FFTs in Graphics and Vision

Rotational and Reflective
Symmetry Detection

Outline

Representation Theory

Symmetry Detection

- Rotational Symmetry
- Reflective Symmetry

Representation Theory

Recall:

A group is a set of elements G with a binary operation (often denoted "·") such that for all $f,g,h \in G$, the following properties are satisfied:

- Closure:

$$g \cdot h \in G$$

- Associativity:

$$f \cdot (g \cdot h) = (f \cdot g) \cdot h$$

- Identity: There exists an identity element $1 \in G$ s.t.:

$$1 \cdot g = g \cdot 1 = g$$

- Inverse: Every element g has an inverse g^{-1} s.t.:

$$g \cdot g^{-1} = g^{-1} \cdot g = 1$$

Representation Theory

Observation 1:

Given a group $G=\{g_1,\ldots,g_n\}$, for any $g \in G$, the (set-theoretic) map that multiplies the elements of G on the left by g is invertible.

(The inverse is the map multiplying the elements of G on the left by g^{-1}.)

Representation Theory

Observation 1:

In particular, this implies that the set $\{g \cdot g_1, \ldots, g \cdot g_n\}$ is just a re-ordering of the set $\{g_1, \ldots, g_n\}$.

Or more simply, $gG = G$.

Representation Theory

Observation 1:

In particular, this implies that the set $\{g \cdot g_1, \ldots, g \cdot g_n\}$ is just a re-ordering of the set $\{g_1, \ldots, g_n\}$.

Or more simply, $gG=G$.

Similarly, the set $\{(g_1)^{-1}, \ldots, (g_n)^{-1}\}$ is just a re-ordering of the set $\{g_1, \ldots, g_n\}$.

Or more simply, $G^{-1}=G$.

Representation Theory

Recall:

A Hermitian inner product is a map from *V*x*V* into the complex numbers that is:

1. Linear: For all $u,v,w \in V$ and any real scalar λ

$$\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$$
$$\langle \lambda v, w\rangle = \lambda\langle v, w\rangle$$

2. Conjugate Symmetric: For all $u,v \in V$

$$\langle v, w\rangle = \overline{\langle w, v\rangle}$$

3. Positive Definite: For all $v \in V$:

$$\langle v, v\rangle \geq 0$$
$$\langle v, v\rangle = 0 \quad \Leftrightarrow \quad v = 0$$

Representation Theory

Observation 2:

Given a Hermitian inner-product space V, and given a set of vectors $\{v_1,\ldots,v_n\}\subset V$, the vector minimizing the sum of squared distances is the average of $\{v_1,\ldots,v_n\}$:

$$\frac{1}{n}\sum_{k=1}^{n} v_k = \arg\min_{v\in V}\left(\sum_{k=1}^{n}\left\|v-v_k\right\|^2\right)$$

Representation Theory

Recall:

A unitary representation of a group G on a Hermitian inner-product space V is a map ρ that sends every element in G to an orthogonal transformation on V, satisfying:

$$\rho(g \cdot h) = \rho(g) \cdot \rho(h)$$

for all $g,h \in G$.

Representation Theory

Definition:

We say that a vector $v \in V$ is invariant under the action of G if G sends v back to itself:

$$\rho_g(v) = v$$

for all $g \in G$.

Representation Theory

Notation:

We denote by V_G the set of vectors in V that are invariant under the action of G:

$$V_G = \{ v \in V \mid \rho_g(v) = v, \forall g \in G \}$$

Representation Theory

Observation 3:

Note that the set V_G is a vector sub-space of V.

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If $v,w \in V_G$, then for any $g \in G$, we have:

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Representation Theory

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If $v,w \in V_G$, then for any $g \in G$, we have:

$$\rho_g(v) = v \quad \text{and} \quad \rho_g(w) = w$$

And for all scalars α and β we have:

$$\rho_g(\alpha v + \beta w) = \alpha \rho_g(v) + \beta \rho_g(w)$$

$$= \alpha v + \beta w$$

Representation Theory

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And for all scalars α and β we have:

$$\rho_g(\alpha v + \beta w) = \alpha\rho_g(v) + \beta\rho_g(w)$$

$$= \alpha v + \beta w$$

So $\alpha v + \beta w \in V_G$ as well.

Representation Theory

<u>Observation 4</u>:

Given a finite group G and given vector $v \in V$, the vector obtained by averaging over G:

$$\text{Average}(v, G) = \frac{1}{|G|} \sum_{g \in G} \rho_g(v)$$

is invariant under the action of G.

Representation Theory

Observation 4:

To see this, let *h* be any element in *G*.

We would like to show that *h* maps the average back to itself:

$$\rho_h\left(\text{Average}(v,G)\right)=\text{Average}(v,G)$$

Representation Theory

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We would like to show that *h* maps the average back to itself:

$$\rho_h\left(\text{Average}(v,G)\right) = \text{Average}(v,G)$$

Expanding the right hand side we get:

$$\rho_h\left(\text{Average}(v,G)\right) = \rho_h\left(\frac{1}{|G|}\sum_{g\in G}\rho_g(v)\right)$$

Representation Theory

Observation 4:

$$\rho_h\left(\text{Average}(v,G)\right) = \rho_h\left(\frac{1}{|G|}\sum_{g\in G}\rho_g(v)\right)$$

By the linearity of the representation, we get:

$$\rho_h\left(\text{Average}(v,G)\right) = \frac{1}{|G|}\sum_{g\in G}\rho_h\left(\rho_g(v)\right)$$

Representation Theory

Observation 4:

$$\rho_h\left(\mathrm{Average}(v,G)\right)=\frac{1}{|G|}\sum_{g\in G}\rho_h\left(\rho_g(v)\right)$$

Since the representation preserves the group structure, we get:

$$\rho_h\left(\mathrm{Average}(v,G)\right)=\frac{1}{|G|}\sum_{g\in G}\rho_{h\cdot g}(v)$$

Representation Theory

Observation 4:

$$\rho_h\left(\text{Average}(v,G)\right)=\frac{1}{|G|}\sum_{g\in G}\rho_{h\cdot g}(v)$$

But this can be re-written as a summation over the set *hG*:

$$\rho_h\left(\text{Average}(v,G)\right)=\frac{1}{|G|}\sum_{g\in hG}\rho_{g}(v)$$

Representation Theory

Observation 4:

$$\rho_h\left(\text{Average}(v,G)\right)=\frac{1}{|G|}\sum_{g\in hG}\rho_g(v)$$

And since *hG*=*G*, this implies that:

$$\rho_h\left(\text{Average}(v,G)\right)=\frac{1}{|G|}\sum_{g\in G}\rho_g(v)$$

$$=\text{Average}(v,G)$$

Representation Theory

Observation 5:

Given a finite group *G* and given a vector $v \in V$, the average of *v* over *G* is the closest *G*-invariant vector to *v*:

$$\text{Average}(v,G) = \arg\min_{v_0 \in V_G} \left(\|v_0 - v\|^2 \right)$$

Representation Theory

Observation 5:

$$\text{Average}(v,G) = \arg\min_{v_0 \in V_G} \left(\|v_0 - v\|^2 \right)$$

Since v_0 is invariant under the action of G, we can write out the squared distances as:

$$\|v_0 - v\|^2 = \frac{1}{|G|} \sum_{g \in G} \|\rho_g(v_0) - v\|^2$$

Representation Theory

Observation 5:

$$\|v_0 - v\|^2 = \frac{1}{|G|}\sum_{g\in G}\|\rho_g(v_0) - v\|^2$$

Since the representation is unitary, we can re-write this as:

$$\|v_0 - v\|^2 = \frac{1}{|G|}\sum_{g\in G}\|v_0 - (\rho_g)^{-1}(v)\|^2$$

Representation Theory

Observation 5:

$$\|v_0 - v\|^2 = \frac{1}{|G|}\sum_{g\in G}\left\|v_0 - (\rho_g)^{-1}(v)\right\|^2$$

