FFTs in Graphics and Vision

Representing Rotations

Outline

- Math Review
  - Polynomials
  - Eigenvectors
  - Orthogonal Transformations
  - Classifying the 2D Orthogonal Transformations
- Representing 3D Rotations

Math Review

Polynomials:

Let $P(x)$ be a polynomial of degree d:

$$P(x) = a_0 + a_1 x + \cdots + a_d x^d$$

Math Review

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Let *P*(*x*) be a polynomial of degree *d*:

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Claim:

If *d* is odd, the polynomial *P*(*x*) must have at least one real root.

Math Review

<u>Polynomials</u>:

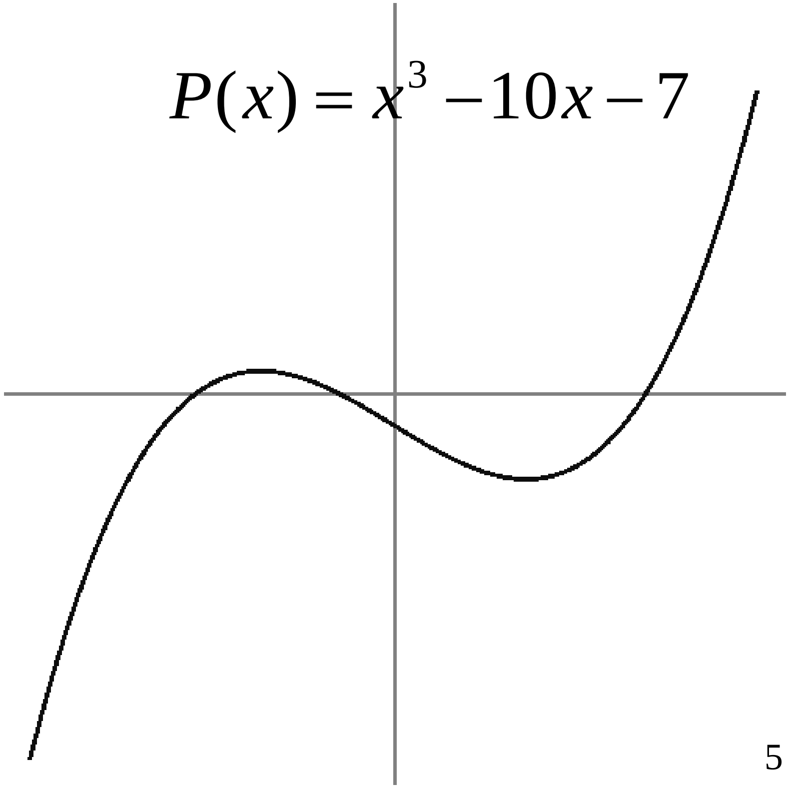
Let $P(x)$ be a polynomial of degree d:

$$P(x) = a_0 + a_1 x + \cdots + a_d x^d$$

<u>Proof</u>:

Consider the sign of a_d:

- If a_d is positive:
  - As $x \to -\infty$: $P(x) \to -\infty$
  - As $x \to \infty$: $P(x) \to \infty$



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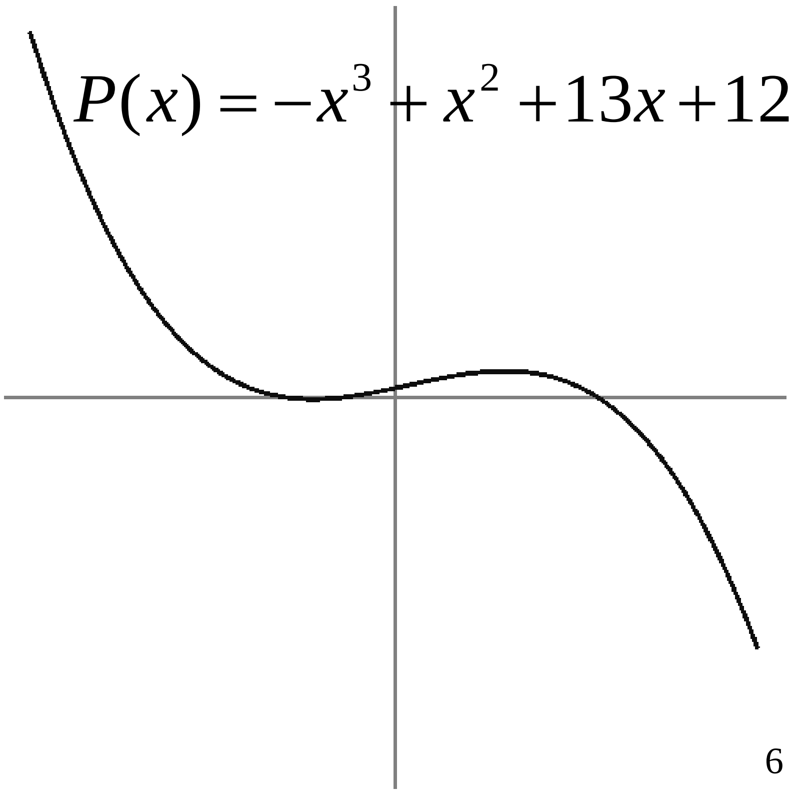
  - » As $x \to \infty$: $P(x) \to \infty$
- If a_d is negative:
  - » As $x \to -\infty$: $P(x) \to \infty$
  - » As $x \to \infty$: $P(x) \to -\infty$



Math Review

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Let *P*(*x*) be a polynomial of degree *d*:

$$P(x) = a_0 + a_1 x + \cdots + a_d x^d$$

Proof:

In either case, the value of *P*(*x*) changes signs so it must have a zero-crossing somewhere.

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Eigenvectors:

Given a vector space V and an invertible linear operator A, if v is an eigenvector of A with eigenvalue λ then v is also an eigenvector of A^{-1} with eigenvalue $1/\lambda$.

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$$\Downarrow$$

$$\frac{1}{\lambda} v = A^{-1} v$$

Math Review

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For a real inner-product space, an orthogonal transformation is a linear transformation preserving the inner product.

