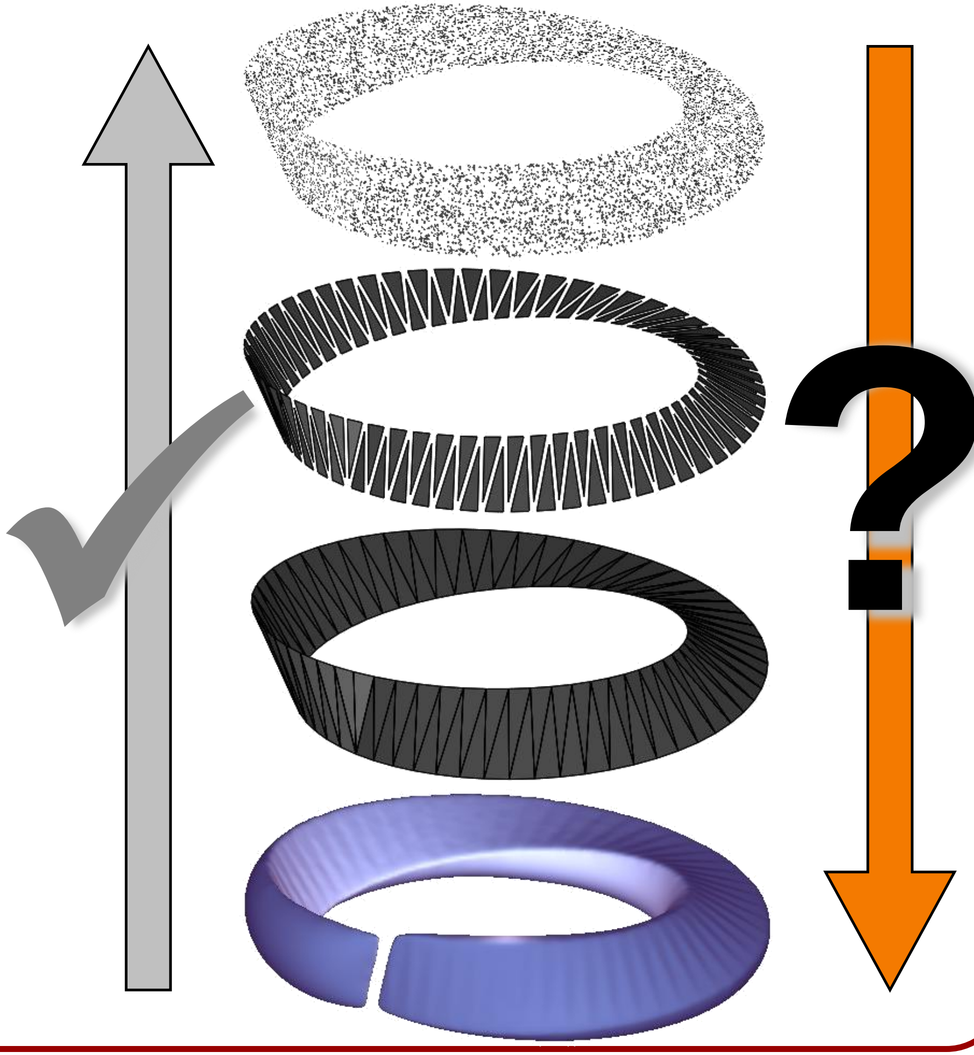
In one direction, the transition between representations is straight-forward

The challenge is to transition in the other direction



Equivalence of Representations

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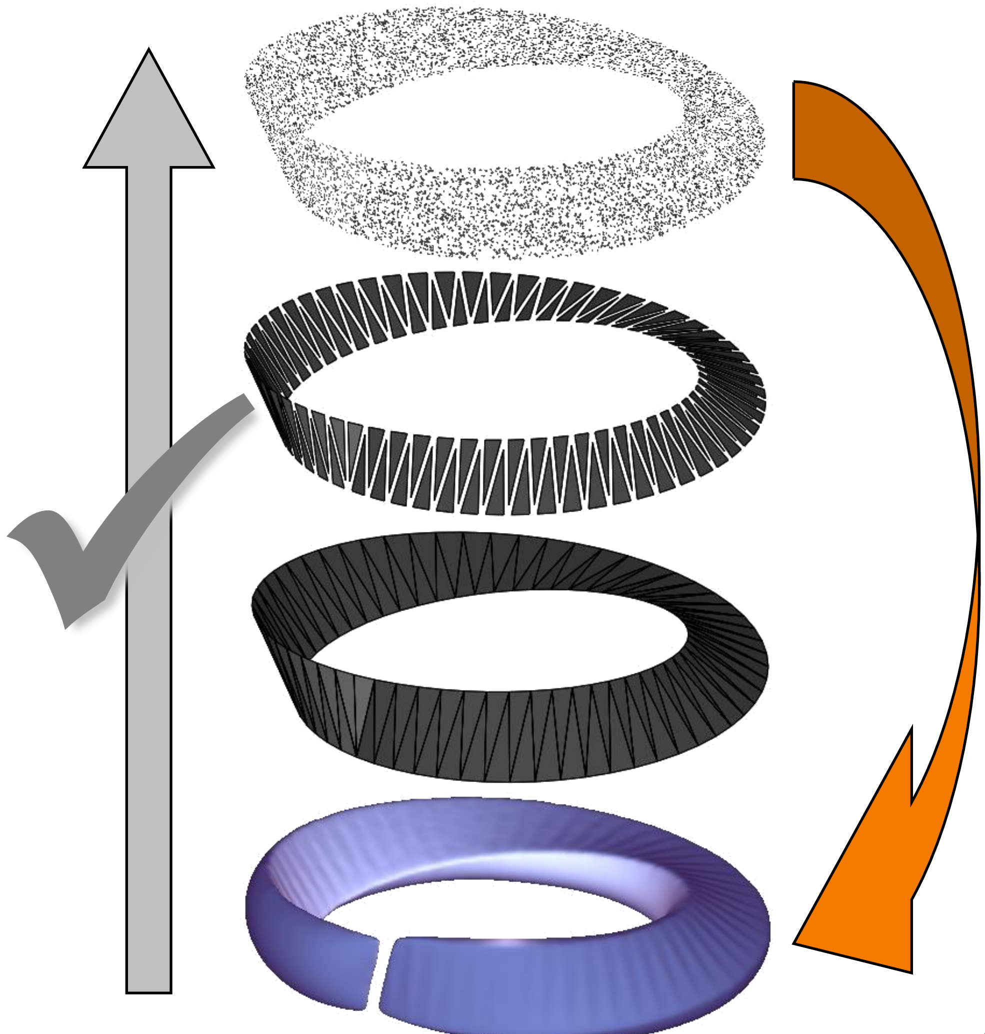
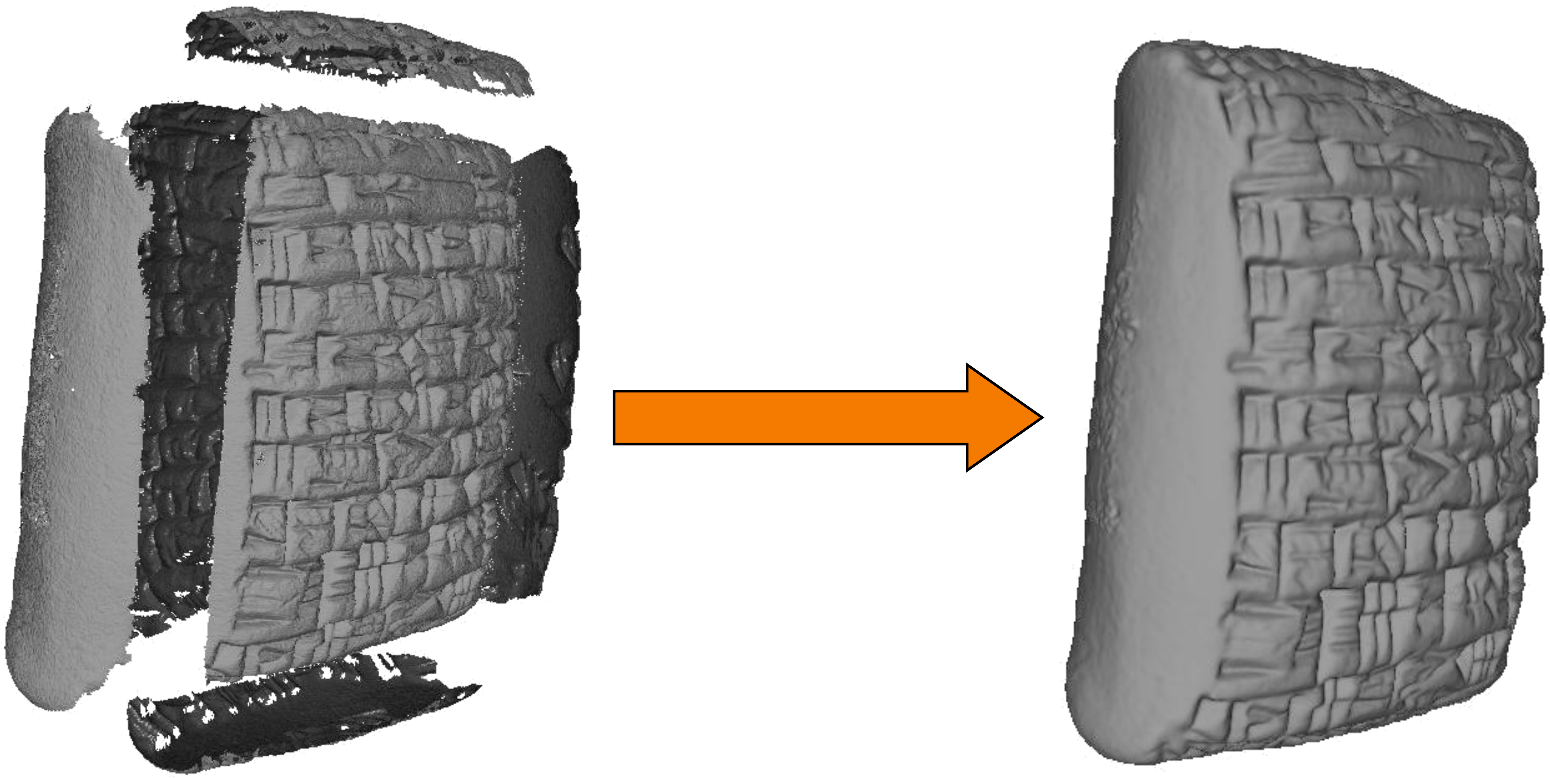


The goal of this work is to define a method for computing solid models from oriented point sets.

