FFTs in Graphics and Vision

Characters of Irreducible Representations

Outline

- Uniqueness of Eigenvalue Decomposition
- Simultaneous Diagonalizability
- Characters of (Irreducible) Representations

Uniqueness of E. Decomposition

Consider the decomposition of a vector space into 1D eigenspaces with respect to a self-adjoint operator *A*:

$$V = \bigoplus_i V_i$$

with V_i=Span{v_i} and $A(v_i)=\lambda_i v_i$.

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with V_i=Span{v_i} and $A(v_i)=\lambda_i v_i$.

This decomposition is unique only as long as the eigenvalues are all different.

Uniqueness of E. Decomposition

If $\lambda_i=\lambda_j$ for some $i\neq j$, then for any θ setting:

$$w_i = \cos(\theta)v_i + \sin(\theta)v_j$$

$$w_j = -\sin(\theta)v_i + \cos(\theta)v_j$$

we get a different decomposition of *V* into 1D eigenspaces.

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Simultaneous Diagonalizability

If we have a finite-dimensional vector space V and a self-adjoint operator $A:V\rightarrow V$, we know that there exists an orthonormal basis $\{v_1,\ldots,v_n\}$ with:

$$Av_i = \lambda_i v_i$$

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What if we have two self-adjoint operators?

Is there an orthonormal basis $\{v_1,\ldots,v_n\}$ with:

$$Av_i = \lambda_i v_i \quad \text{and} \quad Bv_i = \mu_i v_i$$

Simultaneous Diagonalizability

Schur's Lemma:

If A and B commute, (i.e. $AB=BA$) then the operators are simultaneously diagonalizable.

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Let λ be an eigenvalue of A, and consider the operator $(A- \lambda \mathrm{Id})$.

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Proof:

Let λ be an eigenvalue of A, and consider the operator (A- λId).

This operator has a (non-trivial) kernel which is the space of all vectors in V which are eigenvectors of A with eigenvalue λ:

$$\text{Kernel}(A - \lambda \text{Id}) = \{ v \in V \mid Av = \lambda v \}$$

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Proof:

Since the operators commute, we know that if v is in the kernel of $(A- \lambda \mathrm{Id})$ then:

$$\left(A-\lambda \mathrm{Id}\right)Bv=0$$

So Bv must also be in the kernel.

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Schur's Lemma:

If A and B commute, (i.e. $AB=BA$) then the operators are simultaneously diagonalizable.

Proof:

If we denote by V_λ the space of eigenvectors of A with eigenvalue λ, then we must have:

$$B(V_\lambda) = V_\lambda$$

Simultaneous Diagonalizability

<u>Schur's Lemma</u>:

If A and B commute, (i.e. $AB=BA$) then the operators are simultaneously diagonalizable.

<u>Proof</u>:

Since B is self-adjoint and maps V_λ to itself, we know that there exists an orthonormal basis $\{v_{1,\lambda},\ldots,v_{n_\lambda,\lambda}\}$ for V_λ such that:

$$B(v_{i,\lambda}) = \mu_{i,\lambda} v_{i,\lambda}$$

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Since the V_λ are orthogonal spaces, the vectors:

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are an orthogonal basis for V.

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are an orthogonal basis for *V*.

But these vectors also satisfy:

$$Av_{i,\lambda} = \lambda v_{i,\lambda} \quad \text{and} \quad Bv_{i,\lambda} = \mu_{i,\lambda} v_{i,\lambda}$$

Uniqueness of Diagonalizability

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No! Consider *A* is some self-adjoint operator and *B* is the identity.

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No! Consider *A* is some self-adjoint operator and *B* is the identity.

The problem is that the eigenvector λ may have non-trivial multiplicity – i.e. the space V_λ may be more than one dimensional.

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Uniqueness of Diagonalizability

<u>Question</u>:

Does this imply that if v is an eigenvector of A, then it is also an eigenvector of B?

<u>Answer</u>:

We can think of V_λ as the sub-space of V consisting of (1D) spaces on which the operator A acts identically.

We can distinguish V_λ from $V_{\lambda'}$, because the A acts on these sub-spaces differently, but we can't distinguish between the 1D components of V_λ.

Outline

- Uniqueness of Eigenvalue Decomposition
- Simultaneous Diagonalizability
- Characters of (Irreducible) Representations

Characters of a Representations

For irreducible representations, there is a concept analogous to “eigenvalue” called the *character*.

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If we choose a basis $\{v_1,\ldots,v_n\}$ for V, we can express $\rho(g)$ as a matrix w.r.t this basis:

$$\rho(g) = \begin{pmatrix} \lambda_{11}(g) & \cdots & \lambda_{n1}(g) \\ \vdots & \ddots & \vdots \\ \lambda_{1n}(g) & \cdots & \lambda_{nn}(g) \end{pmatrix}$$

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The elements of the matrix depend on the basis. The trace does not.

Characters of a Representation

This allows us to associate a (complex valued) function, $\chi_\rho: G \rightarrow \mathbf{C}$ with every representation ρ:

$$\chi_\rho(g) = \text{Trace}\begin{pmatrix} \lambda_{11}(g) & \cdots & \lambda_{n1}(g) \\ \vdots & \ddots & \vdots \\ \lambda_{1n}(g) & \cdots & \lambda_{nn}(g) \end{pmatrix} = \sum_{i=1}^{n} \lambda_{ii}(g)$$

This is the <u>character</u> of the representation.