For a vector space *V*, *R* is an orthogonal transformation if for any $v,w\in V$, we have:

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If the determinant of *R* is 1, the transformation is called a rotation.

Math Review

Orthogonal Transformations (Property 1):

The set of orthogonal transformations is a group.

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To show this we need to show that if *R* and *S* are orthogonal transformations than:

- RS is orthogonal
- R^{-1} is orthogonal

Math Review

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If *R* and *S* are orthogonal transformations, than so is the transformation *RS.*

Using the fact that *R* is orthogonal, we get:

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Thus, we get:

$$\langle RSv, RSw \rangle = \langle v, w \rangle$$

showing that *RS* also preserves the inner product.

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Starting with the identity:

$$\langle v, w \rangle = \langle RR^{-1}v, RR^{-1}w \rangle$$

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Thus, as desired, we get:

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If R is an orthogonal transformation and v_1 and v_2 are eigenvectors of R with eigenvalues λ_1 and λ_2, then if $\lambda_1 \neq \lambda_2$, v_1 and v_2 must be perpendicular.

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Since $\lambda_1 \neq \lambda_2$, this must imply that:

$$\langle v_1, v_2 \rangle = 0$$

Math Review

Classifying the 2D Orthogonal Transformations:

Let *V* be the space of 2D arrays with the standard inner product:

$$\left\langle (x_1, y_1), (x_2, y_2) \right\rangle = x_1 x_2 + y_1 y_2$$

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$$\begin{pmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \end{pmatrix} = \begin{pmatrix} \boxed{a^2+c^2} & ab+cd \\ ab+cd & \boxed{b^2+d^2} \end{pmatrix}$$

The diagonal entries give rise to the equations:

$$1 = a^2 + c^2$$

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For appropriate θ and φ, this gives:

$$a = \cos\theta \qquad c = \sin\theta$$

$$b = \cos\phi \qquad d = \sin\phi$$

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The other equation(s) then becomes:

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Which implies that:

$$\phi = \theta + k\pi + \pi/2$$

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If *R* is an orthogonal transformation, then in matrix form we have one of two cases:

- *k* is even:

$$R \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

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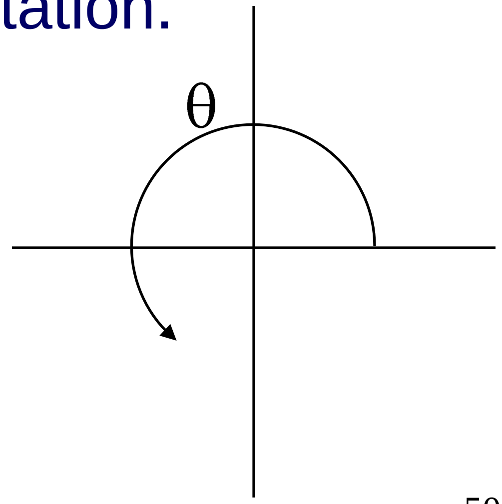
The determinant is 1, and this is a rotation.



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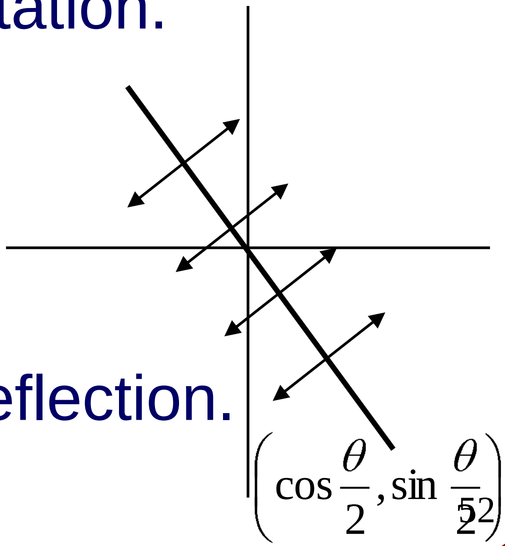
- *k* is odd:

$$R \rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

  The determinant is -1, and this is a reflection.

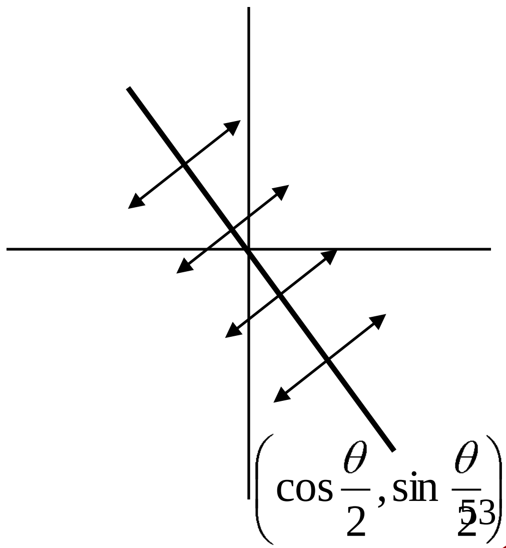


Math Review

Classifying the 2D Orthogonal Transformations:

Claim:

In the case that *k* is odd, the orthogonal transformation has eigenvalues 1 and -1.



Math Review

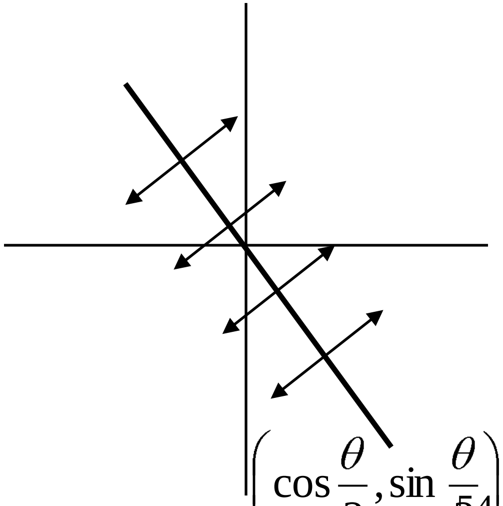
<u>Classifying the 2D Orthogonal Transformations</u>:

<u>Claim</u>:

In the case that *k* is odd, the orthogonal transformation has eigenvalues 1 and -1.