Applications

- Surface Blending

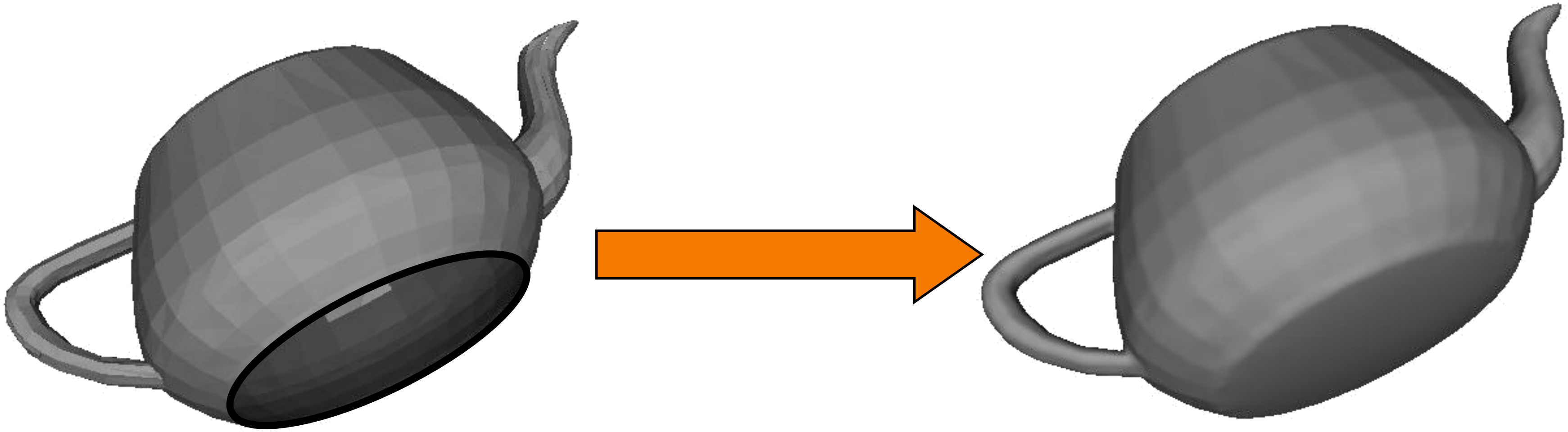


Disjoint Model

"Zippered" Model

Applications

- Surface Blending
- Hole-Filling

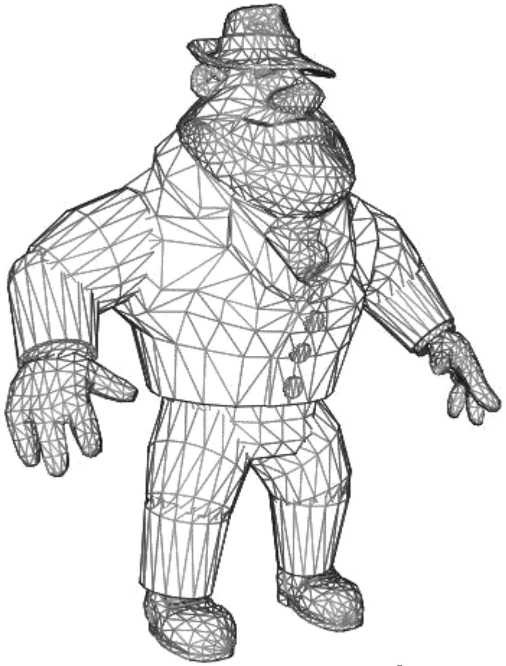


Model with Hole

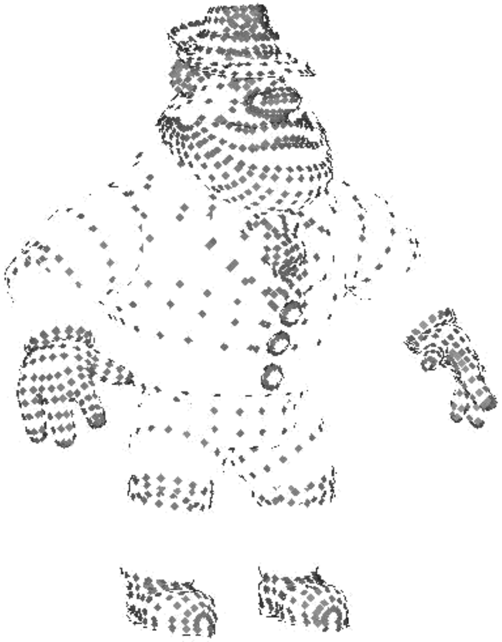
Water-Tight Model

Applications

- Surface Blending
- Hole-Filling
- Compression



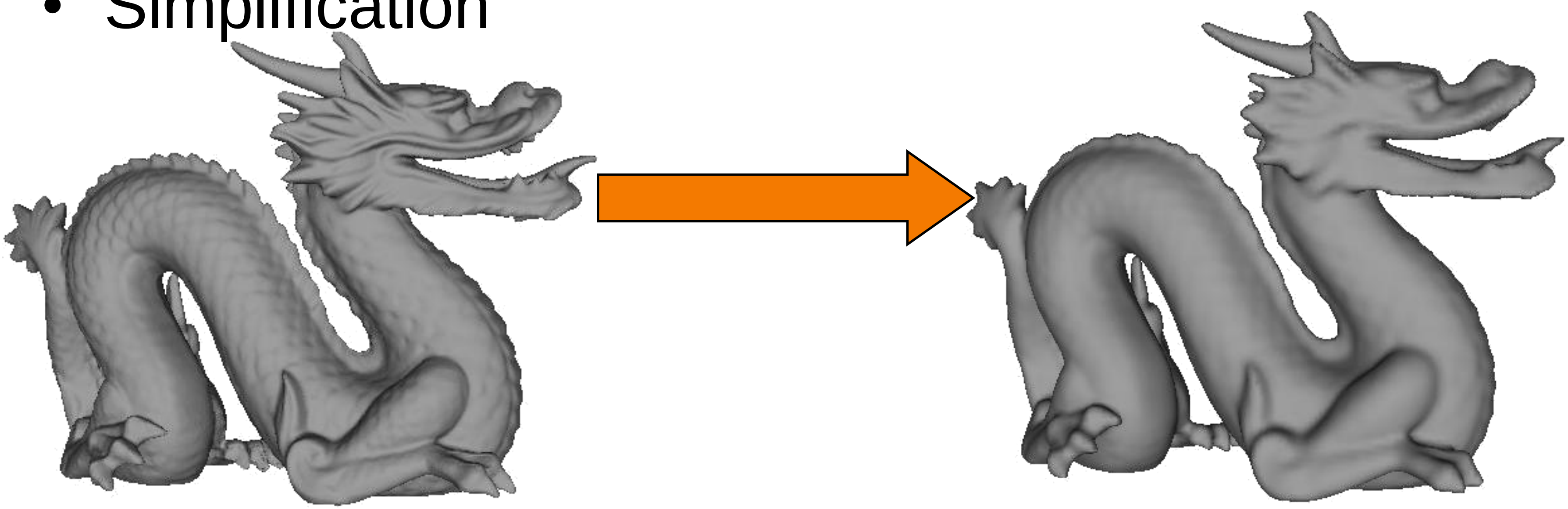
Geometry + Topology
Representation



Geometry
Representation

Applications

- Surface Blending
- Hole-Filling
- Compression
- Simplification



Original Model
871,000 Triangles

Simplified Model
95,000 Triangles

Related Work

Three general approaches:

1. Computational Geometry
2. Surface Fitting
3. Implicit Function Fitting

Related Work

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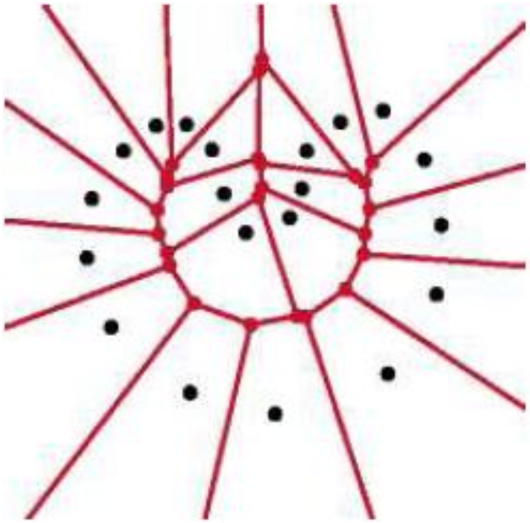
1. Computational Geometry
   - Use the Voronoi diagram / Delaunnay triangulation to extract the surface.



Amenta, Bern and Eppstein. *Graphical Models and Image Processing.* (1998).

Related Work

Three general approaches:

1. Computational Geometry
   - Use the Voronoi diagram / Delaunnay triangulation to extract the surface.

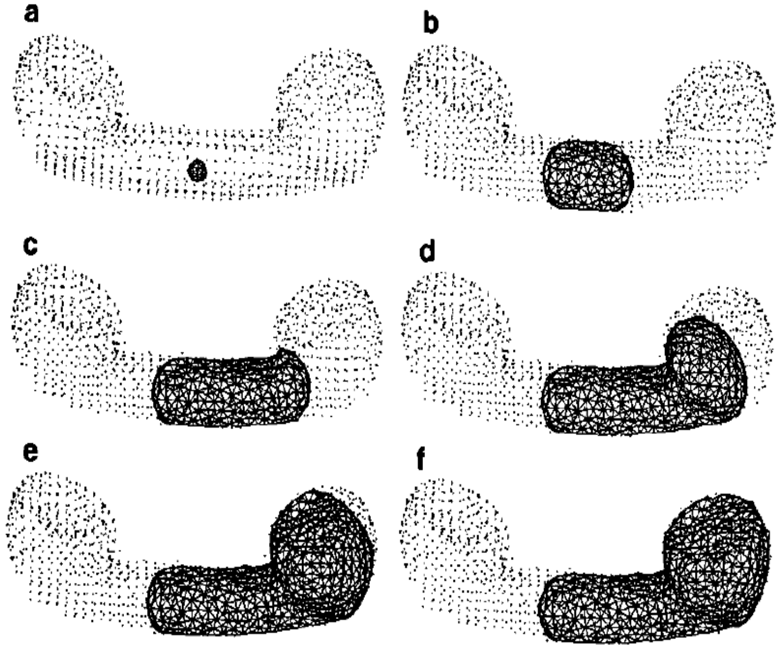
Properties:

- ✓ Complexity of the output is on the order of the complexity of the input.
- ✗ Computing the Voronoi diagram / Delaunnay triangulation can be time consuming.
- ✗ Does not perform well in the presence of noise and non-uniform sampling.

Related Work

Three general approaches:

2. Surface Fitting
   - Deform a base model (represented by a spring system) to fit the input sample points.



Chen and Medioni. *Computer Vision and Image Understanding.* (1995).

Related Work

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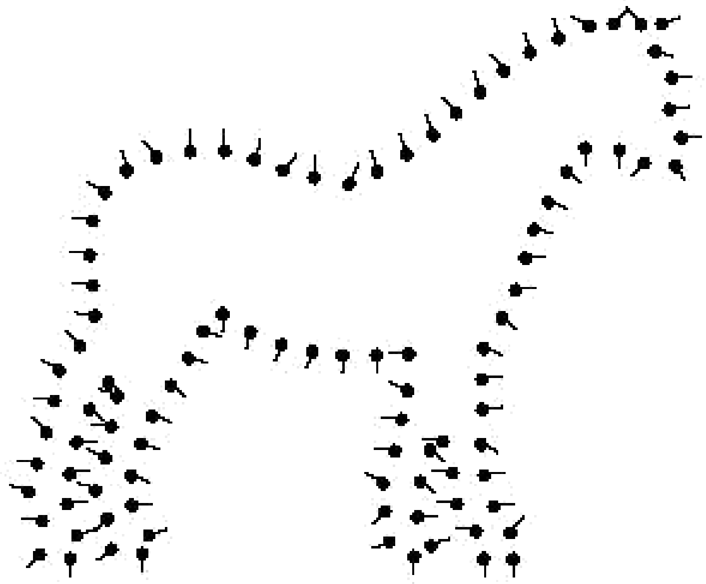
<u>Properties</u>:

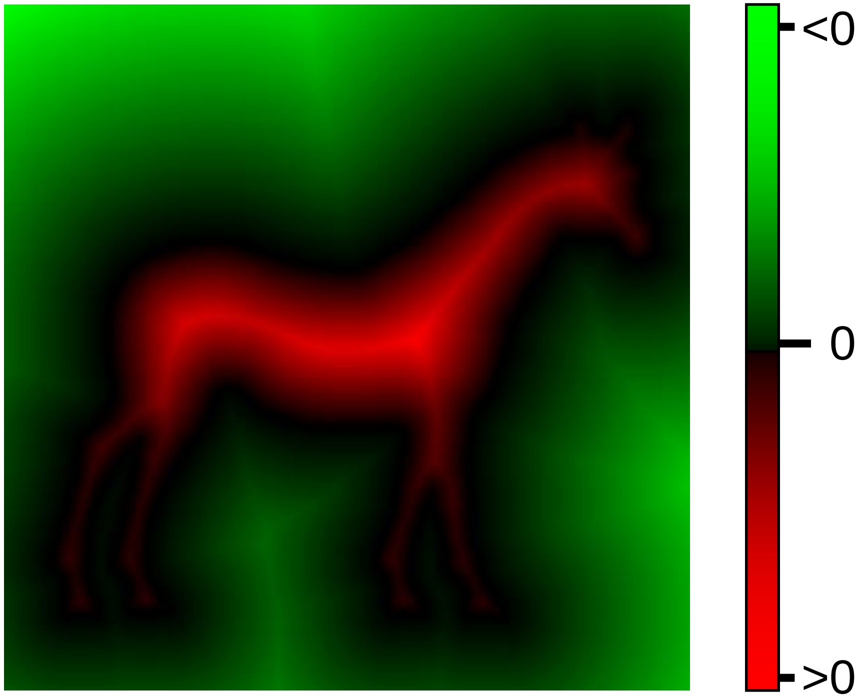
- ✓ Complexity of the output is on the order of the complexity of the input.
- ✗ The reconstructed surface has to have the same topology as the base model.

Related Work

3. Implicit Function Fitting
   - Use the point samples to define an function whose values at the sample positions are zero.



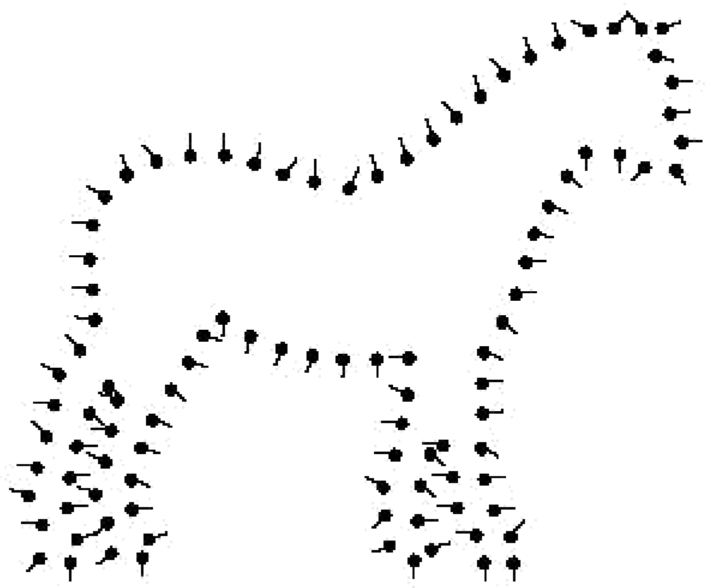
Sample Points



$F(x,y)$

Related Work

3. Implicit Function Fitting
   - Use the point samples to define an function whose values at the sample positions are zero.
   - Extract the iso-surface with iso-value equal to zero.



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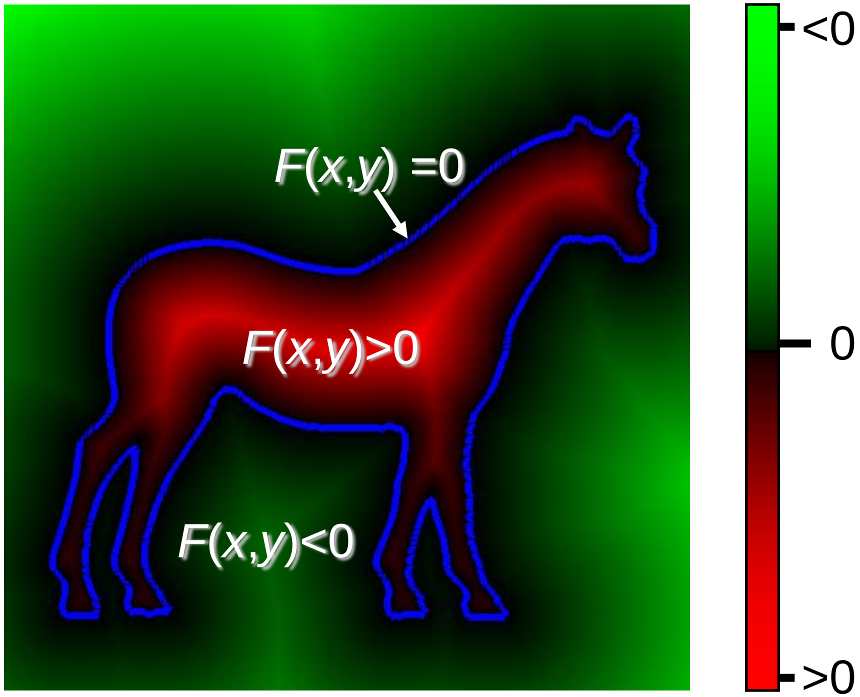

F(*x*,*y*)=0

Related Work

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Properties:

- ✓ No topological restrictions.
- ✓ Noise can be smoothed out.
- ✗ The complexity of the reconstruction depends on the sampling resolution, not the number of input points.