Since the representation preserves the group structure, we get:

$$\|v_0 - v\|^2 = \frac{1}{|G|}\sum_{g\in G}\left\|v_0 - \rho_{g^{-1}}(v)\right\|^2$$

Representation Theory

Observation 5:

$$\left\| v_0 - v \right\|^2 = \frac{1}{|G|} \sum_{g \in G} \left\| v_0 - \rho_{g^{-1}}(v) \right\|^2$$

Re-writing this as a summation over G^{-1}, we get:

$$\left\| v_0 - v \right\|^2 = \frac{1}{|G|} \sum_{g \in G^{-1}} \left\| v_0 - \rho_g(v) \right\|^2$$

Representation Theory

Observation 5:

$$\|v_0 - v\|^2 = \frac{1}{|G|}\sum_{g\in G^{-1}} \|v_0 - \rho_g(v)\|^2$$

And finally, using the fact that the set G^{-1} is just a re-ordering of the set G, we get:

$$\|v_0 - v\|^2 = \frac{1}{|G|}\sum_{g\in G} \|v_0 - \rho_g(v)\|^2$$

Representation Theory

Observation 5:

$$\left\|v_0 - v\right\|^2 = \frac{1}{|G|}\sum_{g\in G}\left\|v_0 - \rho_g(v)\right\|^2$$

Thus, v_0 is the G-invariant vector minimizing the squared distance to v if and only if it minimizes the sum of squared distances to the vectors:

$$\{\rho_{g_1}(v),\ldots,\rho_{g_n}(v)$$

Representation Theory

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$$\|v_0 - v\|^2 = \frac{1}{|G|}\sum_{g\in G}\|v_0 - \rho_g(v)\|^2$$

Thus, v_0 is the G-invariant vector minimizing the squared distance to v if and only if it minimizes the sum of squared distances to the vectors:

$$\{\rho_{g_1}(v),\ldots,\rho_{g_n}(v)$$

So v_0 must be the average of these vectors:

$$v_0 = \frac{1}{|G|}\sum_{g\in G}\rho_g(v) = \text{Average}(v, G)$$

Representation Theory

Note:

Since the average map:

$$\text{Average}(v,G)=\frac{1}{|G|}\sum_{g\in G}\rho_g(v)$$

is a linear map returning the closest *G*-invariant vector to *v*, the average map is just the projection map from *V* to V_G:

$$\pi_G(v)=\text{Average}(v,G)$$

Outline

Representation Theory

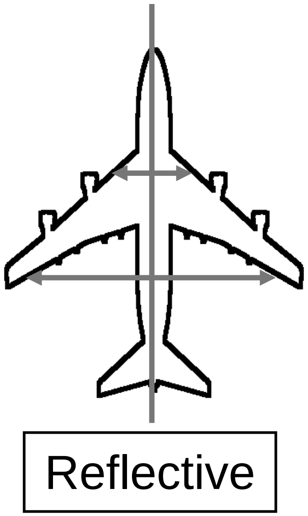
Symmetry Detection

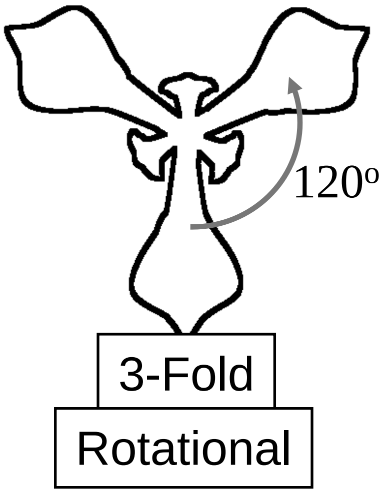
- Rotational Symmetry
- Reflective Symmetry

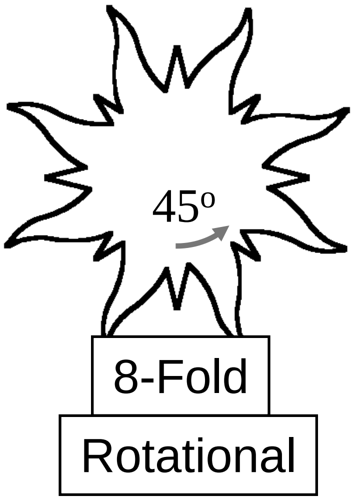
Symmetry Detection

For functions on a circle, we defined measures of:

- Reflective Symmetry: for every axis of reflective symmetry.
- Rotational Symmetry: for every order of rotational symmetry.



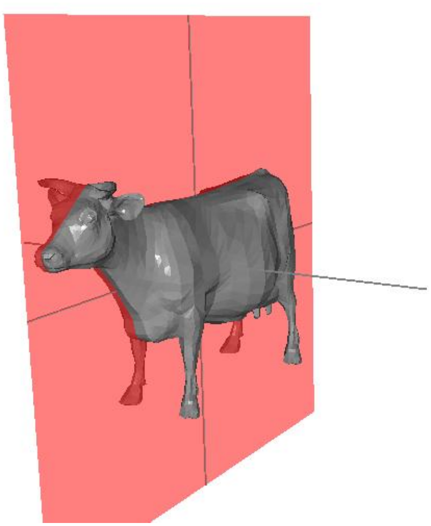




Symmetry Detection

For functions on a sphere, we would like to define a measure of:

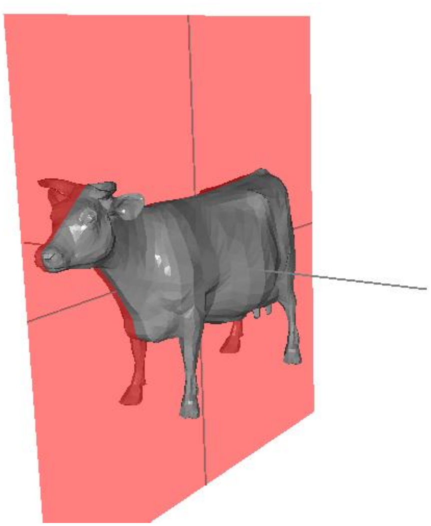
- Reflective Symmetry: for every plane of reflective symmetry.

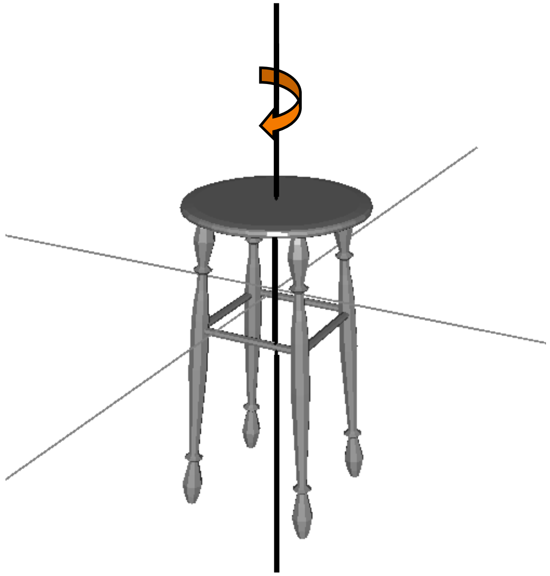


Symmetry Detection

For functions on a sphere, we would like to define a measure of:

- Reflective Symmetry: for every plane of reflective symmetry.
- Rotational Symmetry: for every axis passing through the origin and every order of rotational symmetry.