To compute the eigenvalues, we need to solve for the roots of the polynomial:

$$P_R(\lambda) = \det(R - \lambda \mathrm{Id})$$



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In the case that *k* is odd, the orthogonal transformation has eigenvalues 1 and -1.

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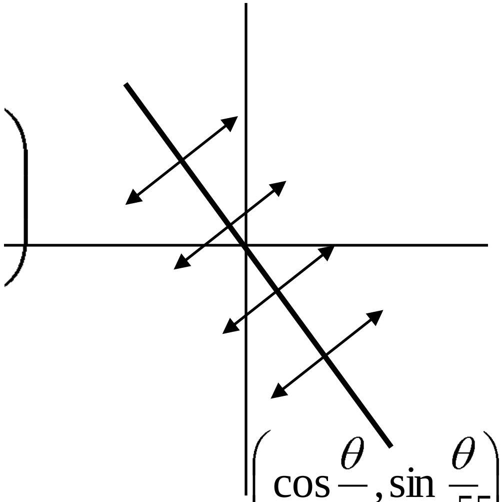
$$P_R(\lambda) = \det\begin{pmatrix} (\cos\theta - \lambda) & \sin\theta \\ \sin\theta & (-\cos\theta - \lambda) \end{pmatrix}$$



$\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)$

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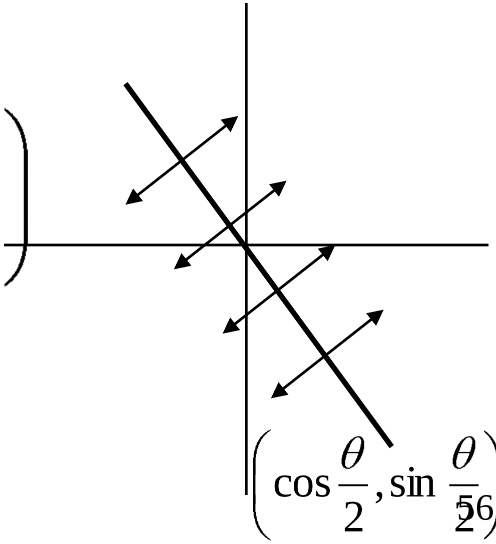

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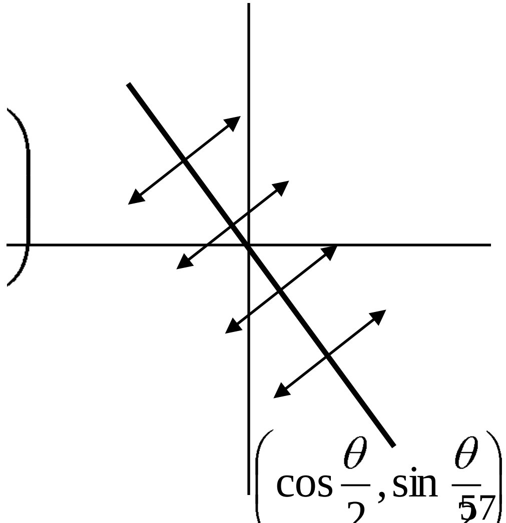
$$= \lambda^2 - 1$$



Math Review

Classifying the 2D Orthogonal Transformations:

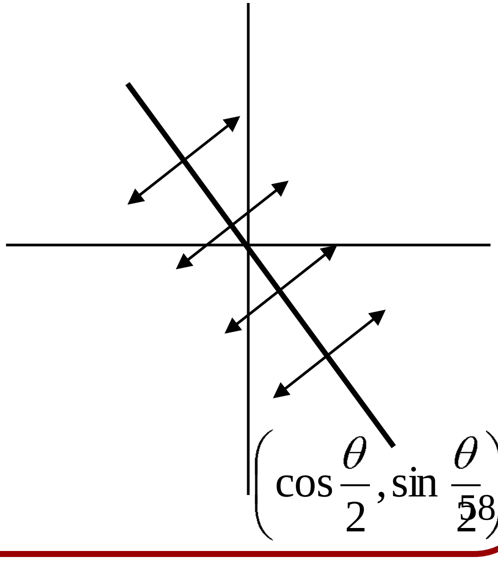
Claim:

In the case that *k* is odd, the orthogonal transformation has eigenvalues 1 and -1.

To compute the eigenvalues, we need to solve for the roots of the polynomial:

$$P_R(\lambda) = \lambda^2 - 1$$

This polynomial has two roots, $\lambda = \pm 1$.



Outline

- Math Review
- Representing 3D Rotations
  - Quaternions
  - Euler Angles

Representing 3D Rotations (Quaternions)

We will show that any rotation R can be thought of as a rotation about some axis.

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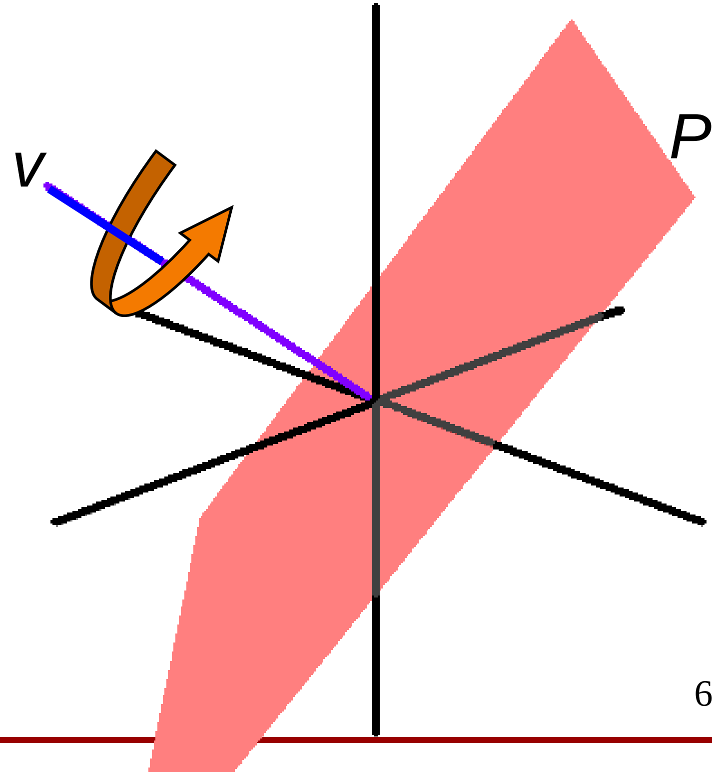
In particular, we need to show that every rotation R fixes some vector v and acts as a rotation in the plane P perpendicular to v.