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How should we define the implicit function so that the reconstruction fits the samples?

Outline

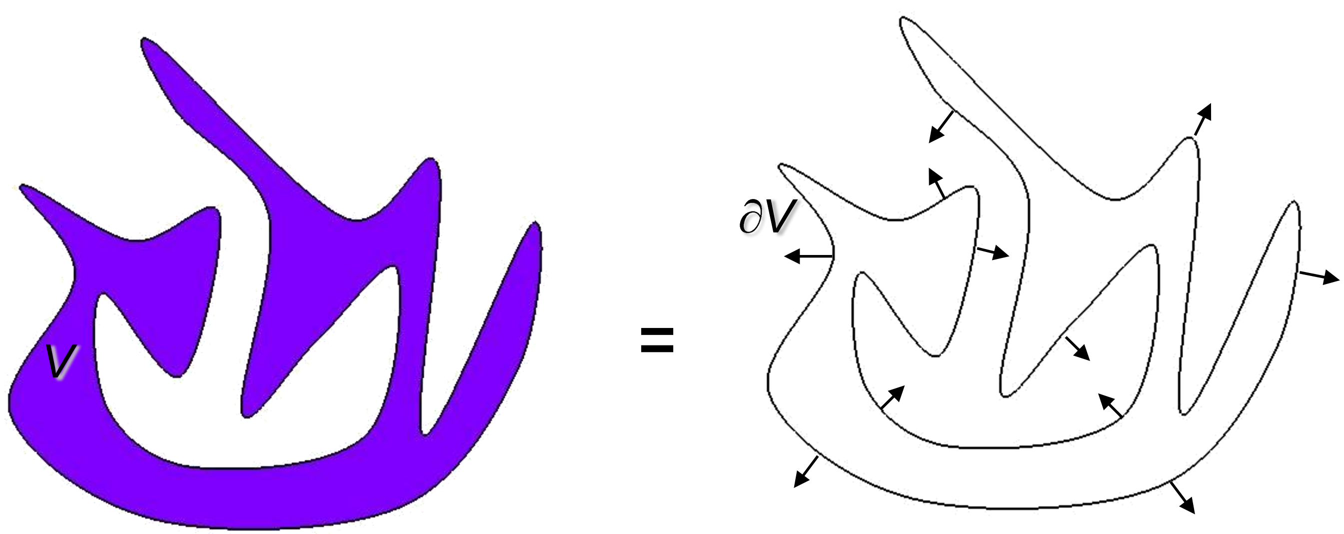
- Introduction
- Related Work
- Approach
    - The Divergence Theorem
    - Reduction to Volume Integration
    - Implementation
- Results
- Conclusion

Divergence (Gauss's) Theorem

Given a vector field $\vec{F}$ and a region V:

The volume integral of $\nabla\cdot\vec{F}$ over V and the surface integral of $\vec{F}$ over ∂V are equal:

$$\int_V \left(\nabla\cdot\vec{F}\right)(p)\,dp = \int_{\partial V}\left\langle \vec{F}(p),\vec{n}(p)\right\rangle dp$$



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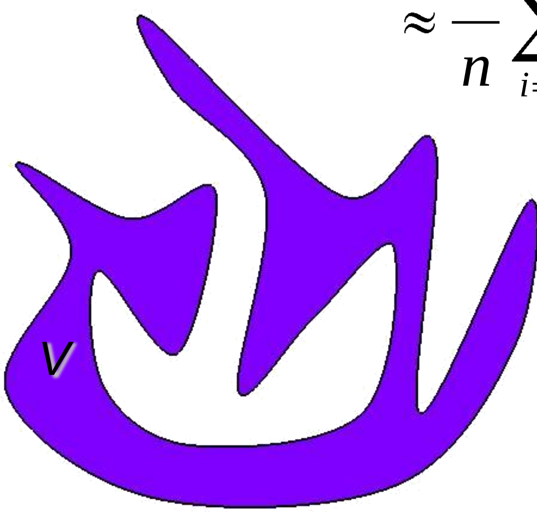
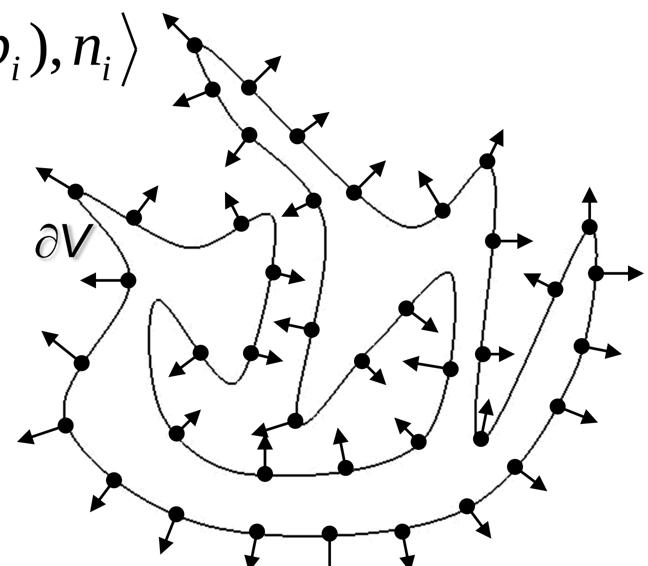
$$\approx \frac{A}{n}\sum_{i=1}^{n}\left\langle F(p_i), n_i \right\rangle$$



$\approx$



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Approach

Reduce surface reconstruction to a volume integration problem:

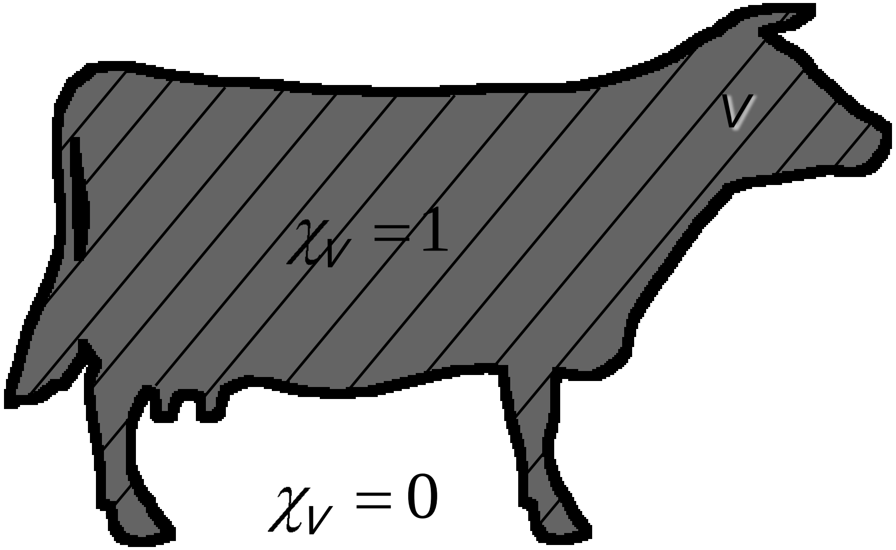
1. Characteristic Function
2. Fourier Coefficients
3. Volume Integrals

Reduction to Volume Integration (Step 1)

Characteristic Function:

The *characteristic function* χ_V of a solid V is the function:

$$\chi_V(x,y,z)=\begin{cases}1 & \text{if } (x,y,z)\in V\\ 0 & \text{otherwise}\end{cases}$$



Reduction to Volume Integration (Step 2)

Fourier Coefficients:

The *Fourier coefficients* of the characteristic function give an expression of χ_V as a sum of complex exponentials:

$$\chi_V(x,y,z) = \frac{1}{\sqrt{2\pi}} \sum_{l,m,n} \hat{\chi}_V(l,m,n) e^{2\pi i(lx+my+nz)}$$

Reduction to Volume Integration (Step 3)

<u>Volume Integration</u>:

The Fourier coefficients of the characteristic function χ_V can be obtained by integrating:

$$\hat{\chi}_V(l,m,n) = \int_{[0,1]^3} \chi_V(x,y,z) \frac{1}{\sqrt{2\pi}} e^{-2\pi i (x+my+nz)} dxdydz$$

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$$= \int_V \frac{1}{\sqrt{2\pi}} e^{-2\pi i (x+my+nz)} dxdydz$$

since the characteristic function is one inside of *V* and zero everywhere else.

Applying the Divergence Theorem

<u>Surface Integration</u>

If $\vec{F}_{lmn}(x,y,z)$ is any function whose divergence is equal to the (l,m,n)-th complex exponential:

$$\left(\nabla\cdot\vec{F}_{lmn}\right)(x,y,z)=\frac{1}{\sqrt{2\pi}}e^{-2\pi i\left(x+my+nz\right)}$$

applying the Divergence Theorem, the volume integral can be expressed as a surface integral:

$$\int_V \frac{1}{\sqrt{2\pi}}e^{-2\pi i\left(x+my+nz\right)}dxdydz=\int_{\partial V}\left\langle \vec{F}_{lmn}(p),n(p)\right\rangle dp$$

Reconstruction Algorithm

Given an oriented point sample $\{(p_i,n_i)\}$:

- Compute a Monte-Carlo approximation of the Fourier coefficients of the characteristic function:

$$\hat{\chi}_V(l,m,n) \approx \sum_{i=1}^{k} \left\langle \vec{F}_{lmn}(p_i), n_i \right\rangle$$

- Apply the inverse Fourier Transform to obtain the characteristic function.
- Extract the reconstruction an iso-surface of the characteristic function

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1. To do this, we need to find a vector valued function $F_{l,m,n}(x,y,z)$ such that:

$$\left(\nabla \cdot F_{l,m,n}\right)(x,y,z) = \frac{1}{\sqrt{2\pi}} e^{-2\pi i (x+my+nz)}$$

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2. Directly computing the Fourier coefficients requires summing over all point samples for each of the $O(n^3)$ coefficients.

Choosing the Functions $F_{l,m,n}$

There are many solutions to the equation:

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<u>Examples</u>:

$$F_{l,m,n}(x,y,z) = \begin{pmatrix} \frac{i}{l(2\pi)^{3/2}} e^{-2\pi i (x+my+nz)} \\ 0 \\ 0 \end{pmatrix} \quad F_{l,m,n}(x,y,z) = \begin{pmatrix} 0 \\ \frac{i}{m(2\pi)^{3/2}} e^{-2\pi i (x+my+nz)} \\ 0 \end{pmatrix} \quad F_{l,m,n}(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ \frac{i}{n(2\pi)^{3/2}} e^{-2\pi i (x+my+nz)} \end{pmatrix}$$

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$$F_{l,m,n}(x,y,z) = \frac{1}{3(2\pi)^{3/2}} \begin{pmatrix} \frac{i}{l} e^{-2\pi i (x+my+nz)} \\ \frac{i}{m} e^{-2\pi i (x+my+nz)} \\ \frac{i}{n} e^{-2\pi i (x+my+nz)} \end{pmatrix} \quad F_{l,m,n}(x,y,z) = \frac{1}{(2\pi)^{3/2}} \begin{pmatrix} \frac{i}{l+m+n} e^{-2\pi i (x+my+nz)} \\ \frac{i}{l+m+n} e^{-2\pi i (x+my+nz)} \\ \frac{i}{l+m+n} e^{-2\pi i (x+my+nz)} \end{pmatrix}$$

Choosing the Functions $F_{l,m,n}$

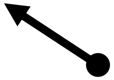
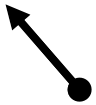
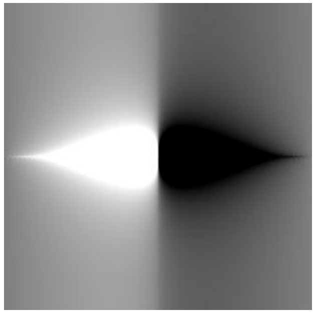
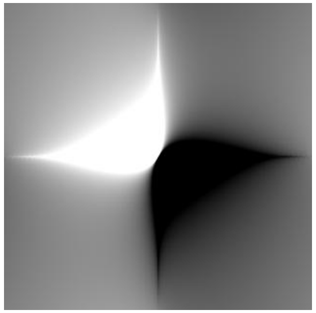
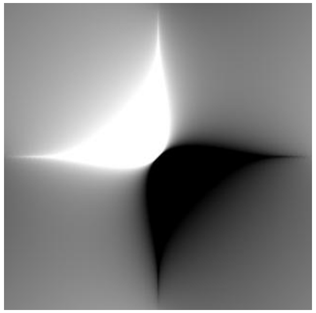
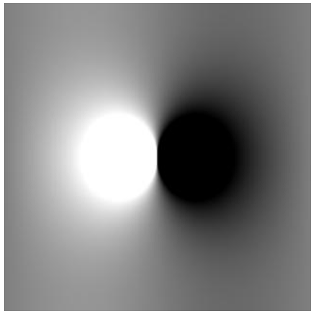
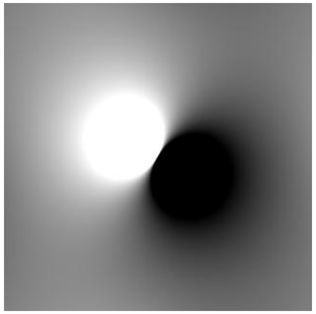
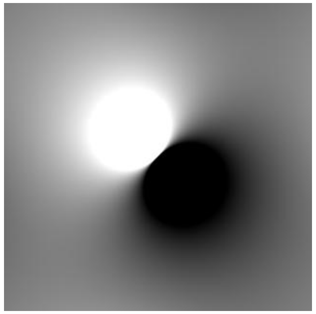
There are many solutions to the equation:

$$\left(\nabla \cdot F_{l,m,n}\right)(x,y,z) = \frac{1}{\sqrt{2\pi}} e^{-2\pi i\left(x+my+nz\right)}$$