Symmetry Detection

Goal:

Reflective Symmetry:

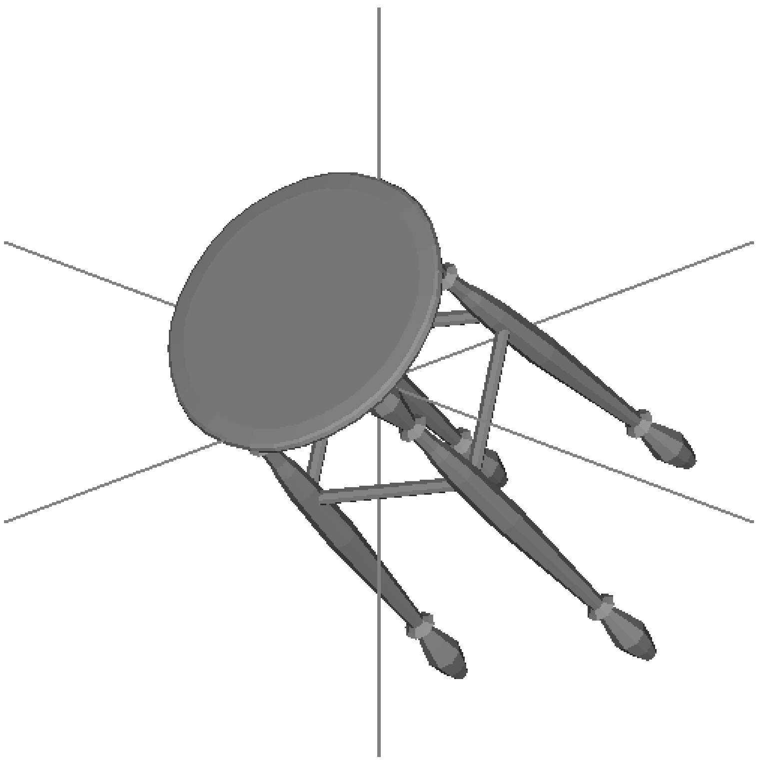
- Compute the spherical function giving the measure of reflective symmetry about the plane perpendicular to every axis through the origin.

Rotational Symmetry:

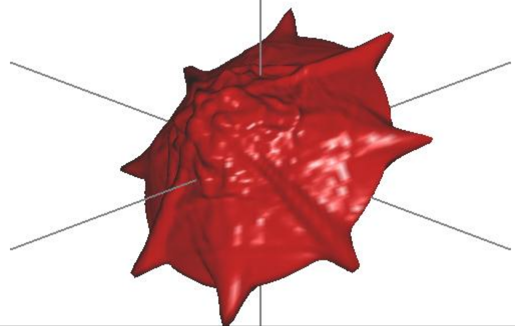
- For every order of rotational symmetry *k*:
  - » Compute the spherical function giving the measure of *k*-fold symmetry about every axis through the origin.

Symmetry Detection

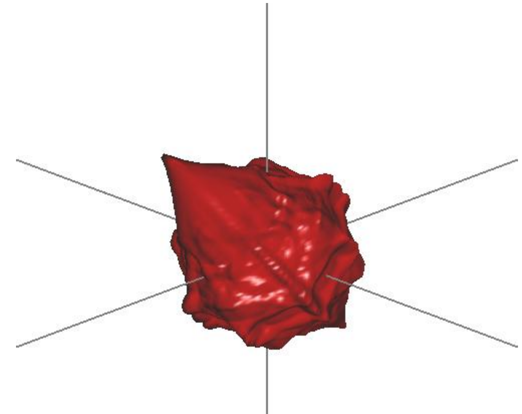
Goal:



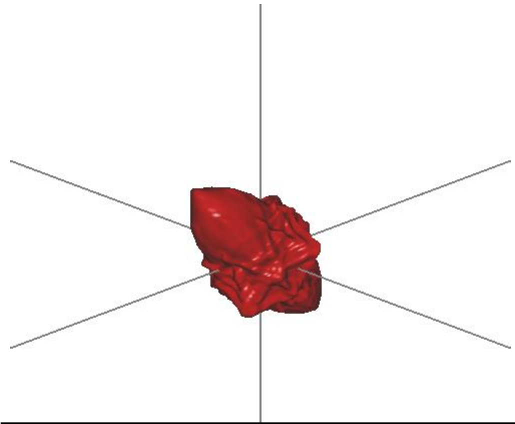
Model



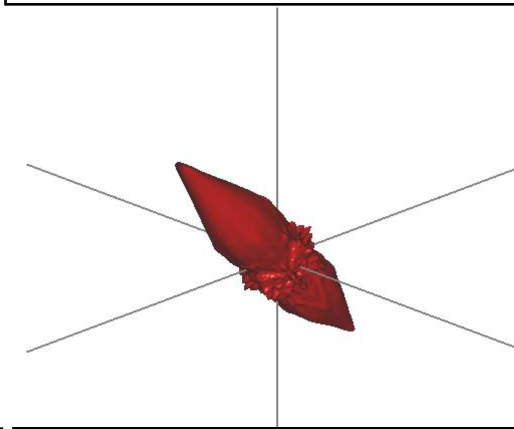
Reflective Symmetries



2-Fold Rotational Symmetries



3-Fold Rotational Symmetries

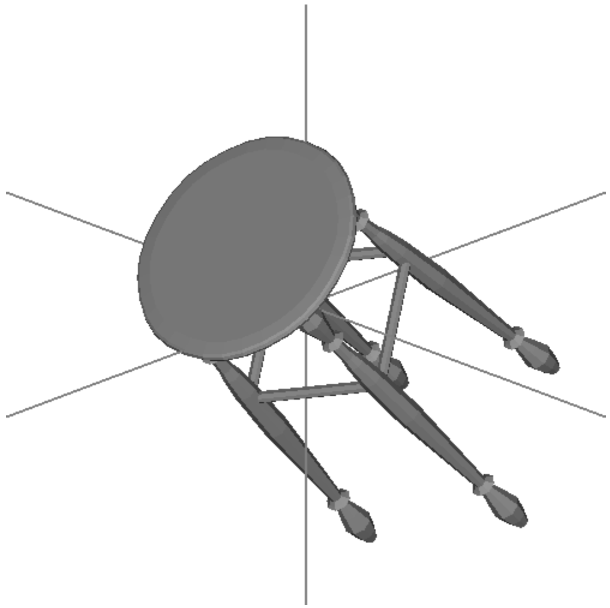


4-Fold Rotational Symmetries

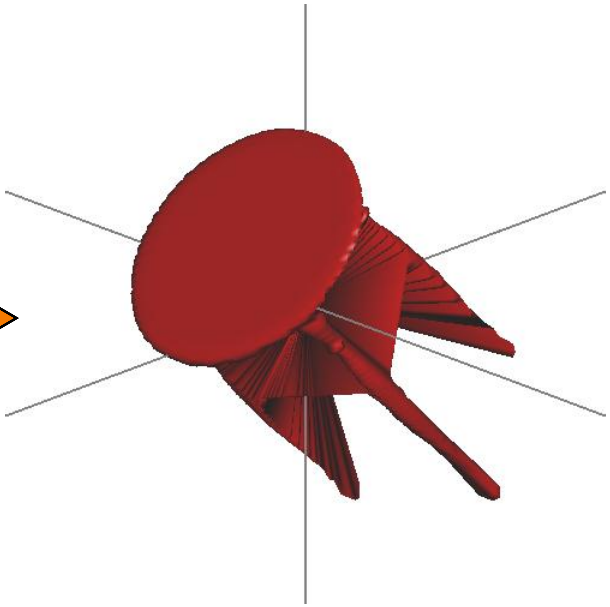
Symmetry Detection

Approach:

As in the 1D case, we will compute the symmetries of a shape by representing the shape by a spherical function.