Representing 3D Rotations (Quaternions)

Let *V* be the space of 3D arrays with the standard inner product:

$$\langle (x_1, y_1, z_1), (x_2, y_2, z_2) \rangle = x_1 x_2 + y_1 y_2 + z_1 z_2$$

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We can compute the eigenvalues of *R* by finding the roots of the determinant:

$$P_R(\lambda) = \det(R - \lambda \mathrm{Id}) = \det \begin{pmatrix} (a-\lambda) & b & c \\ d & (e-\lambda) & f \\ g & h & (i-\lambda) \end{pmatrix}$$

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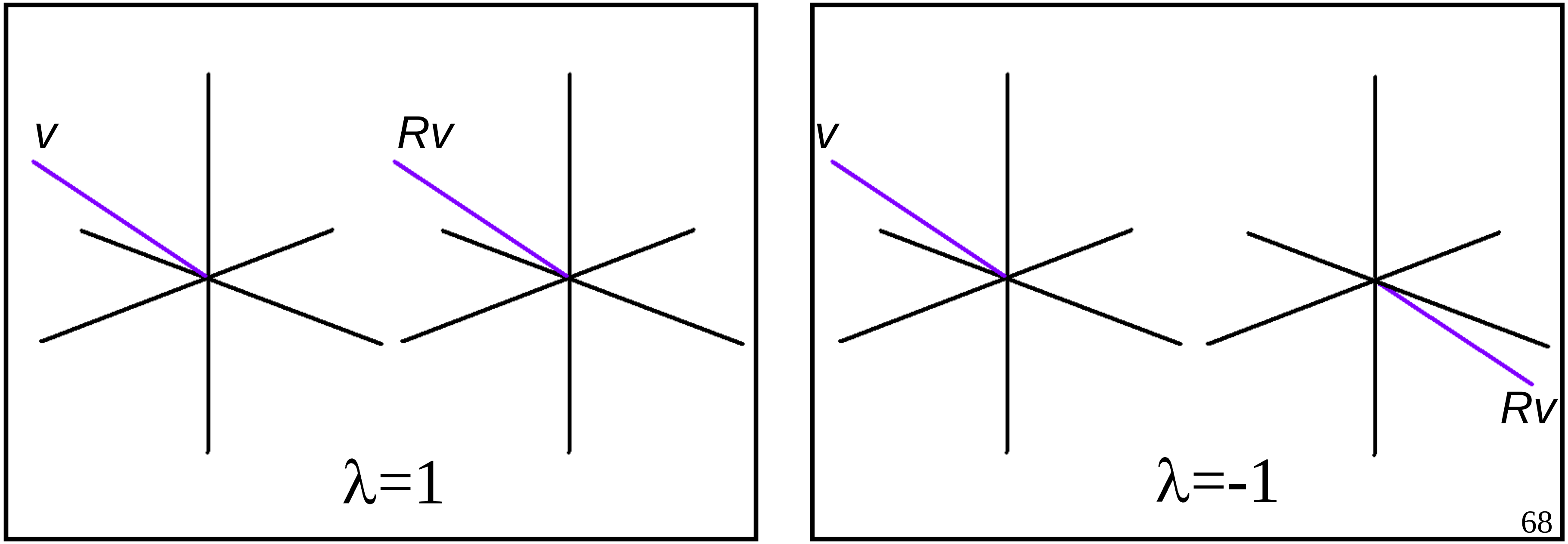
Thus, R must have an eigenvector v with real eigenvalue λ.

When R is orthogonal, we know that $\lambda=\pm 1$.

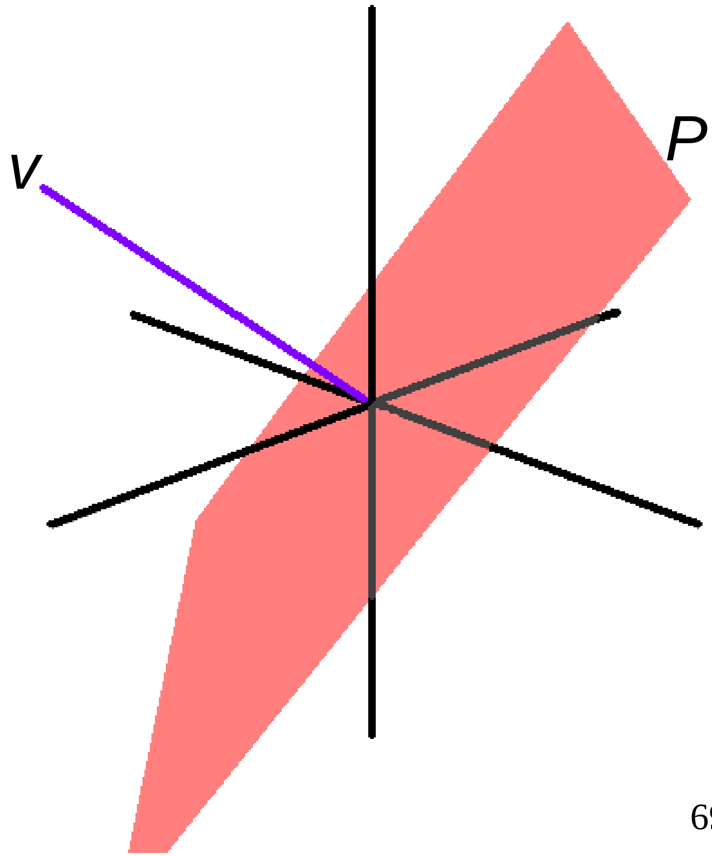
Representing 3D Rotations (Quaternions)

Thus, for every orthogonal transformation *R*, there must exist a vector *v* that is either fixed by *R* or mapped to its antipode.