In our implementation, we choose the functions $F_{l,m,n}$ to be the unique vector fields that commute with rotation:

$$F_{l,m,n}(x,y,z) = \frac{1}{(2\pi)^{3/2}} \begin{pmatrix} \dfrac{il}{l^2+m^2+n^2} e^{-2\pi i\left(x+my+nz\right)} \\ \dfrac{im}{l^2+m^2+n^{2^2}} e^{-2\pi i\left(x+my+nz\right)} \\ \dfrac{in}{l^2+m^2+n^2} e^{-2\pi i\left(x+my+nz\right)} \end{pmatrix}$$

Choosing the Functions $F_{l,m,n}$

	 0^o	 30^o	 45^o
Does not Commute: $$F_{l,m,n}(x,y,z)=\frac{1}{(2\pi)^{3/2}}\begin{pmatrix}\frac{i}{l+m+n}e^{-2\pi i(x+my+nz)}\\ \frac{i}{l+m+n}e^{-2\pi i(x+my+nz)}\\ \frac{i}{l+m+n}e^{-2\pi i(x+my+nz)}\end{pmatrix}$$			
Commutes: $$F_{l,m,n}(x,y,z)=\frac{1}{(2\pi)^{3/2}}\begin{pmatrix}\frac{il}{l^2+m^2+n^2}e^{-2\pi i(x+my+nz)}\\ \frac{im}{l^2+m^2+n^2}e^{-2\pi i(x+my+nz)}\\ \frac{in}{l^2+m^2+n^2}e^{-2\pi i(x+my+nz)}\end{pmatrix}$$			

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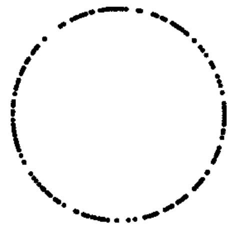
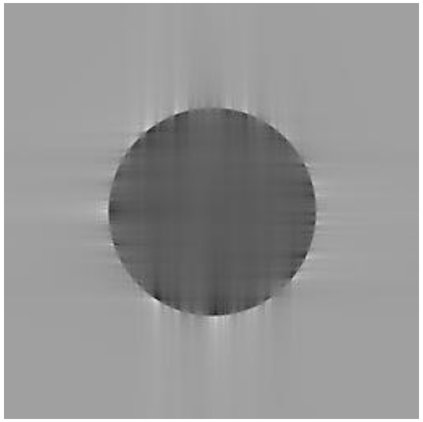
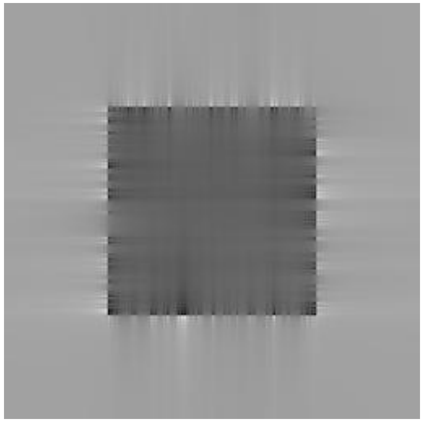
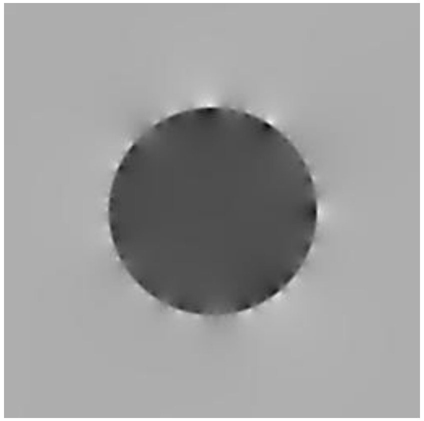
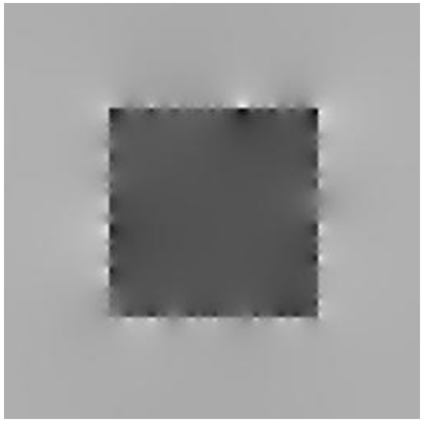
		
Does not Commute: $F_{l,m,n}(x,y,z)=\frac{1}{(2\pi)^{3/2}}\begin{pmatrix}\frac{i}{l+m+n}e^{-2\pi i(lx+my+nz)}\\ \frac{i}{l+m+n}e^{-2\pi i(lx+my+nz)}\\ \frac{i}{l+m+n}e^{-2\pi i(lx+my+nz)}\end{pmatrix}$		
Commutes: $F_{l,m,n}(x,y,z)=\frac{1}{(2\pi)^{3/2}}\begin{pmatrix}\frac{il}{l^2+m^2+n^2}e^{-2\pi i(lx+my+nz)}\\ \frac{im}{l^2+m^2+n^2}e^{-2\pi i(lx+my+nz)}\\ \frac{in}{l^2+m^2+n^2}e^{-2\pi i(lx+my+nz)}\end{pmatrix}$		

Choosing the Functions $F_{l,m,n}$

Convolution:

In general, to convolve with a filter *f*:

We take the sum of different scales of f at different translations :

$$(f * g)(t) = \int g(s) f(t-s)\, ds$$

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When the functions $F_{l,m,n}$ commute with rotation, we extend the notion of convolution:

We take the sum of different rotations of F, at different scales, and different translations.

Choosing the Functions $F_{l,m,n}$

<u>Extended Convolution</u>:

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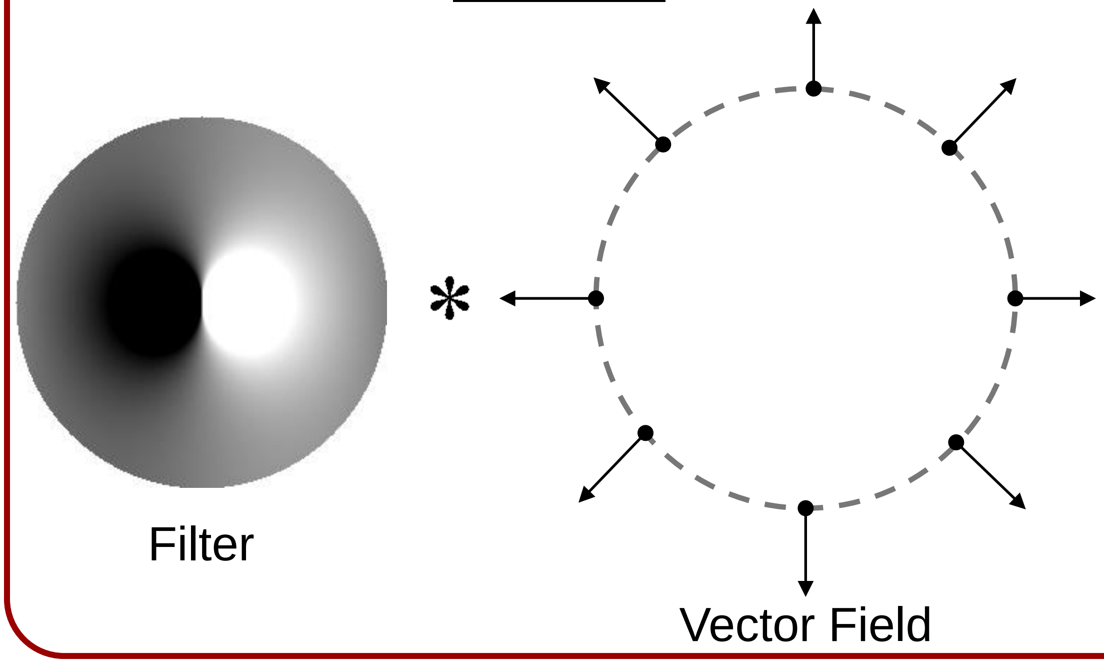

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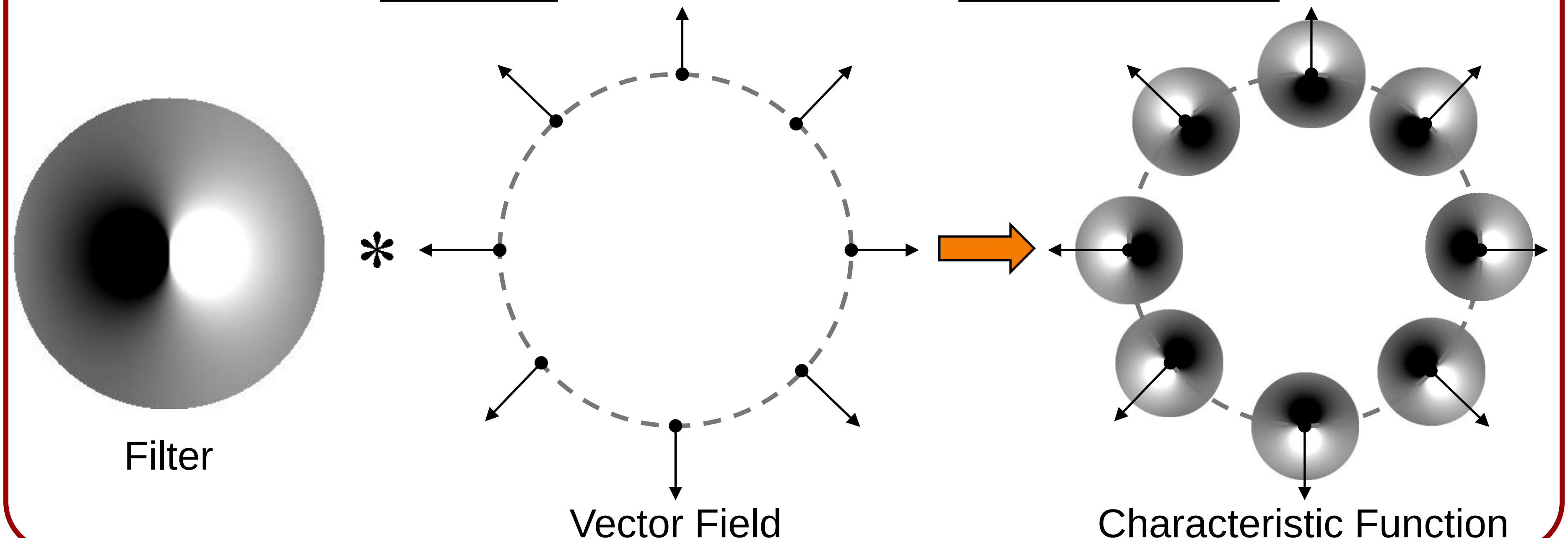

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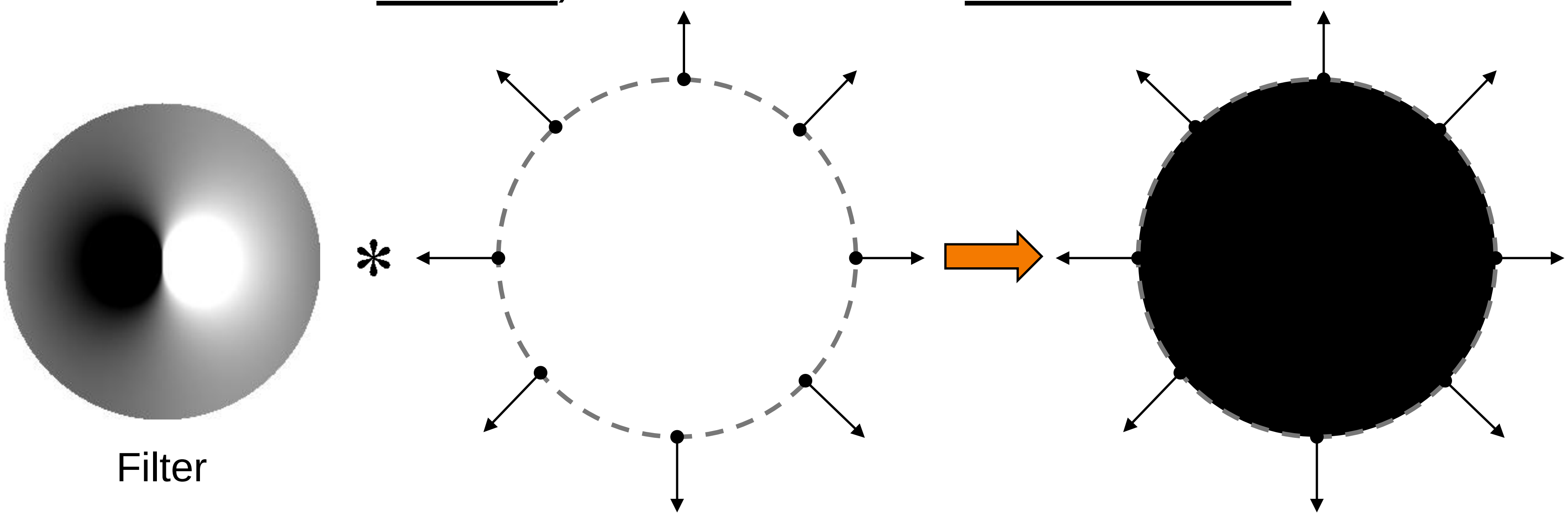
When the functions $F_{l,m,n}$ commute with rotation, we extend the notion of convolution:
We take the sum of different rotations of F, at different scales, and different translations.