Model



Spherical Extent Function

Symmetry Detection

Recall:

To compute the measure of symmetry of a function, we:

- Associated a group *G* of transformations to each type of symmetry

Symmetry Detection

Recall:

To compute the measure of symmetry of a function, we:

- Associated a group *G* of transformations to each type of symmetry
- Defined the measure of symmetry as the size of the closest *G*-invariant function:

$$\mathrm{Sym}^2(f,G) = \left\|\pi_G(f)\right\|^2$$

Symmetry Detection

Recall:

Using the fact that nearest symmetric function was the average of the function under the image of the group, we got:

$$\text{Sym}^2(f,G) = \left\| \frac{1}{|G|} \sum_{g \in G} \rho_g(f) \right\|^2$$

Outline

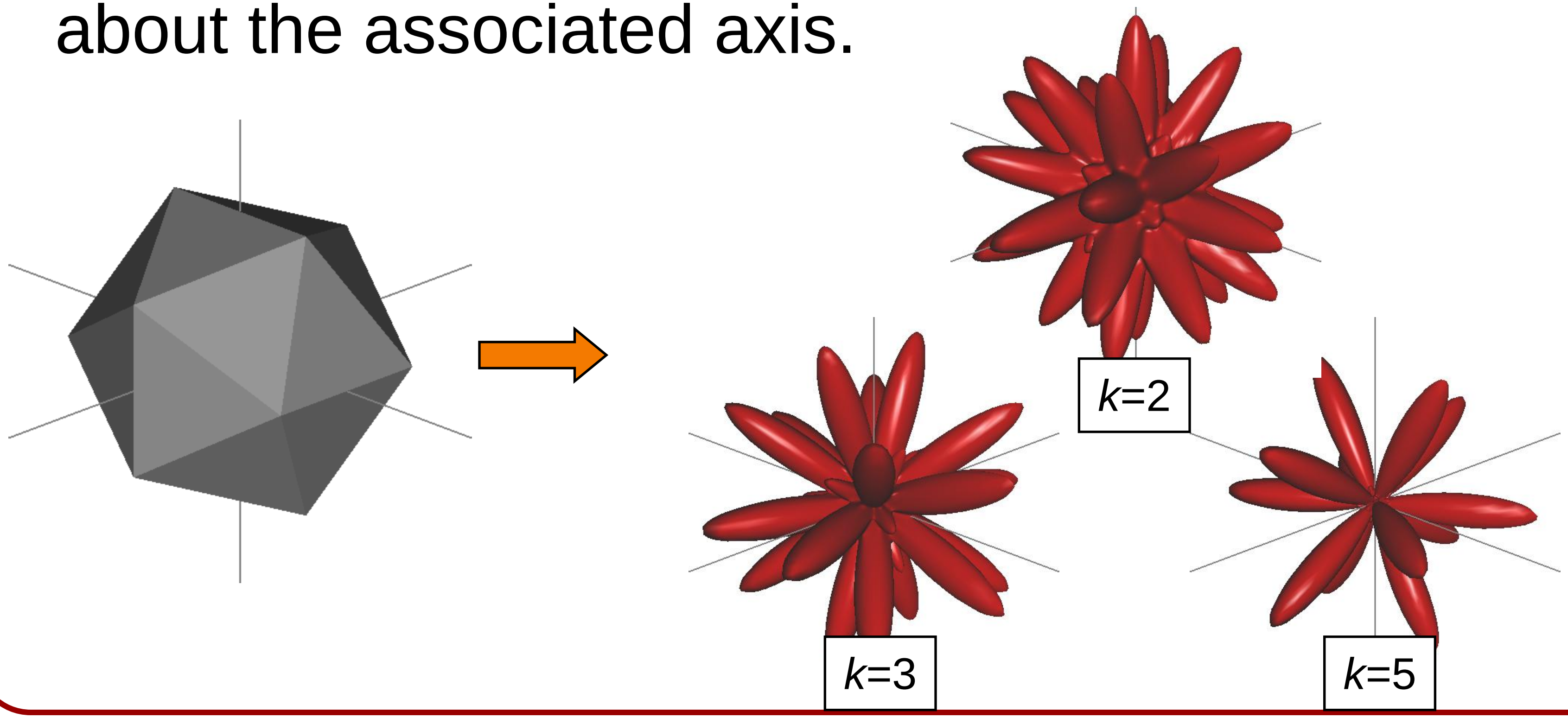
Representation Theory

Symmetry Detection

- Rotational Symmetry
- Reflective Symmetry

Rotational Symmetry

Given a function on the sphere, and given a fixed order of rotational symmetry *k*, we would like to define a function whose value at every point point is the measure of *k*-fold rotational symmetry about the associated axis.



Rotational Symmetry

To do this, we need to associate a group to every axis passing through the origin.

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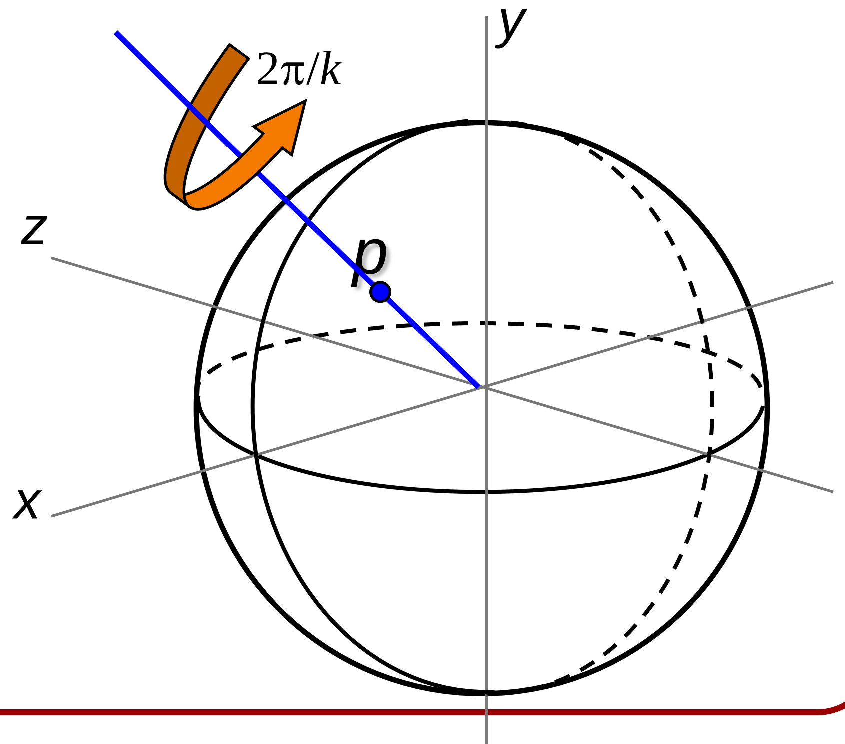
In particular, if we denote by $G_{p,k}$ the group of k-fold rotations about the axis through p:

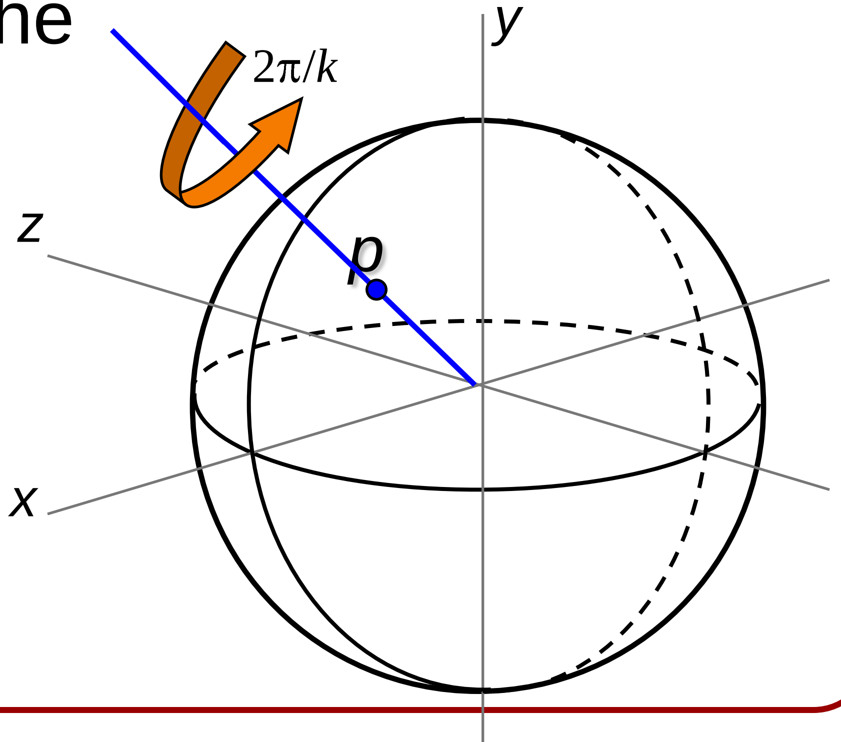


Rotational Symmetry

To do this, we need to associate a group to every axis passing through the origin.