Representing 3D Rotations (Quaternions)

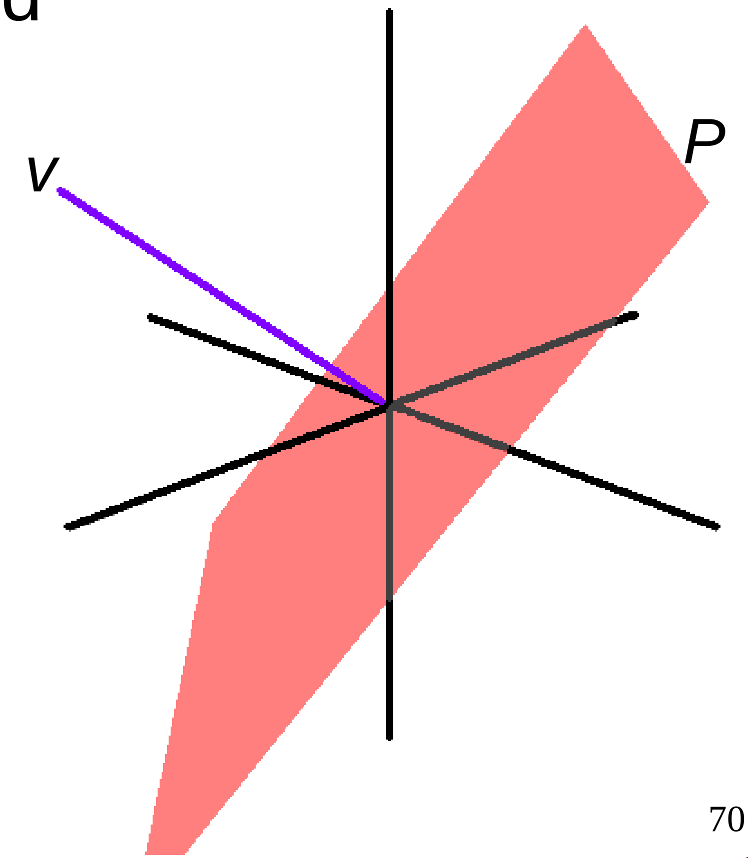
What happens to the plane *P* that is orthogonal to the eigenvector *v*?



Representing 3D Rotations (Quaternions)

What happens to the plane P that is orthogonal to the eigenvector v?

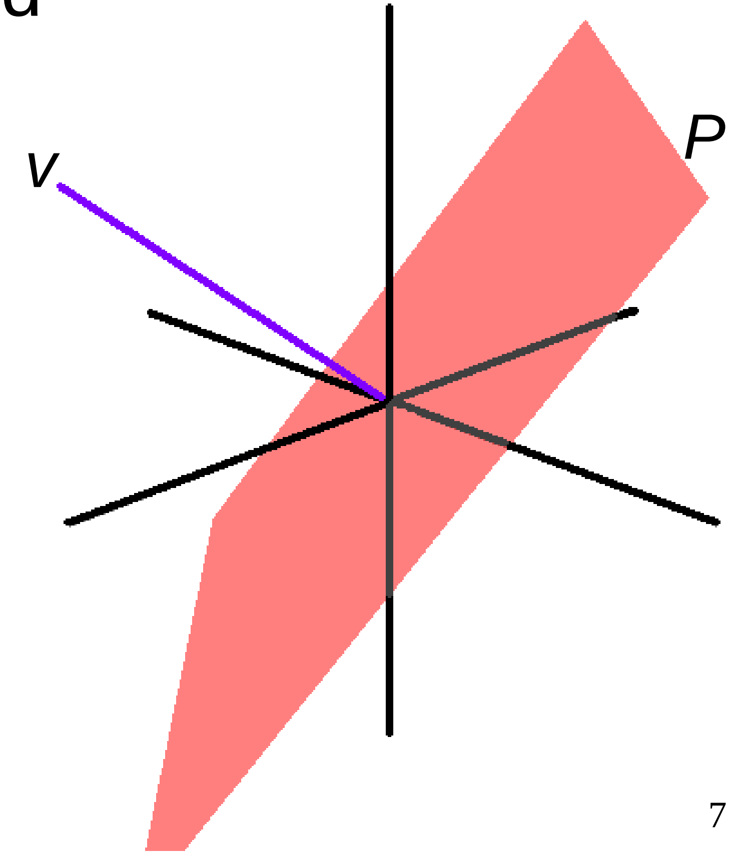
Since R maps the line spanned by v back into itself, and since R is orthogonal, R must map the plane P back into itself.



Representing 3D Rotations (Quaternions)

What happens to the plane *P* that is orthogonal to the eigenvector *v*?

Since *R* maps the line spanned by *v* back into itself, and since *R* is orthogonal, *R* must map the plane *P* back into itself.



Since *R* must preserve the inner product within *P*, the restriction of *R* to *P* is a 2D orthogonal operator.