Characters of a Representation

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Otherwise, their characters are orthogonal.

Similarly, the decomposition into irreducible representations is only unique up to:

$$V = \bigoplus_{\chi} W_{\chi}$$

where W_{χ} is the sum of all the irreducible representations with the same character χ.

Irreducible Representations

Example:

We know that if G is the group of 2D rotations, $G=SO(2)$, and V is the space of functions on a circle, we have:

$$V = \bigoplus_{k=-\infty}^{\infty} V_k$$

where V_k is the 1D space of functions spanned by the complex exponentials:

$$V_k = \text{Span}\left\{ e^{ik\theta} \right\}$$

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Denoting by g_θ the rotation by θ degrees, we get:

$$\rho_k(g_\theta) = (e^{-ik\theta})$$

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So the character of this representation is:

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Thus, each irreducible representation only occurs once and the decomposition into irreducible representations is unique.

Irreducible Representations

Example:

Similarly, the decomposition of the space of spherical functions into irreducible representations spanned by the spherical harmonics is also unique.

Relationship to Symmetric Operators

Suppose we have a representation ρ of a group G on a vector space V, the decomposition of V into irreducible representations is unique, and we have a self-adjoint operator A commuting with the action of G:

$$\rho(g)A = A\rho(g)$$

Relationship to Symmetric Operators

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Since A is self-adjoint, we can decompose V into eigenspaces:

$$V = \bigoplus_{\lambda} V_{\lambda}$$

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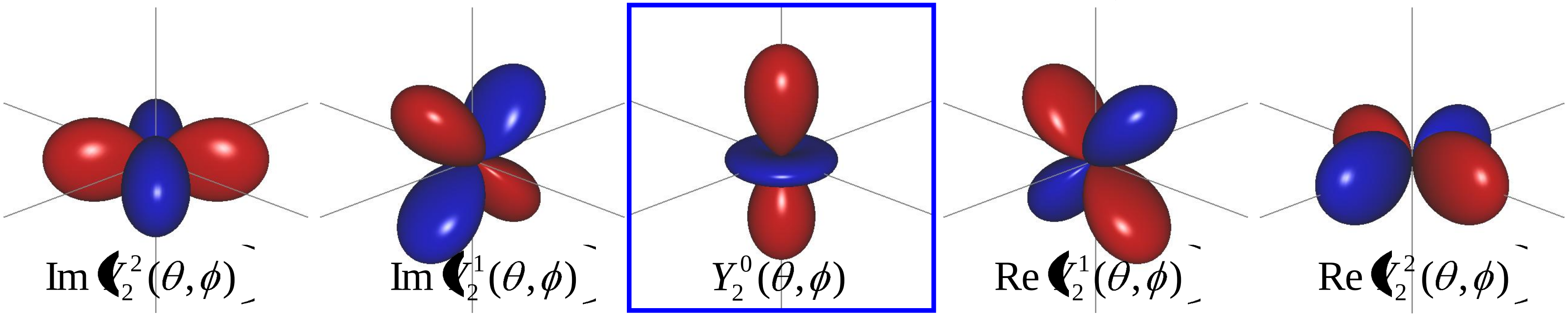
Since this gives a decomposition of V into irreducible representations, and we assumed the decomposition was unique, this implies that the irreducible representations are eigenspaces of A.

Relationship to Symmetric Operators

In particular, if we define the operator C_l to be the spherical convolution with the axially symmetric l-th frequency harmonic:

$$\left(C_l(f)\right)(\theta,\phi) = \left\langle f, \rho_{R(\theta,\phi)} Y_l^0 \right\rangle$$



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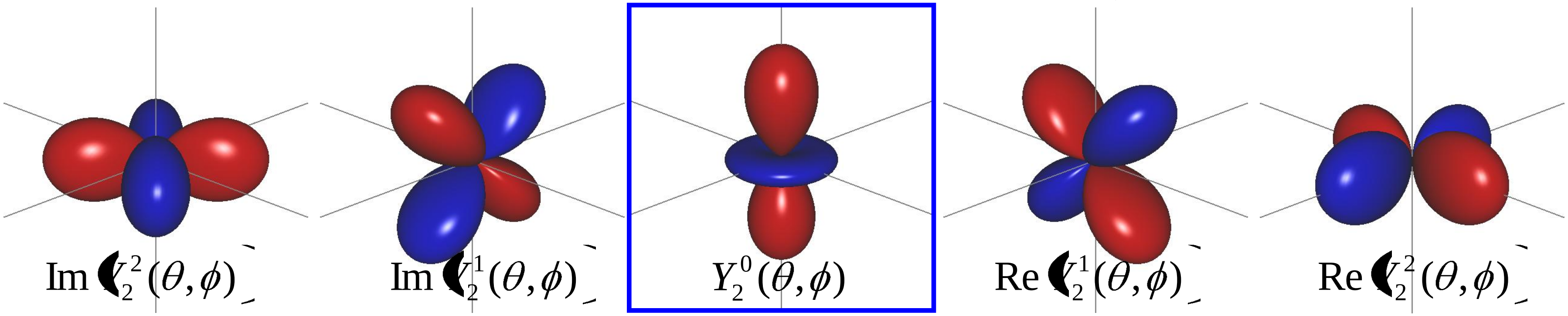

then the symmetry of C_l, combined with the fact that it commutes with rotation, implies that:

$$\big(C_l(Y_l^m)\big)(\theta,\phi) = \lambda_l Y_l^m(\theta,\phi)$$