In particular, if we denote by $G_{p,k}$ the group of k-fold rotations about the axis through p, the elements of the group are the rotations:

$$g_j = R\left(p, \frac{2j\pi}{k}\right)$$



corresponding to rotations about p by the angle $2j\pi/k$.

Rotational Symmetry

Using this notation, the equation for the measure of *k*-fold symmetry of a function *f* about the axis *p* becomes:

$$\mathrm{Sym}^2(f, G_{p,k}) = \left\| \frac{1}{k} \sum_{j=0}^{k-1} \rho_{g_j}(f) \right\|^2$$

Rotational Symmetry

Using this notation, the equation for the measure of *k*-fold symmetry of a function *f* about the axis *p* becomes:

$$\mathrm{Sym}^2\left(f, G_{p,k}\right) = \left\| \frac{1}{k} \sum_{j=0}^{k-1} \rho_{g_j}(f) \right\|^2$$

Expanding this expression in terms of dot-products, we get:

$$\mathrm{Sym}^2\left(f, G_{p,k}\right) = \frac{1}{k^2} \left\langle \sum_{i=0}^{k-1} \rho_{g_i}(f), \sum_{j=0}^{k-1} \rho_{g_j}(f) \right\rangle$$

Rotational Symmetry

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$$\mathrm{Sym}^2(f, G_{p,k}) = \frac{1}{k^2}\left\langle \sum_{i=0}^{k-1}\rho_{g_i}(f), \sum_{j=0}^{k-1}\rho_{g_j}(f)\right\rangle$$

Using the linearity of the inner-product, we get:

$$\mathrm{Sym}^2(f, G_{p,k}) = \frac{1}{k^2}\sum_{i,j=0}^{k-1}\left\langle \rho_{g_i}(f), \rho_{g_j}(f)\right\rangle$$

Rotational Symmetry

Using this notation, the equation for the measure of *k*-fold symmetry of a function *f* about the axis *p* becomes:

$$\mathrm{Sym}^2\left(f, G_{p,k}\right) = \frac{1}{k^2}\sum_{i,j=0}^{k-1}\left\langle \rho_{g_i}(f), \rho_{g_j}(f)\right\rangle$$

And using the fact that the representation is unitary we get:

$$\mathrm{Sym}^2\left(f, G_{p,k}\right) = \frac{1}{k^2}\sum_{i,j=0}^{k-1}\left\langle f, \rho_{g_{j-i}}(f)\right\rangle$$

Rotational Symmetry

We can further simplify this expression by observing that the rotation g_{j-i} only depends on the difference between the index *j* and *i*.

$$\mathrm{Sym}^2(f, G_{p,k}) = \frac{1}{k^2} \sum_{i,j=0}^{k-1} \left\langle f, \rho_{g_{j-i}}(f) \right\rangle$$

Rotational Symmetry

We can further simplify this expression by observing that the rotation g_{j-i} only depends on the difference between the index *j* and *i*.

In particular, using the fact that every index in the range $[0,k)$ can be expressed in exactly *k* different ways as the difference *j-i* with $j,i \in [0,k)$, we get :

$$\mathrm{Sym}^2(f, G_{p,k}) = \frac{1}{k^2} \sum_{i,j=0}^{k-1} \left\langle f, \rho_{g_{j-i}}(f) \right\rangle$$

$$= \frac{1}{k} \sum_{j=0}^{k-1} \left\langle f, \rho_{g_j}(f) \right\rangle$$

Rotational Symmetry

Thus, the measure of *k*-fold rotational symmetry about the axis *p* can be computed by taking the average of the dot-products of the function *f* with its *k* rotations about the axis *p.*

$$\mathrm{Sym}^2\left(f, G_{p,k}\right) = \frac{1}{k}\sum_{j=0}^{k-1}\left\langle f, \rho_{g_j}(f)\right\rangle$$

Rotational Symmetry

$$\mathrm{Sym}^2\left(f, G_{p,k}\right) = \frac{1}{k}\sum_{j=0}^{k-1}\left\langle f, \rho_{g_j}(f)\right\rangle$$

So computing the measures of rotational symmetry reduces to the problem of computing the correlation of *f* with itself:

$$\mathrm{Dot}_{f,f}(R) = \left\langle f, \rho_R f\right\rangle$$

Rotational Symmetry

$$\mathrm{Sym}^2\left(f, G_{p,k}\right) = \frac{1}{k}\sum_{j=0}^{k-1}\left\langle f, \rho_{g_j}(f)\right\rangle$$

So computing the measures of rotational symmetry reduces to the problem of computing the correlation of *f* with itself:

$$\mathrm{Dot}_{f,f}(R) = \left\langle f, \rho_R f\right\rangle$$

And this is something that we can do using the Wigner D-transform from last lecture.

Rotational Symmetry

$$\mathrm{Sym}^2(f, G_{p,k}) = \frac{1}{k}\sum_{j=0}^{k-1}\left\langle f, \rho_{g_j}(f)\right\rangle$$

<u>Algorithm</u>:

Given a function *f*:

- Compute the auto-correlation of *f*.
- For each order of symmetry *k*:
  - » Compute the spherical function whose value at *p* is the average of the correlation values at rotations *R*(*p*,2π*j*/*k*), 0≤*j*≤*k*.

Rotational Symmetry

$$\mathrm{Sym}^2\left(f, G_{p,k}\right) = \frac{1}{k}\sum_{j=0}^{k-1}\left\langle f, \rho_{g_j}(f)\right\rangle$$

Complexity:

- Compute the auto-correlation: $O(n^3 \log^2 n)$
- For each order of symmetry k:
  - Compute the spherical function: $O(n^2 k)$

For a total computational complexity of $O(n^4)$ to compute all rotational symmetries.

Rotational Symmetry

$$\text{Sym}^2(f, G_{p,k}) = \frac{1}{k}\sum_{j=0}^{k-1}\left\langle f, \rho_{g_j}(f)\right\rangle$$

<u>Complexity:</u>

- Comput

<u>Note</u>: There is a lot of redundancy in this computation.

- For each order of symmetry *k*:
  - Compute the spherical function: O(n^2k)

For a total computational complexity of O(n^4) to compute all rotational symmetries.

Rotational Symmetry

$$\mathrm{Sym}^2(f, G_{p,k}) = \frac{1}{k}\sum_{j=0}^{k-1}\left\langle f, \rho_{g_j}(f)\right\rangle$$

<u>Complexity:</u>

- Comput

<u>Note</u>: There is a lot of redundancy in this computation.

- Fo

<u>Example</u>: Computing the 2*k*-fold symmetries, we re-compute the *k*-fold symmetry, allowing for a 2-fold improvement in efficiency

For a total computational complexity of $O(n^4)$ to compute all rotational symmetries.