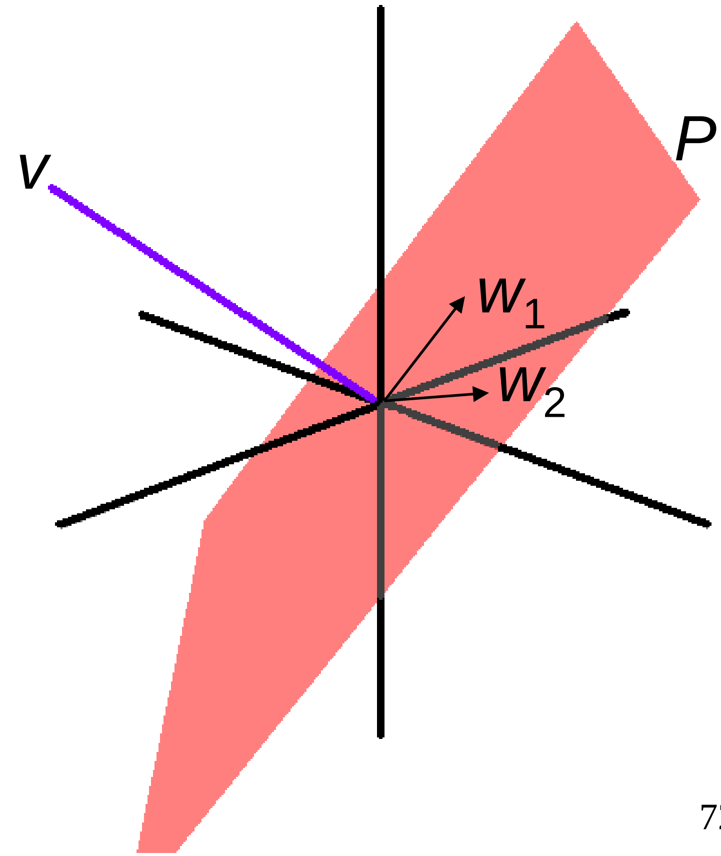
Representing 3D Rotations (Quaternions)

Thus, if we let w_1 and w_2 be an orthonormal basis for the plane P, then with respect to the basis $\{v,w_1,w_2\}$, we can express R in matrix form as either:

$$R \rightarrow \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin & \cos\theta \end{pmatrix}$$

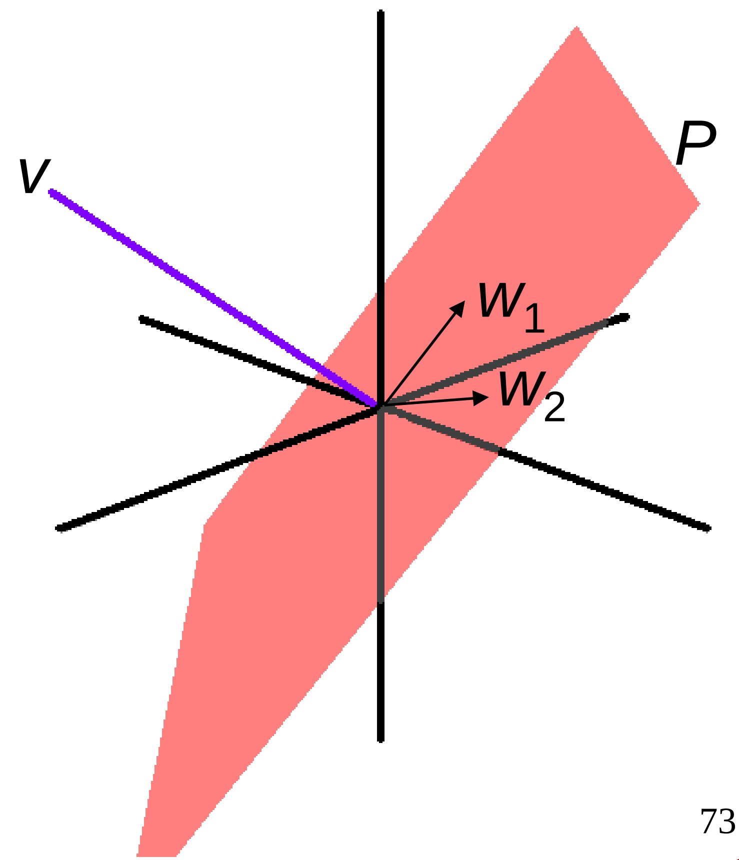
or:

$$R \rightarrow \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin & -\cos\theta \end{pmatrix}$$



Representing 3D Rotations (Quaternions)

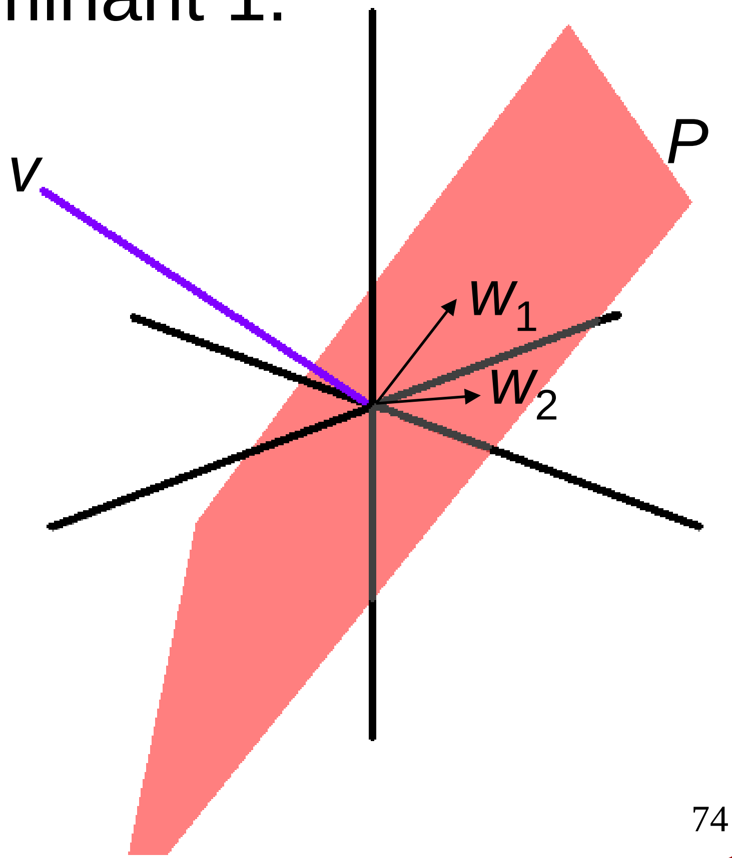
What happens in the case that *R* is a rotation?