Choosing the Functions $F_{l,m,n}$

Extended Convolution:

When the functions $F_{l,m,n}$ commute with rotation, we extend the notion of convolution:

By turning the surface integral into a convolution, we can compute the characteristic function efficiently:

$$\{p_i, n_i\} \rightarrow \hat{\chi}_V(l,m,n) \approx \sum_{i=1}^{k} \left\langle \vec{F}_{lmn}(p_i), n_i \right\rangle \rightarrow \chi_V(x,y,z)$$

$$O(n^5)$$

$$\{p_i, n_i\} \rightarrow V(x,y,z) \approx \sum_{i=1}^{k} \delta_{p_i}(x,y,z) n_i \rightarrow \chi_V(x,y,z) = V * F$$

$$O(n^3 \log n)$$

Vector Field

Characteristic Function

Outline

- Introduction
- Related Work
- Approach
  - The Divergence Theorem
  - Reduction to Volume Integration
  - Implementation
    - Defining the vector fields
    - Weighting non-uniform samples
- Results
- Conclusion

Non-Uniform Samples

Challenge:

In a direct implementation of Monte-Carlo integration, it is assumed that the samples are uniformly distributed:

$$\int f(x)dx \approx \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

Non-Uniform Samples

Challenge:

In a direct implementation of Monte-Carlo integration, it is assumed that the samples are uniformly distributed:

$$\int f(x)dx \approx \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

However, often the oriented point samples may not be uniformly distributed over the surface:

- Parts of scans may overlap
- Faces parallel to the view plane may be more densely sampled
- Compressed representations may store fewer points in regions of low curvature

Non-Uniform Samples

<u>Challenge</u>:

If we have a sampling density ρ_i associated to each sample x_i, we can modify the summation:

$$\int f(x)dx \approx \frac{1}{\left(\sum_{i=1}^{n}\rho_i\right)}\sum_{i=1}^{n} f(x_i)\rho_i$$

Non-Uniform Samples

Challenge:

If we have a sampling density ρ_i associated to each sample x_i, we can modify the summation:

$$\int f(x)dx \approx \frac{1}{\left(\sum_{i=1}^{n}\rho_i\right)}\sum_{i=1}^{n} f(x_i)\rho_i$$

However, when we get an oriented point sample, we usually aren't given the sampling density at each sample.

Non-Uniform Samples

Challenge:

If we have a sampling density ρ_i associated to each sample x_i, we can modify the summation:

$$\int f(x)dx \approx \frac{1}{\sum_{i=1}^{n} \rho_i} \sum_{i=1}^{n} f(x_i)\rho_i$$

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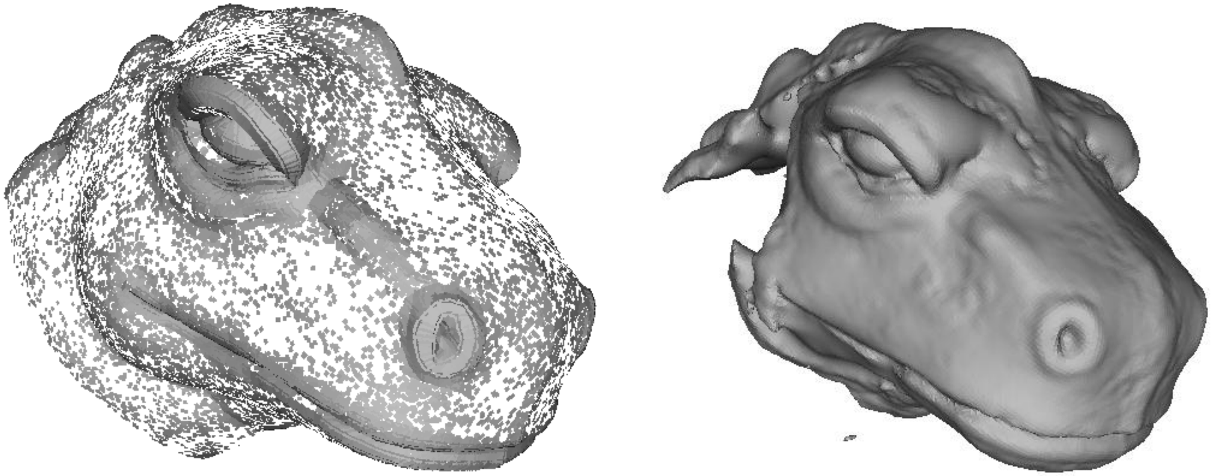
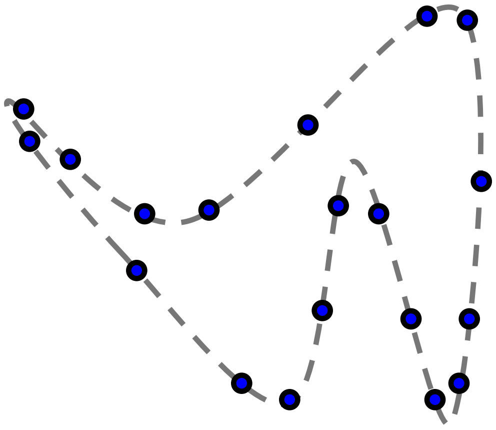


Non-Uniform Samples Unweighted Reconstruction

Non-Uniform Samples

Approach:

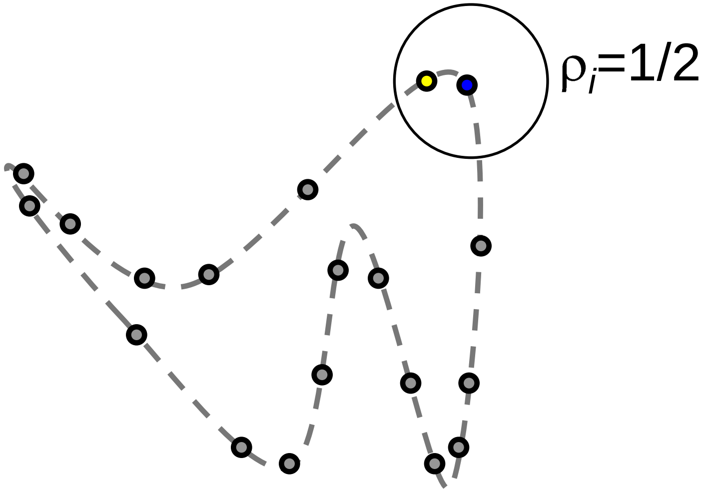
Compute the sampling density at each sample by counting the number of samples around it:



Non-Uniform Samples

Approach:

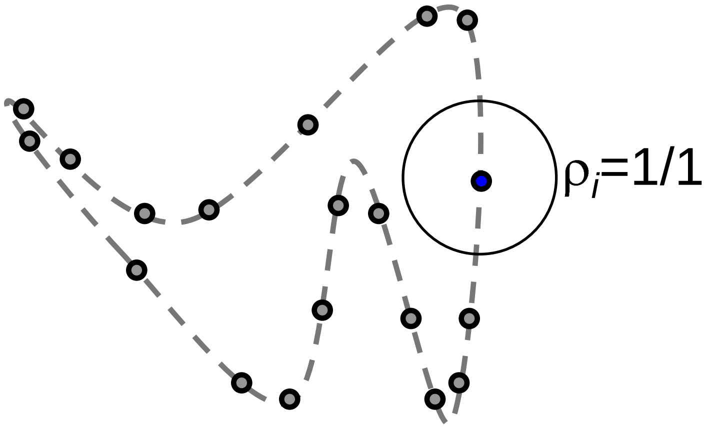
Compute the sampling density at each sample by counting the number of samples around it:



Non-Uniform Samples

<u>Approach</u>:

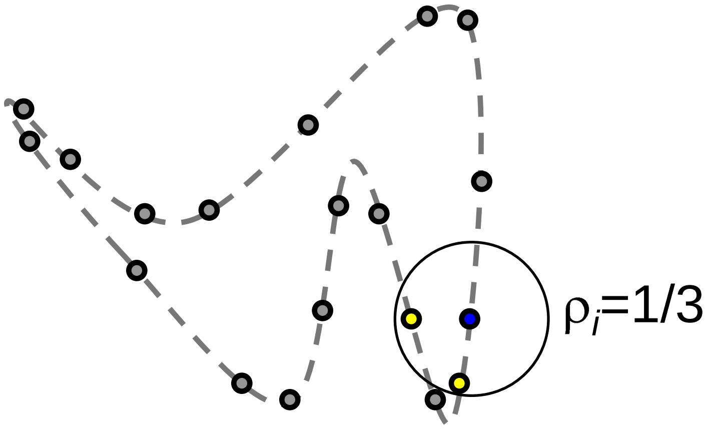
Compute the sampling density at each sample by counting the number of samples around it:



Non-Uniform Samples

<u>Approach</u>:

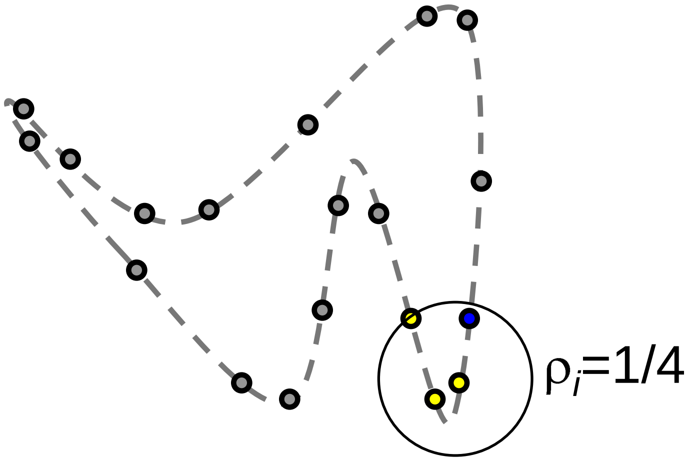
Compute the sampling density at each sample by counting the number of samples around it:



Non-Uniform Samples

Approach:

Compute the sampling density at each sample by counting the number of samples around it:

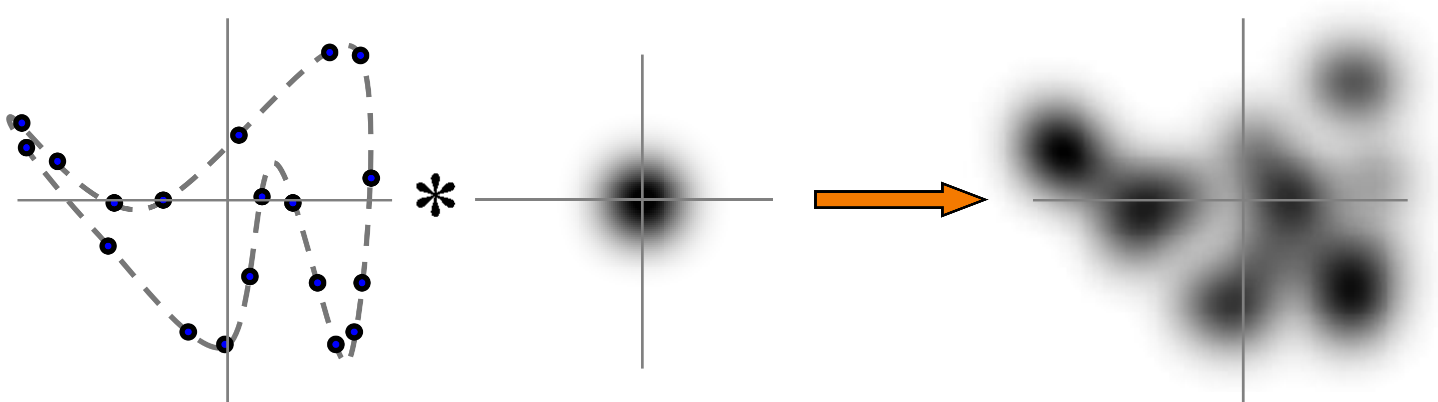


Non-Uniform Samples

<u>Approach</u>:

Compute the sampling density at each sample by counting the number of samples around it.

To do this efficiently, we “splat” the points into a voxel grid and convolve with a low-pass filter.



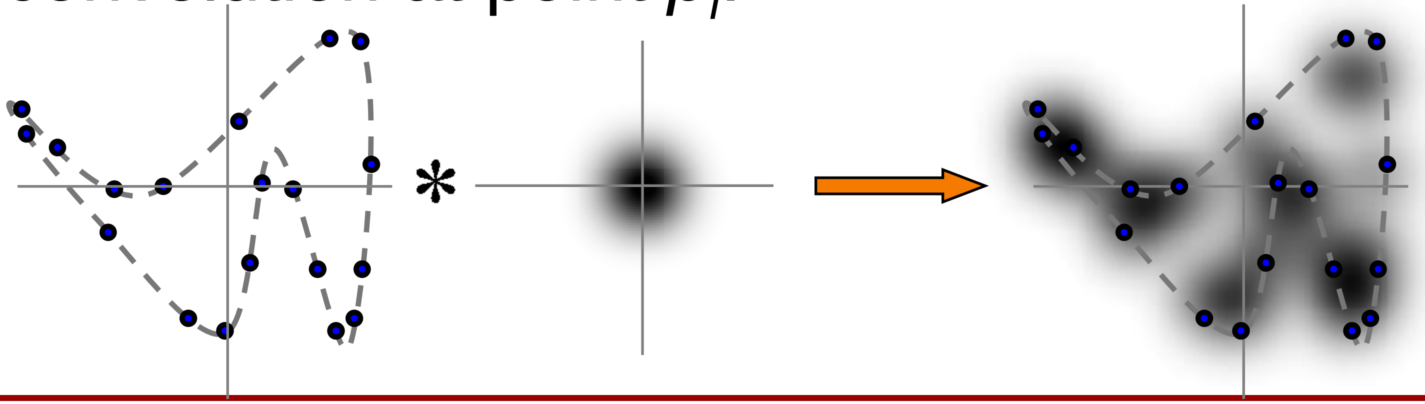
Non-Uniform Samples

<u>Approach</u>:

Compute the sampling density at each sample by counting the number of samples around it.

To do this efficiently, we “splat” the points into a voxel grid and convolve with a low-pass filter.

We set ρ_i equal to the reciprocal of the value convolution at point p_i.