Outline

Representation Theory

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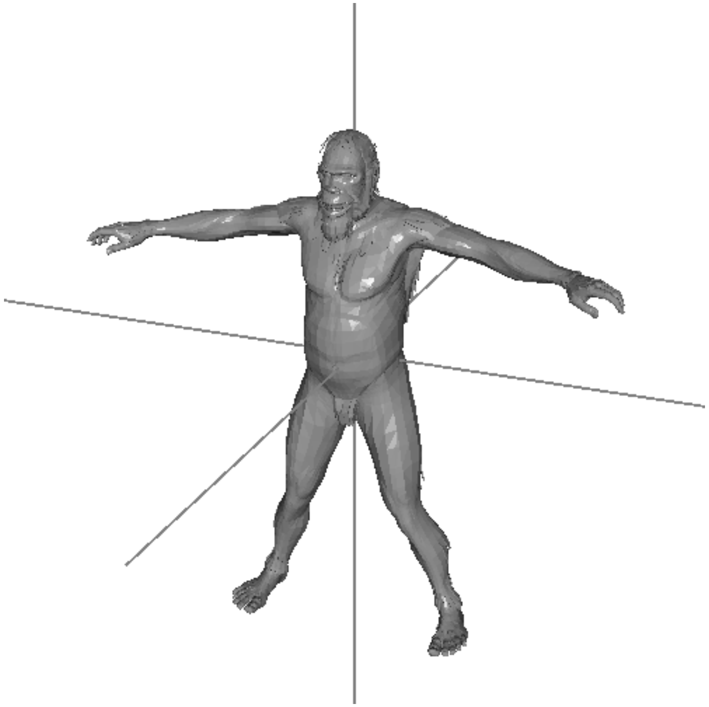
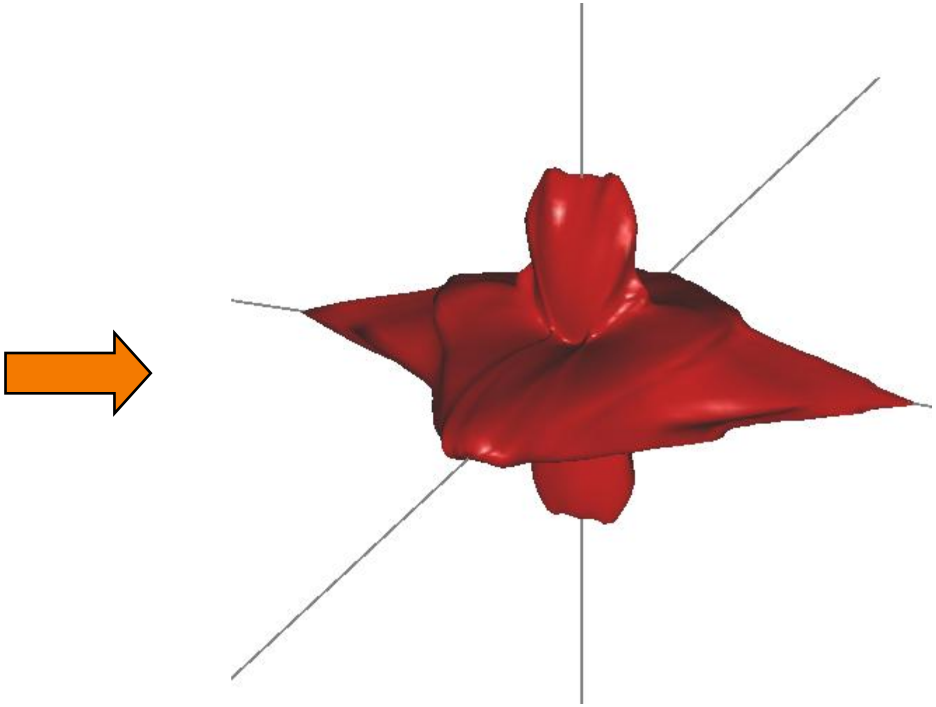
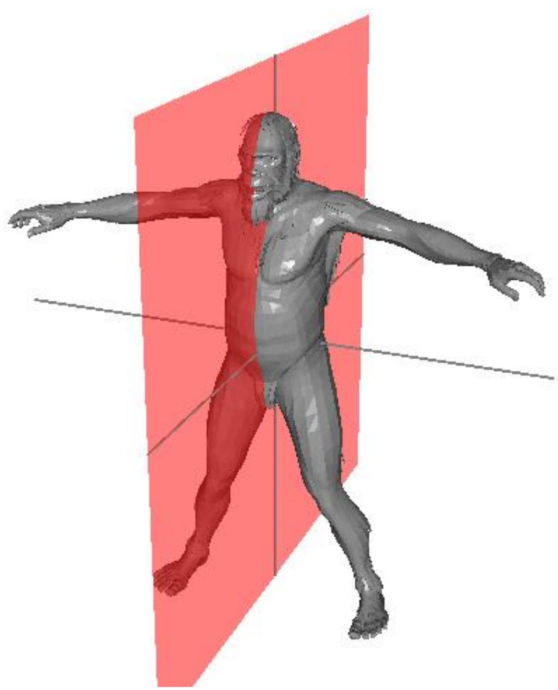
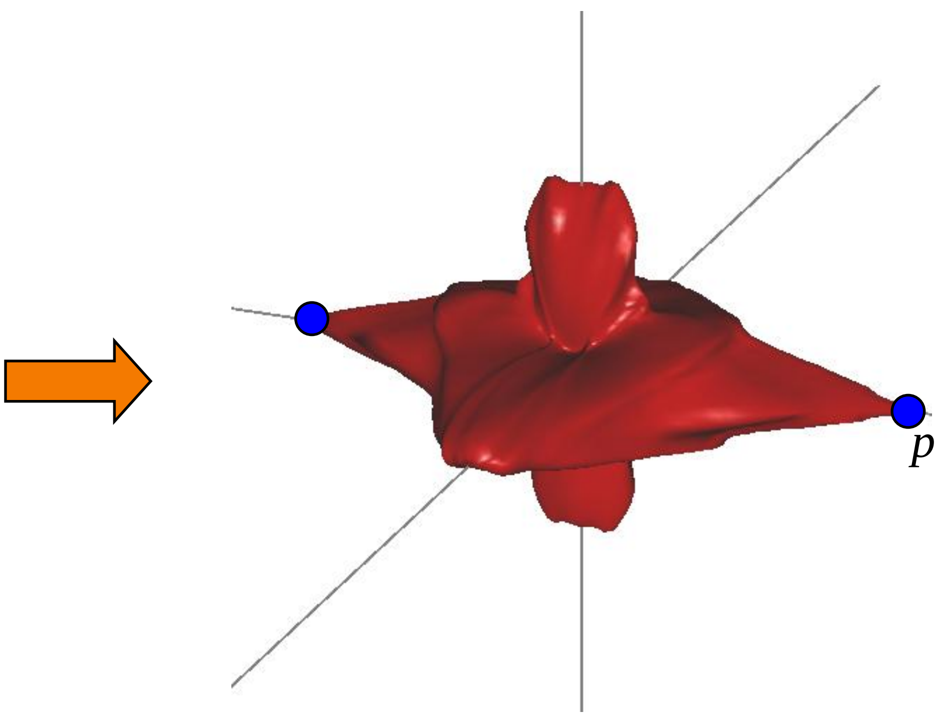




Reflective Symmetry

Given a spherical function *f*, we would like to compute a function whose value at a point *p* is the measure of reflective symmetry with respect to the plane perpendicular to *p*.