Representing 3D Rotations (Quaternions)

What happens in the case that *R* is a rotation?

If *R* is a rotation, then in addition to being orthogonal, it must have determinant 1.



Representing 3D Rotations (Quaternions)

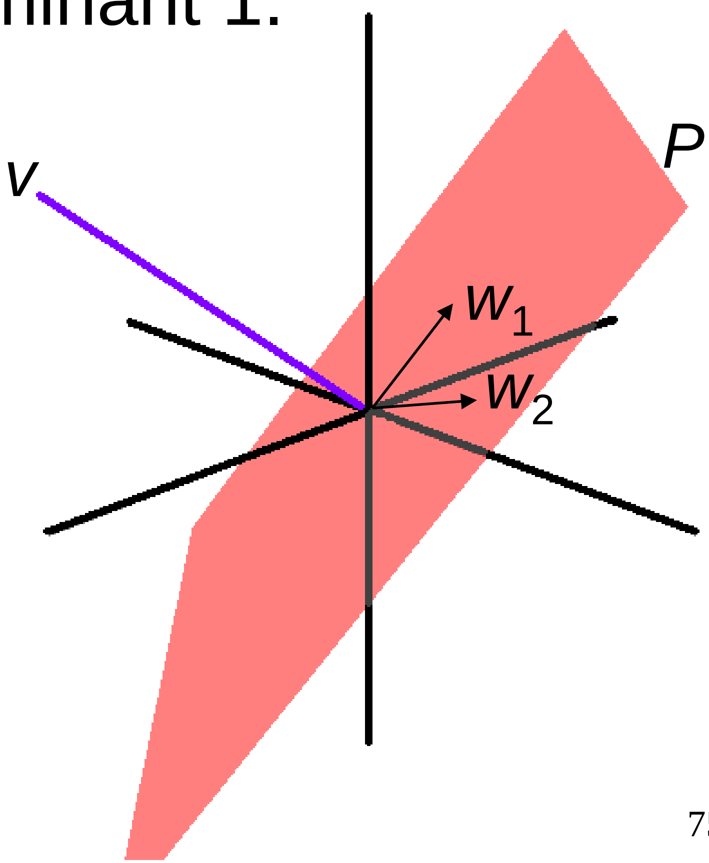
What happens in the case that *R* is a rotation?

If *R* is a rotation, then in addition to being orthogonal, it must have determinant 1.

For the two representations of *R* we get:

$$\det\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin & \cos\theta \end{pmatrix} = \lambda$$

$$\det\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin & -\cos\theta \end{pmatrix} = -\lambda$$



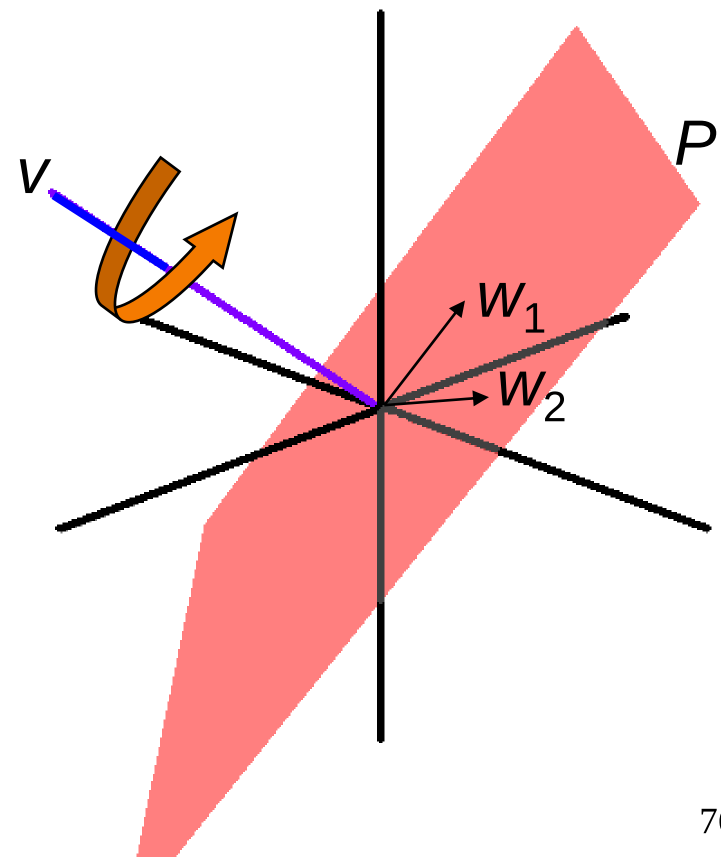
Representing 3D Rotations (Quaternions)

What happens in the case that *R* is a rotation?

Thus, if λ=1, we must have:

$$R \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

and *R* is a rotation in the plane *P* by angle θ.