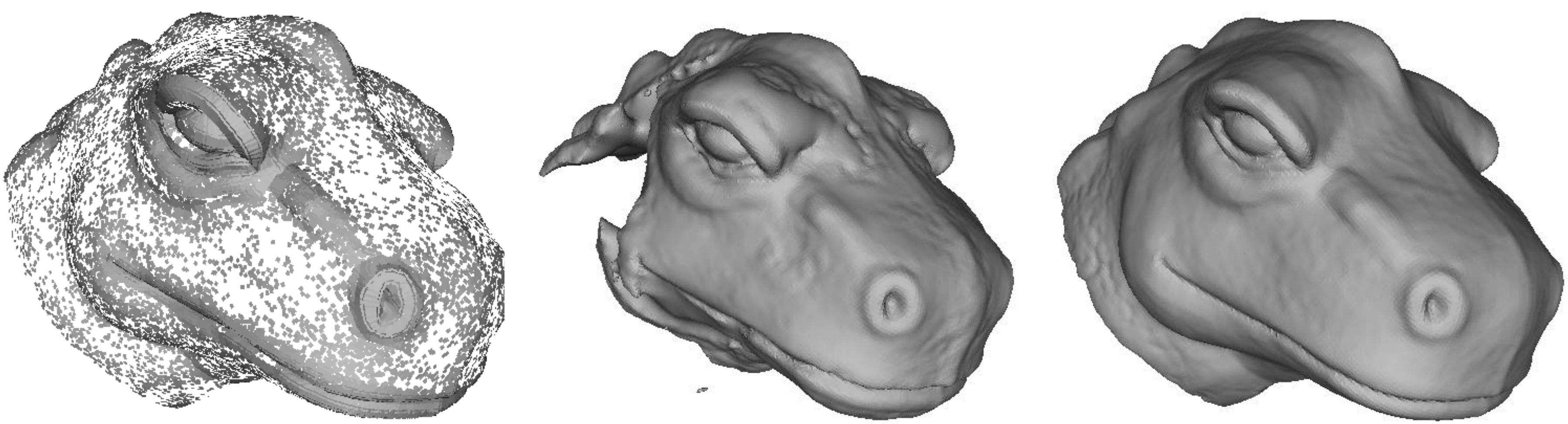


Non-Uniform Samples

Approach:

Compute the sampling density at each sample by counting the number of samples around it.

To do this efficiently, we "splat" the points into a voxel grid and convolve with a low-pass filter.

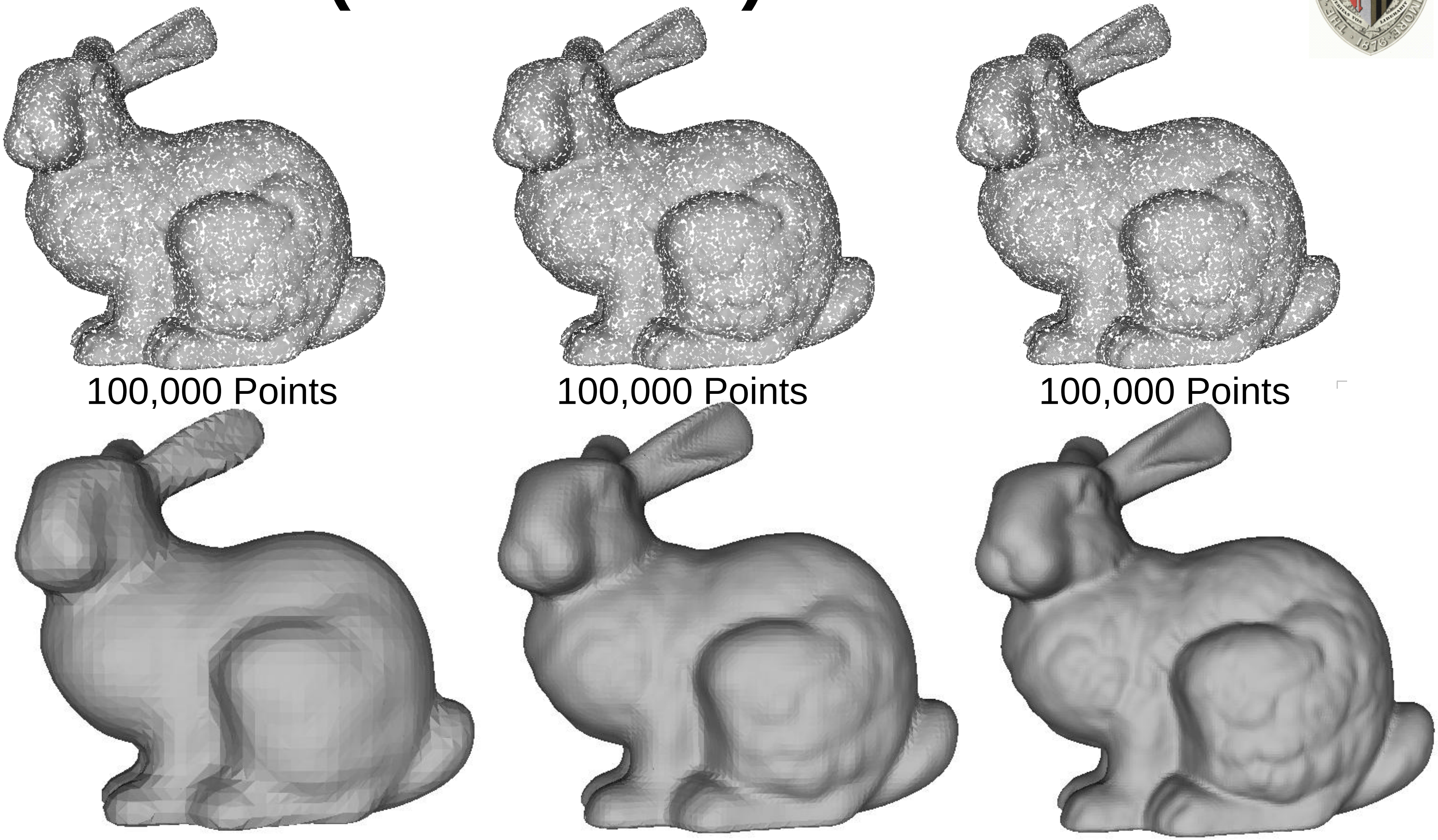


Non-Uniform Samples Unweighted Reconstruction Weighted Reconstruction

Outline

- Introduction
- Related Work
- Approach
- Results
- Conclusion

Results (Resolution)

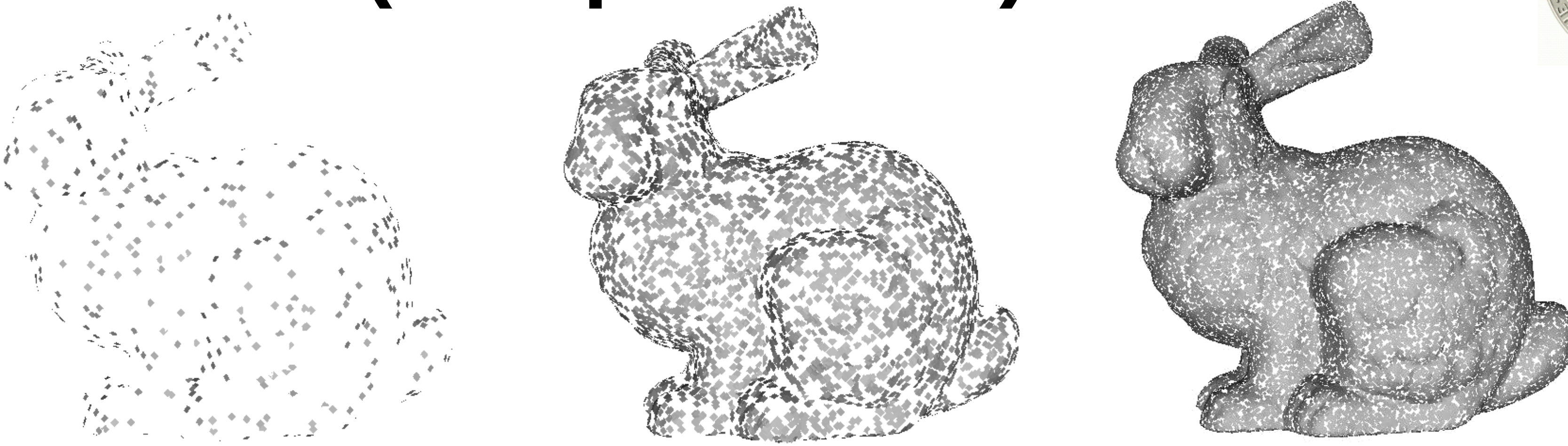


res=64^3
tris=11,900
time=0:01

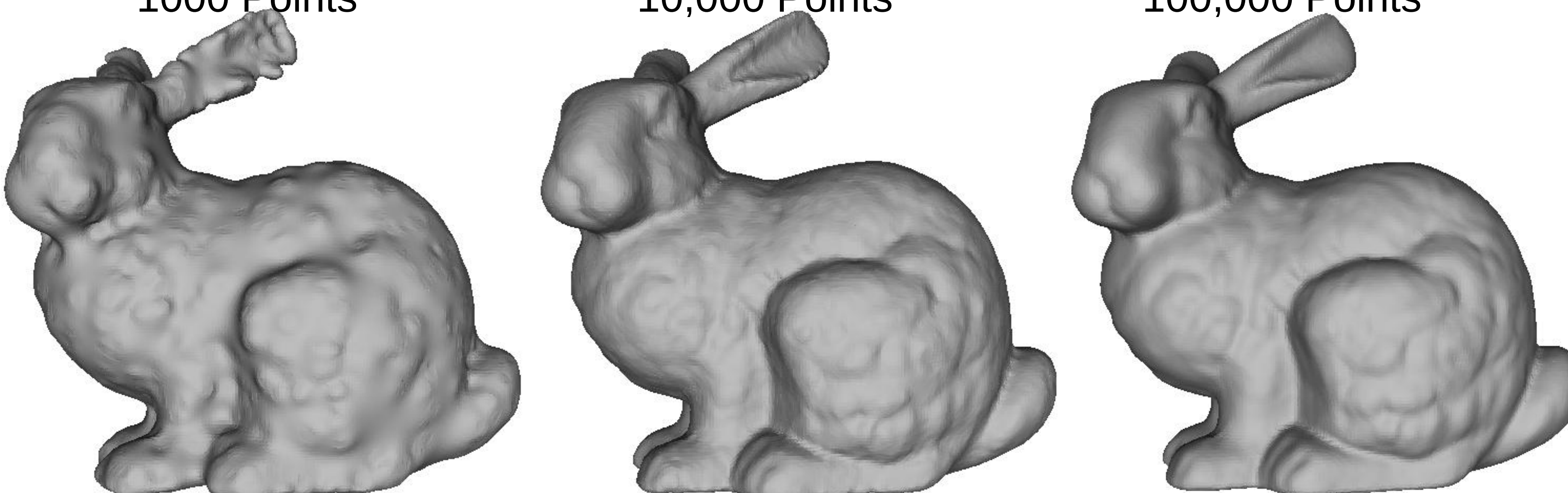
res=128^3
tris=49,556
time=0:03

res=256^3
tris=200,692
time=0:17

Results (Sample Count)



1000 Points | 10,000 Points | 100,000 Points

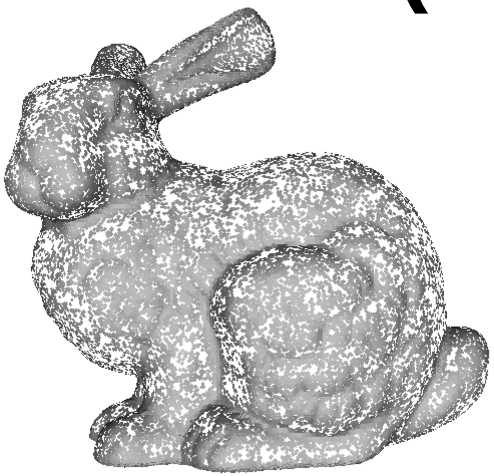


res=256^3
tris=205,440
time=0:17

res=256^3
tris=201,464
time=0:17

res=256^3
tris=200,692
time=0:17

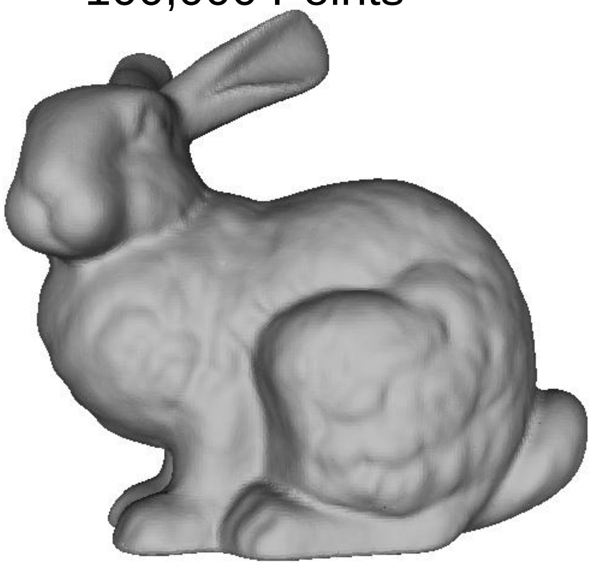
Results (Non-Uniform Sampling)