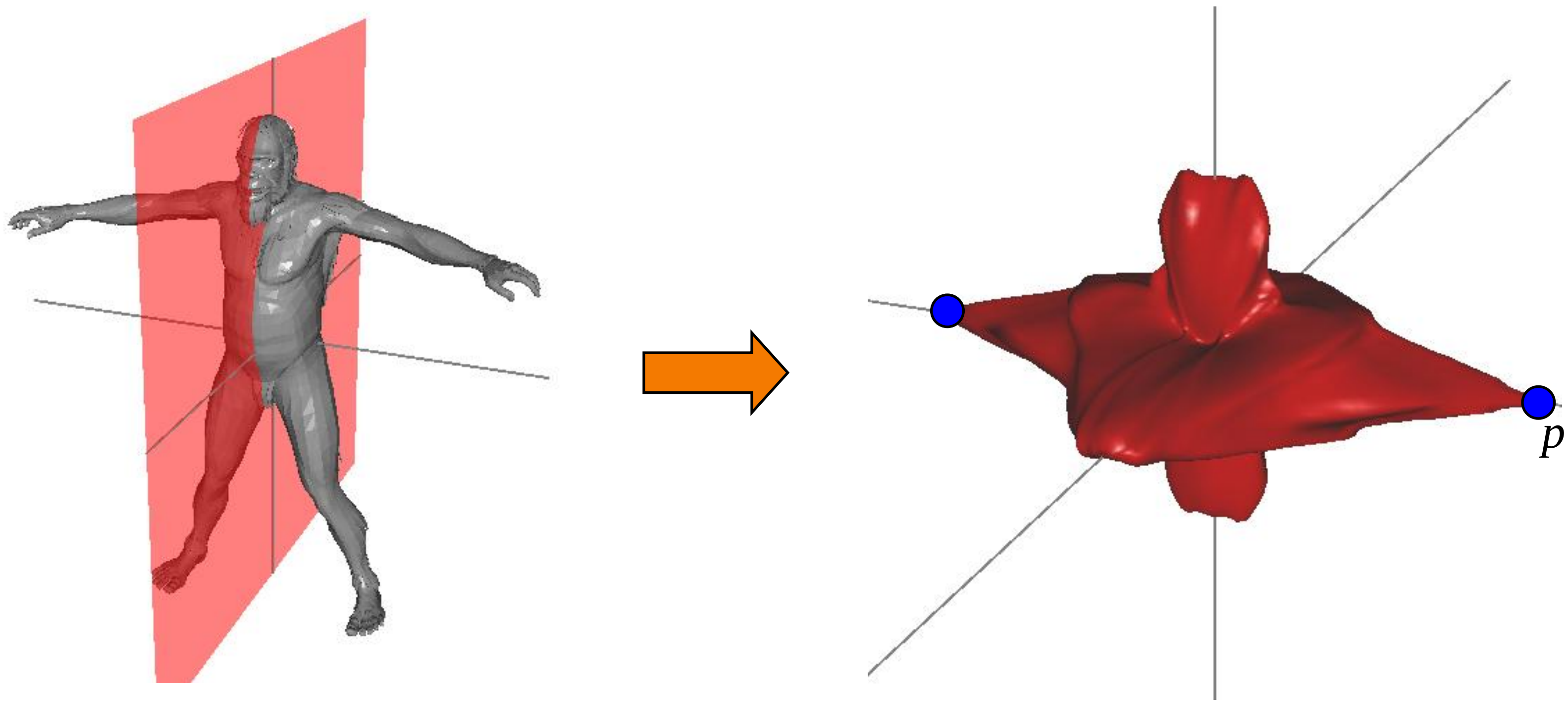




Reflective Symmetry

Reflections through the plane perpendicular to *p* correspond to a group with two elements:

$$G_p = \{\mathrm{Id}, \mathrm{Ref}_p\}$$



Reflective Symmetry

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Reflective Symmetry

How do we compute the dot-product of the function *f* with the reflection of *f* through the plane perpendicular to *p*?

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Since reflections are not rotations (they have determinant -1) we cannot use the values of the autocorrelation.

Reflective Symmetry

General Approach:

If we have two reflections *S* and *T*, we can set *R* to be the transformation:

$$R = TS^{-1}$$

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Since both *S* and *T* have determinant -1, *R* must have determinant 1.

Reflective Symmetry

General Approach:

If we have two reflections *S* and *T*, we can set *R* to be the transformation:

$$R = TS^{-1}$$

Thus, *R* must be a rotation and we have:

$$RS = T$$

so *T* is just the reflection *S* followed by some rotation.

Reflective Symmetry

General Approach:

Thus, if we compute the correlation of *f* with some reflection $\rho_S(f)$:

$$\text{Dot}_{f,\rho_S f}(R) = \left\langle f, \rho_R(\rho_S f) \right\rangle$$

We can obtain the dot product of *f* with its reflection through the plane perpendicular to *p* by evaluating:

$$\left\langle f, \rho_{\text{Ref}_p} f \right\rangle = \text{Dot}_{f,\rho_S f}(\underbrace{\text{Ref}_p \cdot S^{-1}}_{\text{Rotation}})$$

Reflective Symmetry

$$\mathrm{Sym}^2(f, G_p) = \frac{1}{2}\left(\|f\|^2 + \mathrm{Dot}_{f,\rho_S f}(\mathrm{Ref}_p \cdot S^{-1})\right)$$

Algorithm:

Given a function *f*:

- Compute the correlation of *f* with the reflection $\rho_S(f)$
- Compute the spherical function whose value at *p* is the average of the size of *f* and the dot-product of *f* with the rotation of $\rho_S(f)$ by $\mathrm{Ref}_p S^{-1}$.

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Complexity:

- Compute the correlation: O($n^3 \log^2 n$)
- Compute the spherical function: O(n^2)

For a total computational complexity of O($n^3 \log^2 n$) to compute all reflective symmetries.

Reflective Symmetry

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Complexity:

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For computing reflective symmetries, the computation of the correlation is overkill as we don't use most of the correlation values.

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Reflective Symmetry

There are many different choices for the reflection S we use to compute:

$$\text{Dot}_{f,\rho_S f}(T)$$

Reflective Symmetry

There are many different choices for the reflection *S* we use to compute:

$$\mathrm{Dot}_{f,\rho_S f}(T)$$

The simplest reflection we can use is the antipodal map:

$$S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Reflective Symmetry

The advantage of using the antipodal map is that it makes it easy to express $\text{Ref}_p S^{-1}$.

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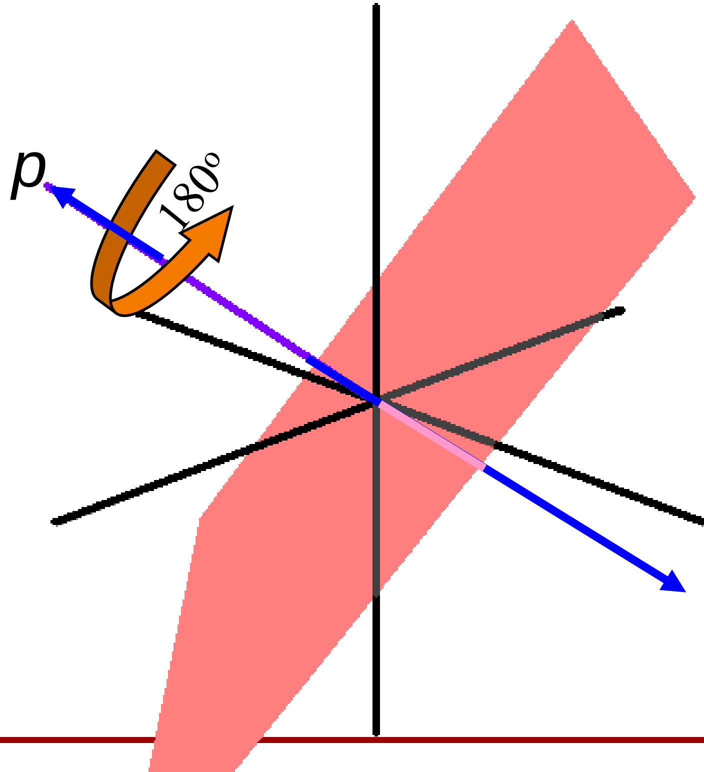
In particular, fixing a point *p*, we can think of *S* as the combination of two maps:

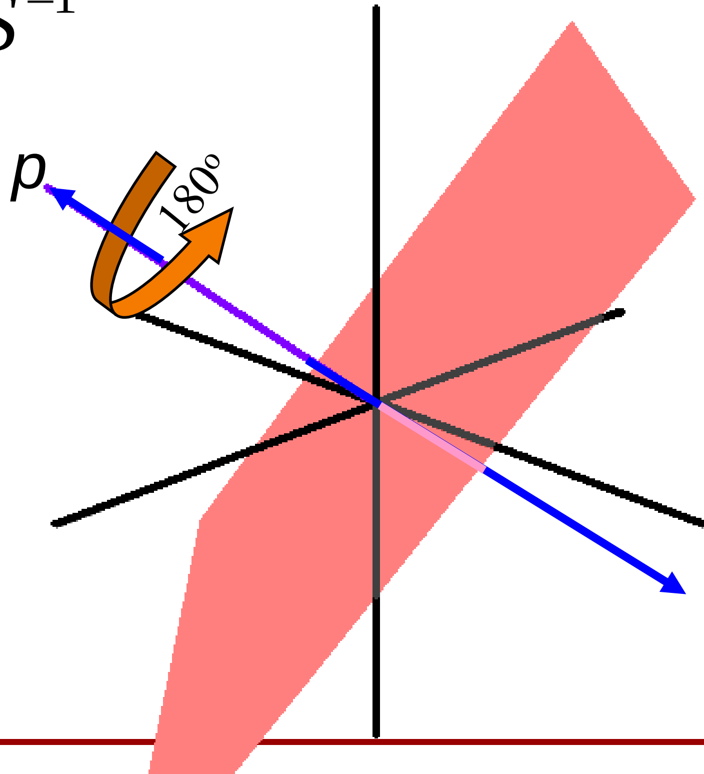
- A reflection through the plane perpendicular to *p*, and
- A rotation by 180° about the axis through *p*.



Reflective Symmetry

Thus, a reflection through the plane perpendicular to *p* can be expressed as the product of the antipodal map and a rotation by 180º around the axis through *p*:

$$\mathrm{Ref}_p = R(p,\pi)S^{-1}$$



Reflective Symmetry

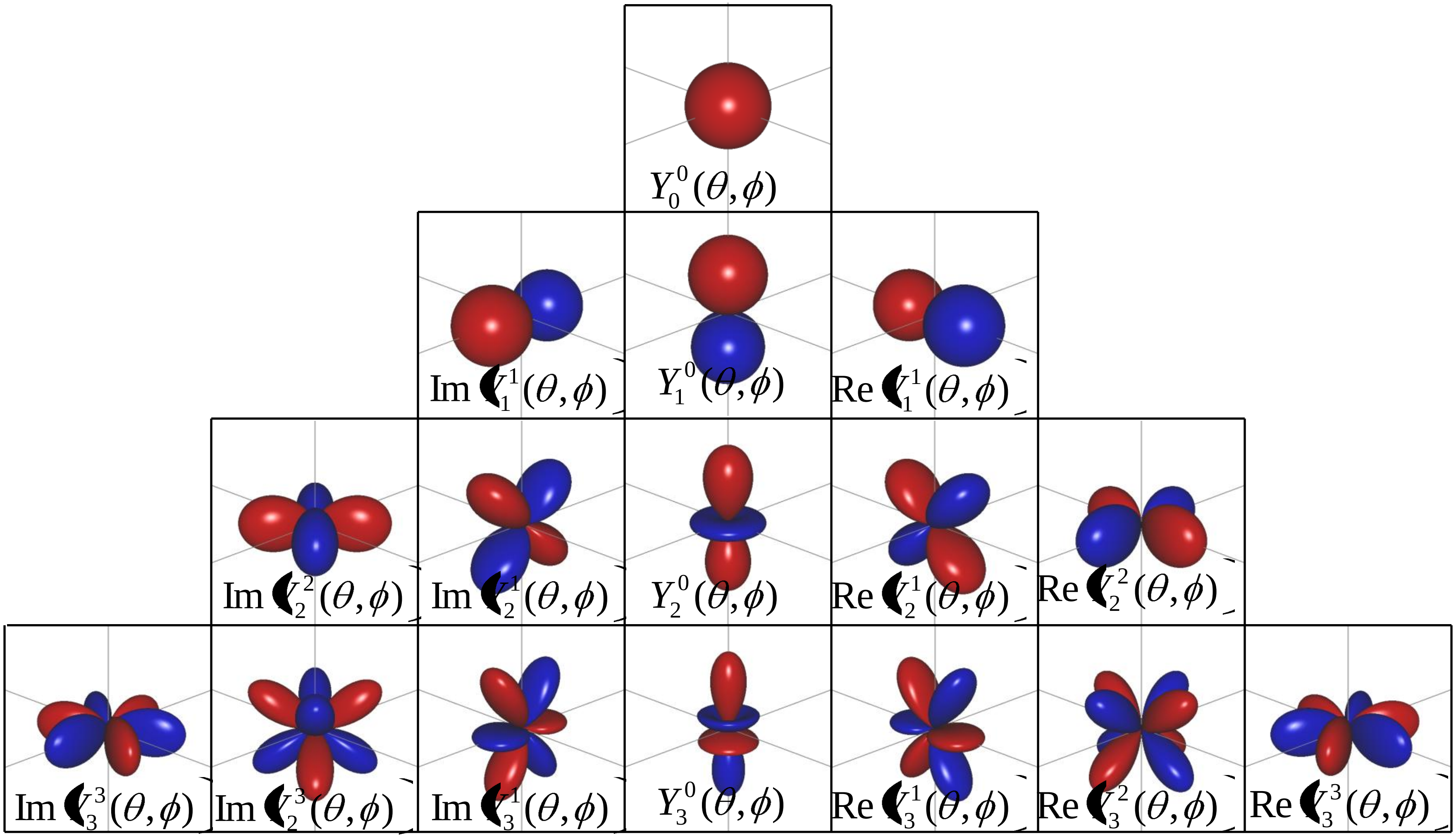
Thus, setting *S* to be the antipodal map, we get:

$$\mathrm{Sym}^2\left(f, G_p\right) = \frac{1}{2}\left(\|f\|^2 + \left\langle f, \rho_{R(p,\pi)}(\rho_S f)\right\rangle\right)$$

Reflective Symmetry

Using the spherical harmonics, we get a simple expression for $\rho_S(f)$:

Reflective Symmetry



Reflective Symmetry

Using the spherical harmonics, we get a simple expression for $\rho_S(f)$:

$$\rho_S f = \sum_l (-1)^l \sum_{m=-l}^{l} \hat{f}(l,m) Y_l^m$$

Reflective Symmetry

Additionally, if the function *f* is antipodally symmetric:

$$\rho_S f = f$$

the equation for the reflective symmetries becomes:

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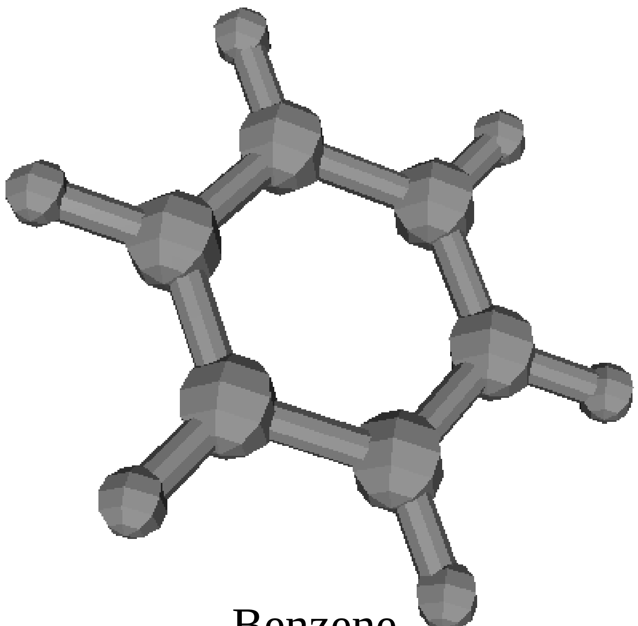
Reflective Symmetry

That is, in the case that *f* is antipodally symmetric, the reflective symmetries of *f* and the 2-fold rotational symmetries of *f* are equal.



Benzene