100,000 Points

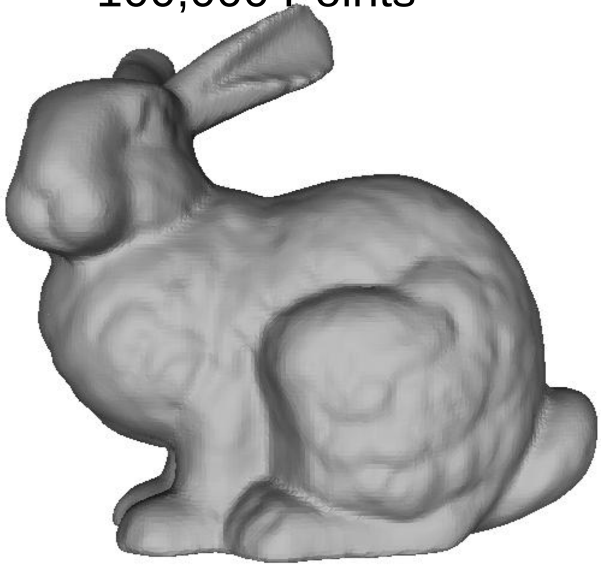
100,000 Points

100,000 Points



res=256^3
tris=209,020
time=0:20

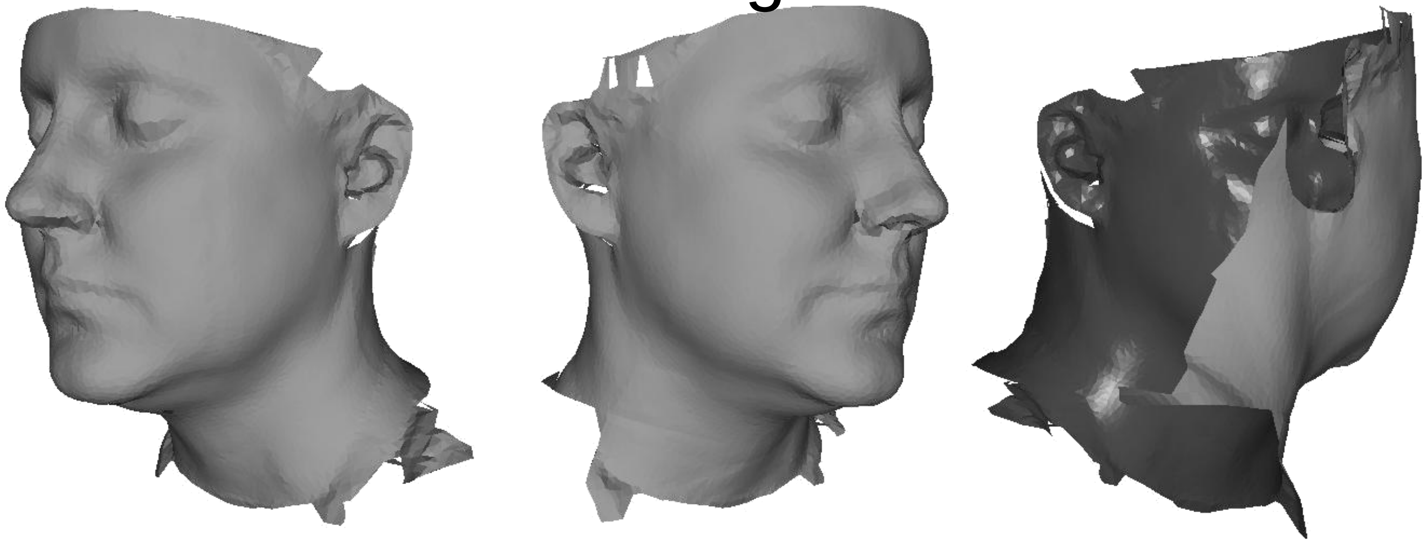
res=256^3
tris=200,692
time=0:17



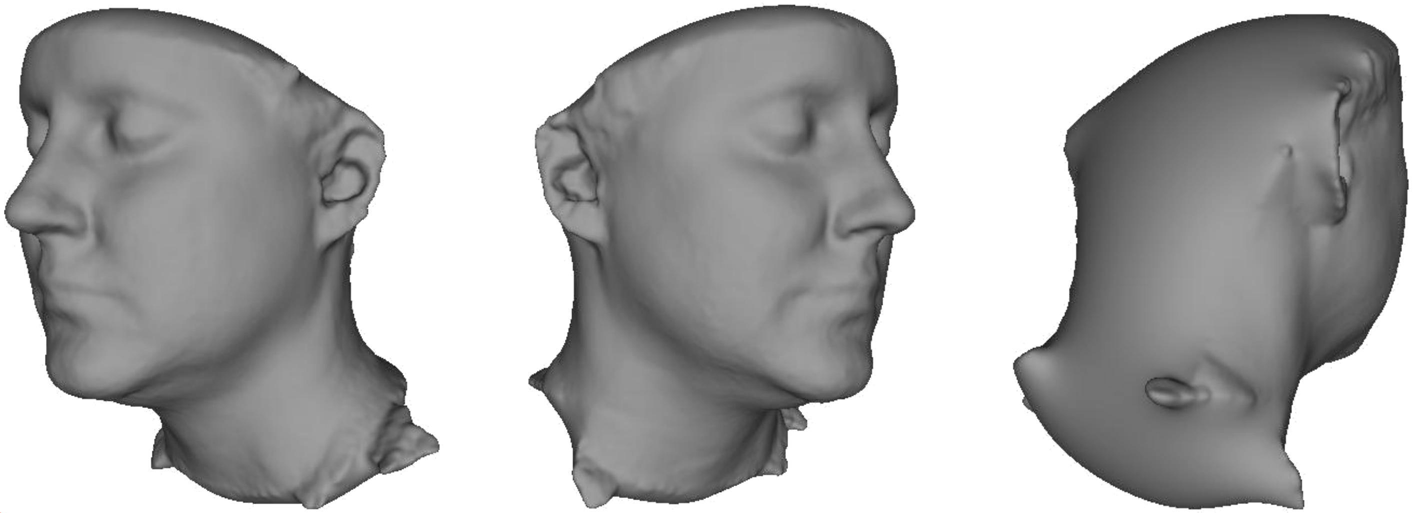
res=256^3
tris=130,440
time=0:19

Results (Holes)

Original

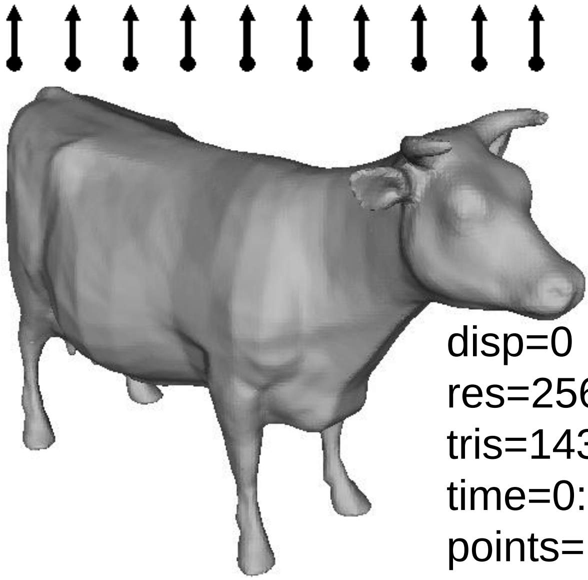


Reconstruction

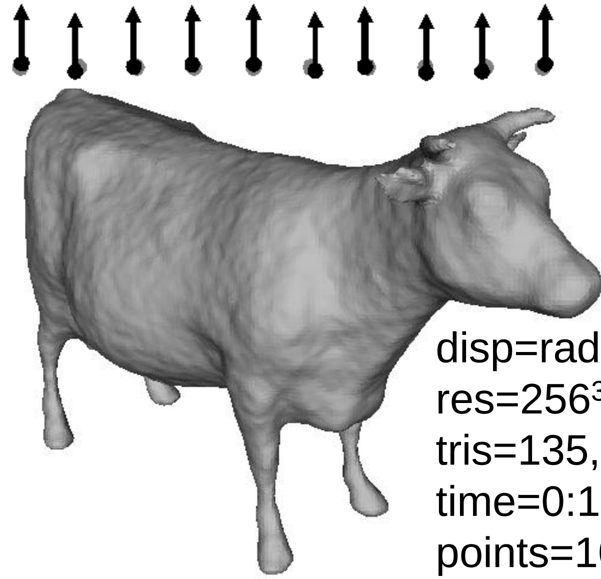


res=256^3
tris=267,736
time=0:07
points=25,000

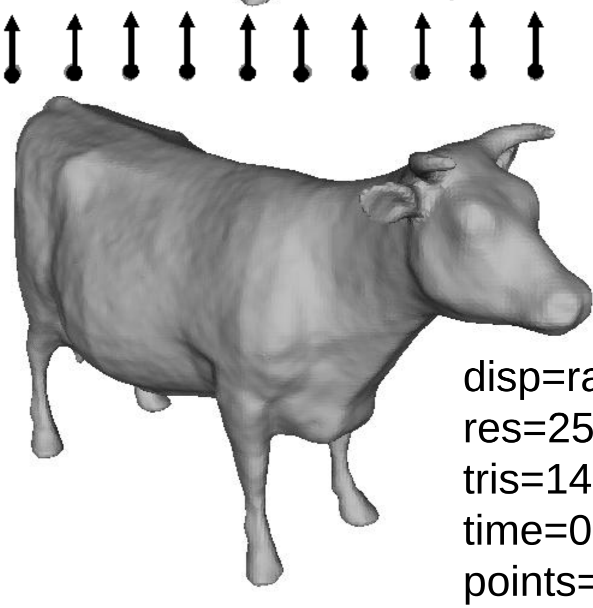
Results (Positional Noise)



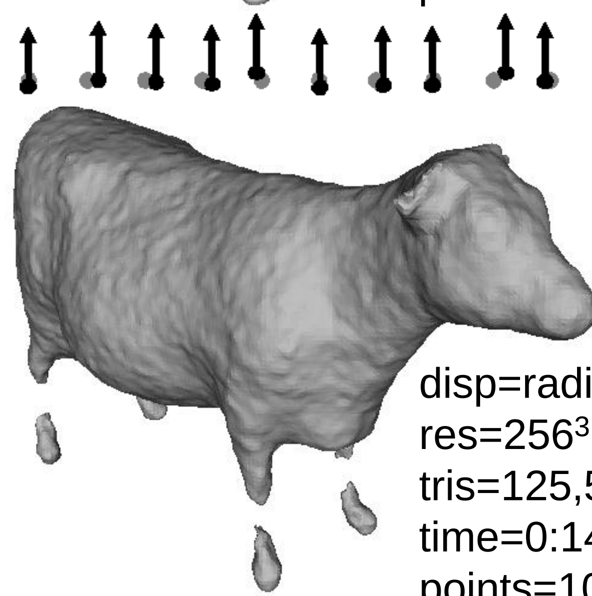
disp=0
res=256^3
tris=143,172
time=0:16
points=100,000



disp=radius/64
res=256^3
tris=135,988
time=0:15
points=100,000

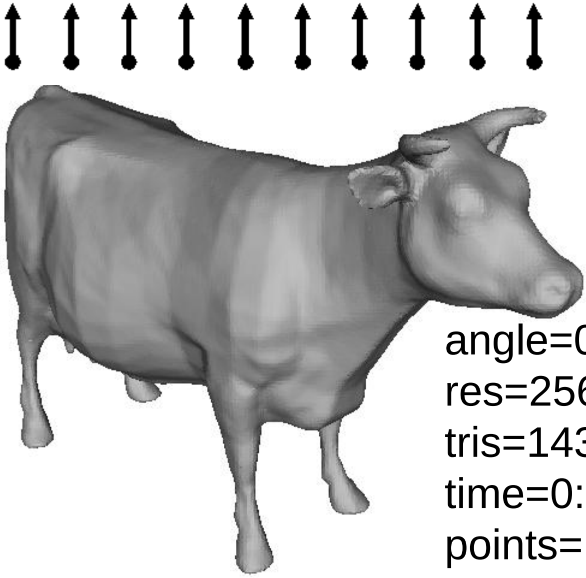


disp=radius/128
res=256^3
tris=140,688
time=0:15
points=100,000

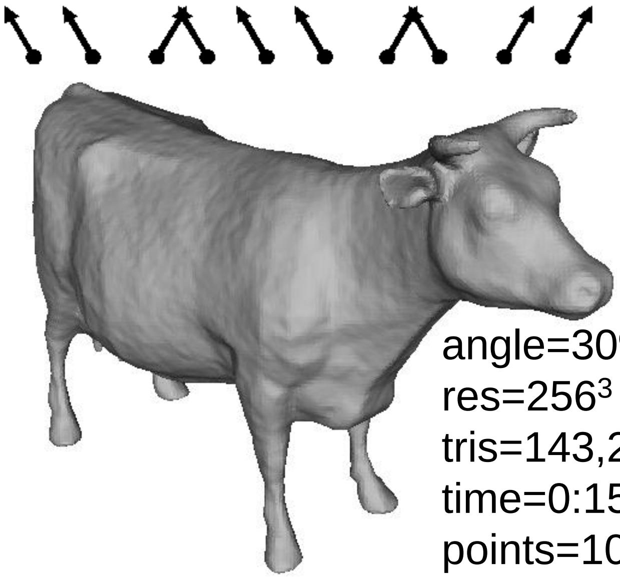


disp=radius/32
res=256^3
tris=125,500
time=0:14
points=100,000

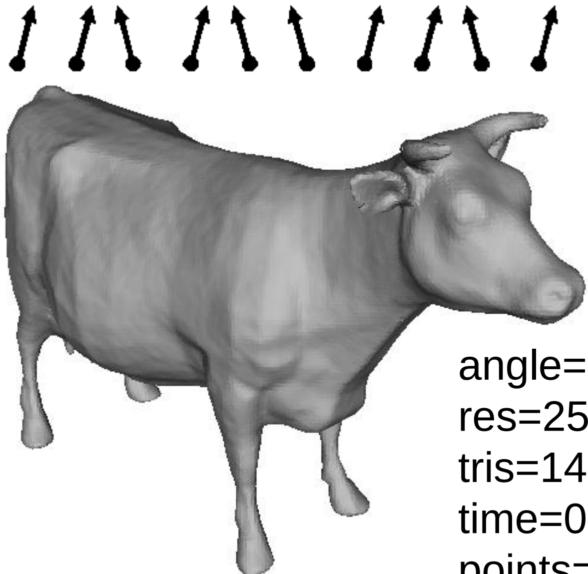
Results (Normal Noise)



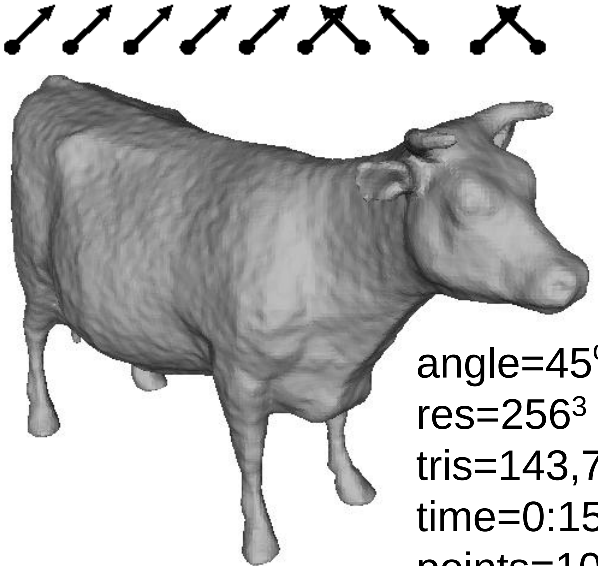
angle=0^o
res=256^3
tris=143,172
time=0:16
points=100,000



angle=30^o
res=256^3
tris=143,268
time=0:15
points=100,000



angle=15^o
res=256^3
tris=143,136
time=0:15
points=100,000



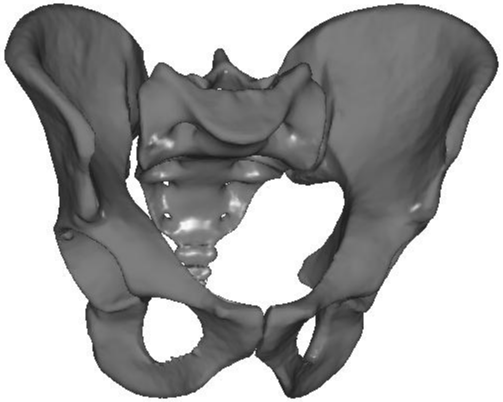
angle=45^o
res=256^3
tris=143,748
time=0:15
points=100,000

Results
(Non-Uniform Sampling / Related Work)

100,000 Points

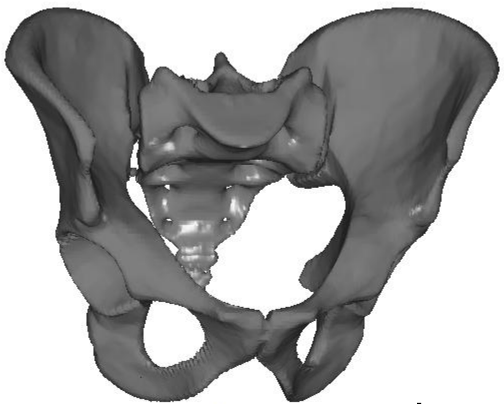
100,000 Points

100,000 Points



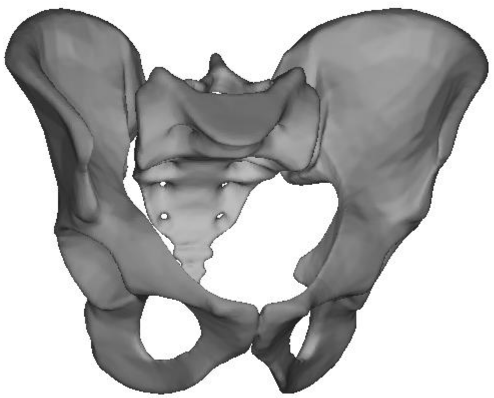
RBF Reconstruction

res=256^3
tris=302,000
time=5:23



MPU Reconstruction

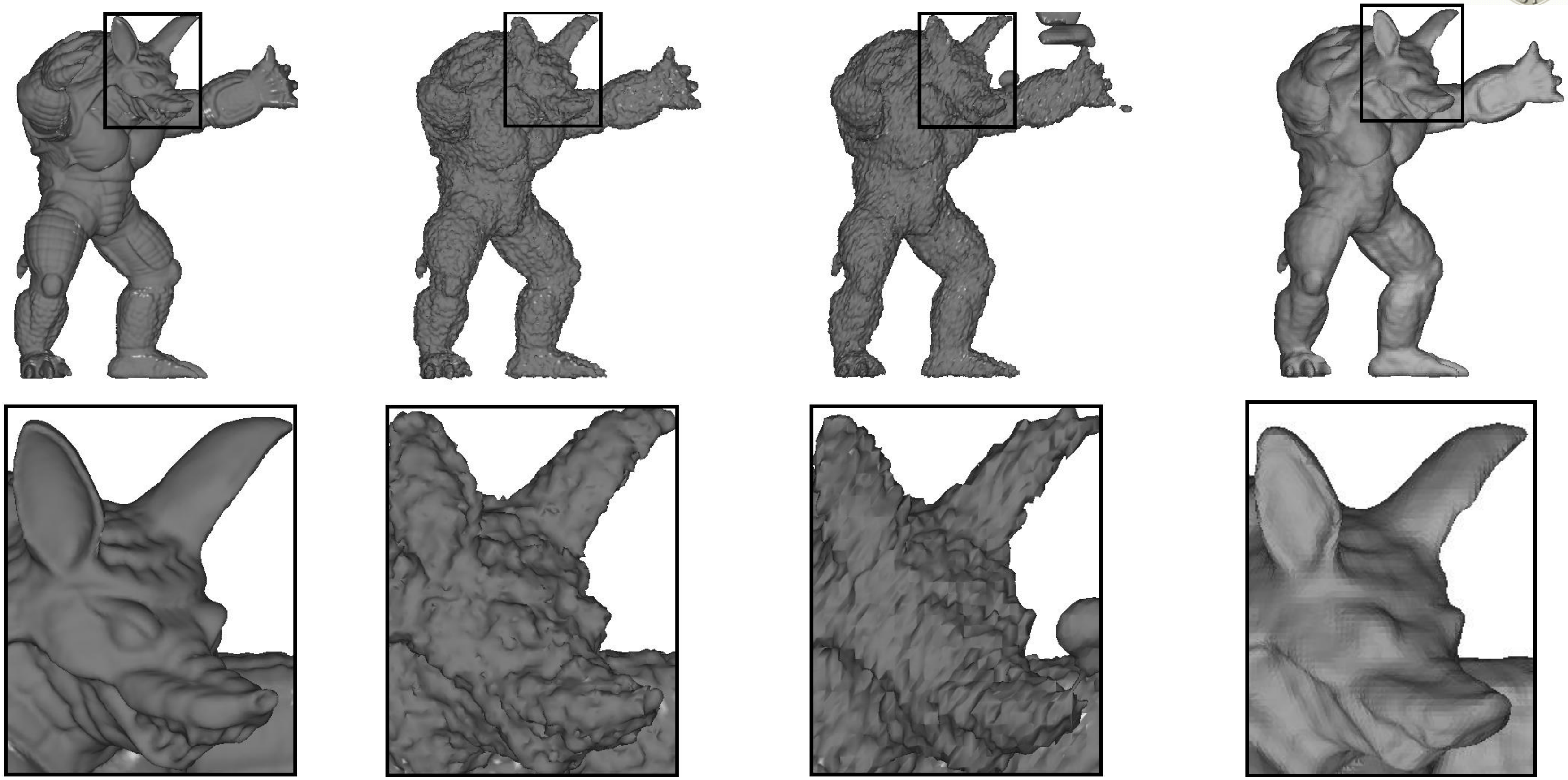
res=256^3
tris=288,000
time=0:39



Our Reconstruction

res=256^3
tris=314,000
time=0:17

Results (Noise / Related Work)



Original	RBF Reconstruction	MPU Reconstruction	Our Reconstruction
	res=256^3 tris=200,000 time=24:10 points=100,000	res=256^3 tris=205,000 time=2:14 points=100,000	res=256^3 tris=177,000 time=0:16 points=100,000

Outline

- Introduction
- Related Work
- Approach
- Results
- Conclusion

Conclusion

Properties:

✓ Fast and simple to compute

✓ Independent of topology

✓ Robust to non-uniform sampling

✓ Robust to noise

✗ O(R^3) memory footprint for O(R^2) reconstruction

Conclusion

Theoretical Contribution:

- Transformed the surface reconstruction problem into a Volume integral
- Used the Divergence Theorem to express the integral as a surface integral
- Used Monte-Carlo integration to approximate the surface integral as a summation over an oriented point sample

Conclusion

We presented an algorithm for reconstruction that proceeds in three simple steps:

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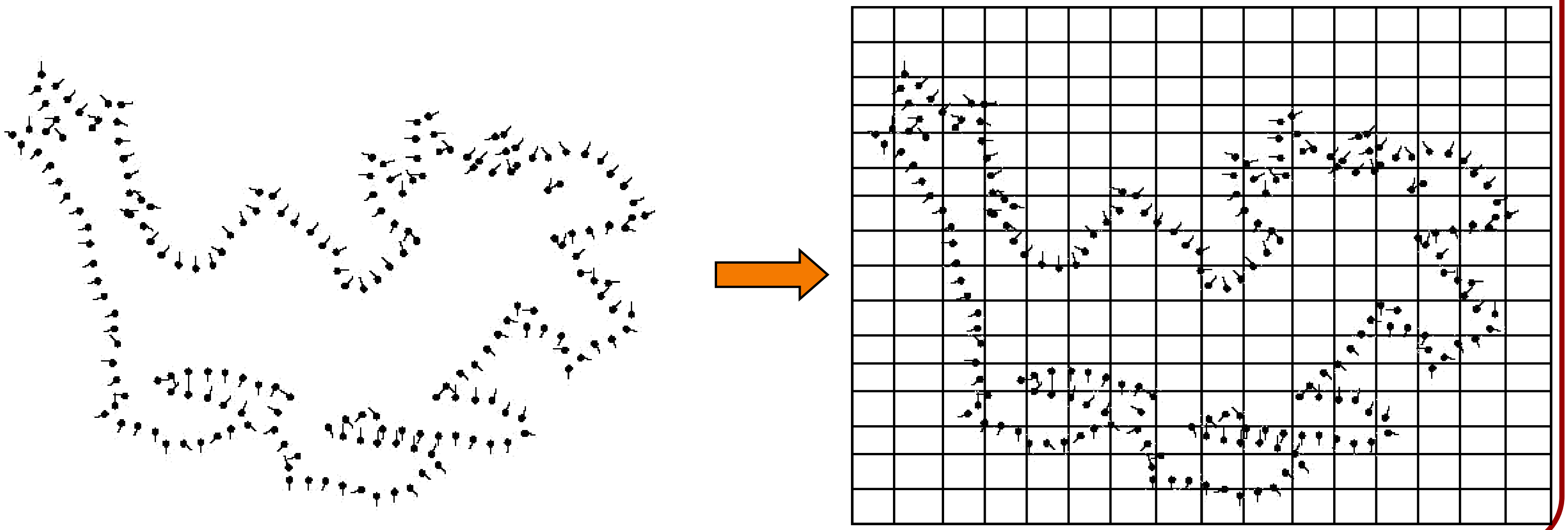
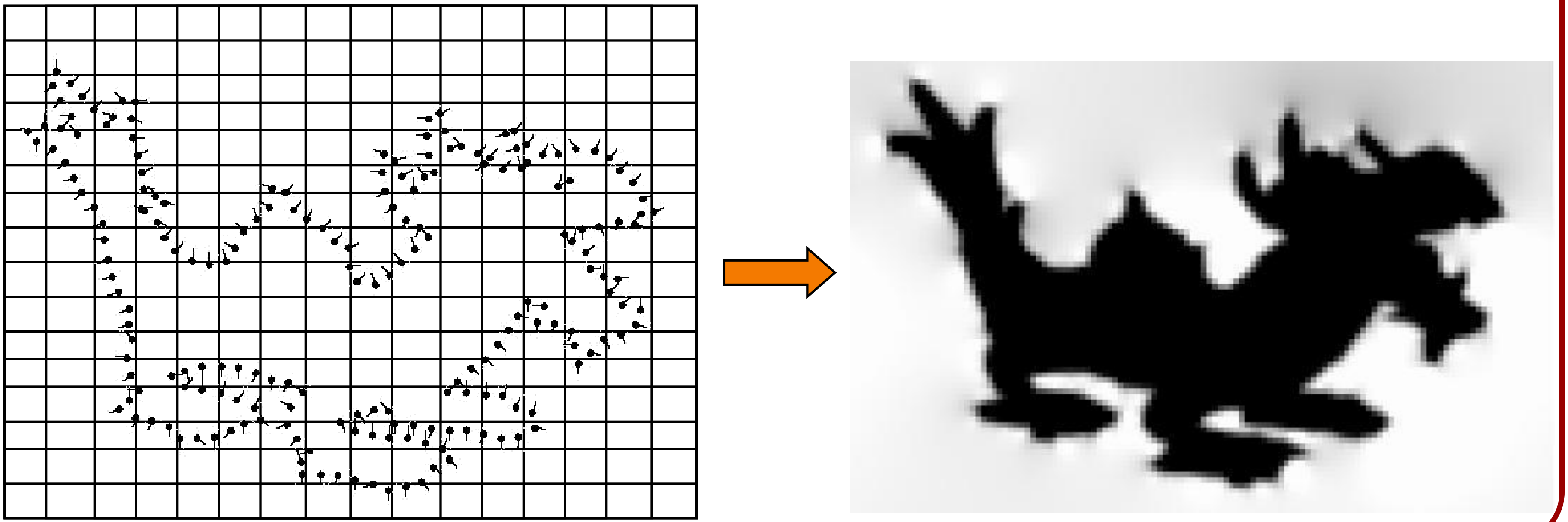
1. Splat the oriented points into a voxel grid



Conclusion

We presented an algorithm for reconstruction that proceeds in three simple steps:

1. Splat the oriented points into a voxel grid
2. Convolve with a fixed filter



Conclusion

We presented an algorithm for reconstruction that proceeds in three simple steps:

1. Splat the oriented points into a voxel grid
2. Convolve with a fixed filter
3. Extract the iso-surface