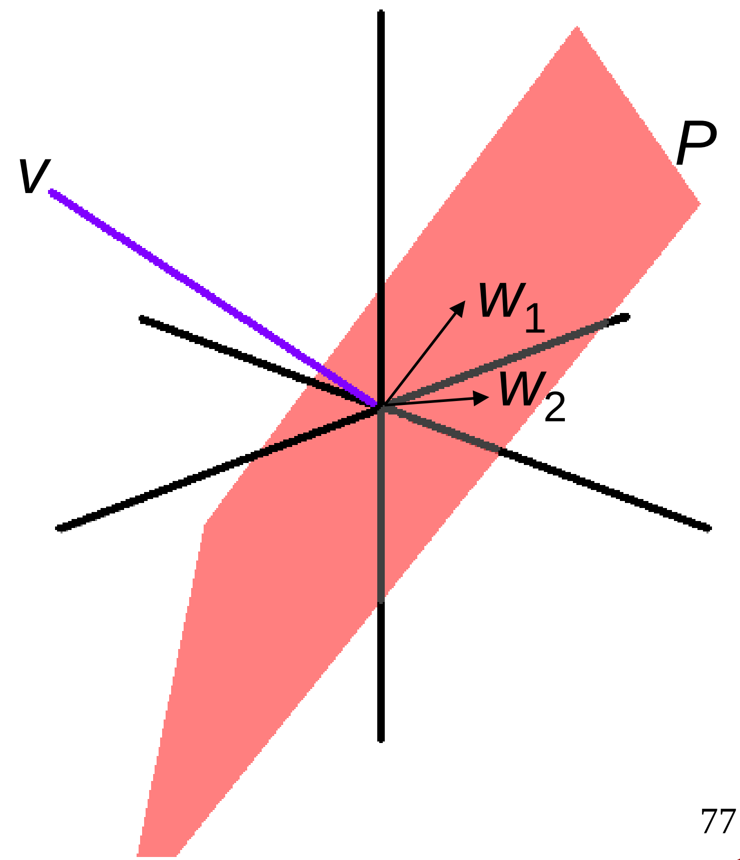
Representing 3D Rotations (Quaternions)

What happens in the case that *R* is a rotation?

On the other hand, if λ=-1, we get:

$$R \rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & -\cos\theta \end{pmatrix}$$

and *R* is the composition of a reflection in the plane *P* and a flip about the line spanned by *v*.



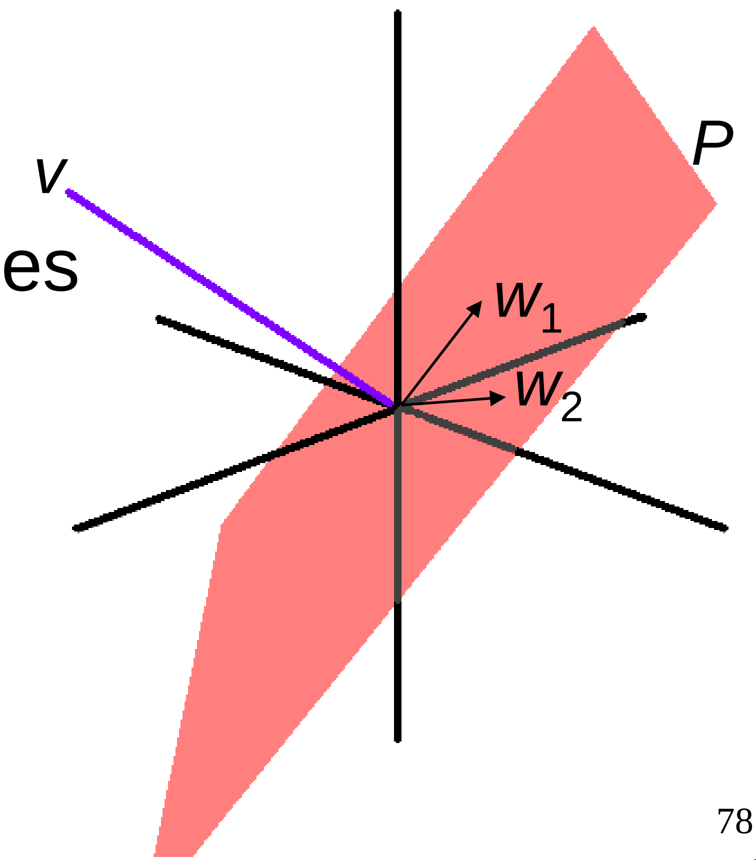
Representing 3D Rotations (Quaternions)

What happens in the case that *R* is a rotation?

Restricting *R* to the plane *P*, we get:

$$R_P \rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

which we know has eigenvalues -1 and 1.

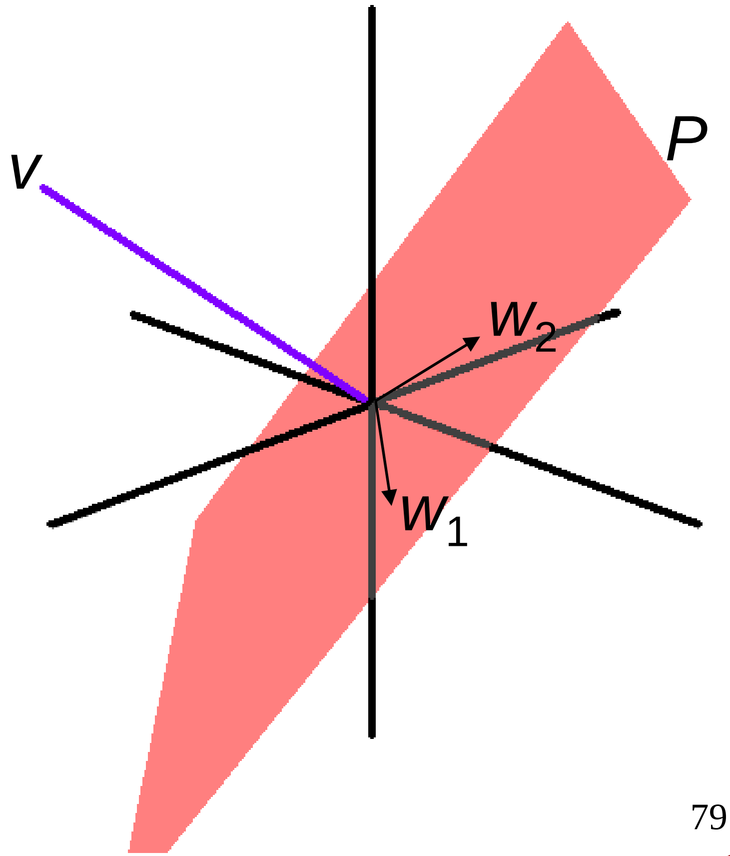


Representing 3D Rotations (Quaternions)

What happens in the case that R is a rotation?

Thus, if we set w_1 and w_2 to be the corresponding eigenvectors, we get:

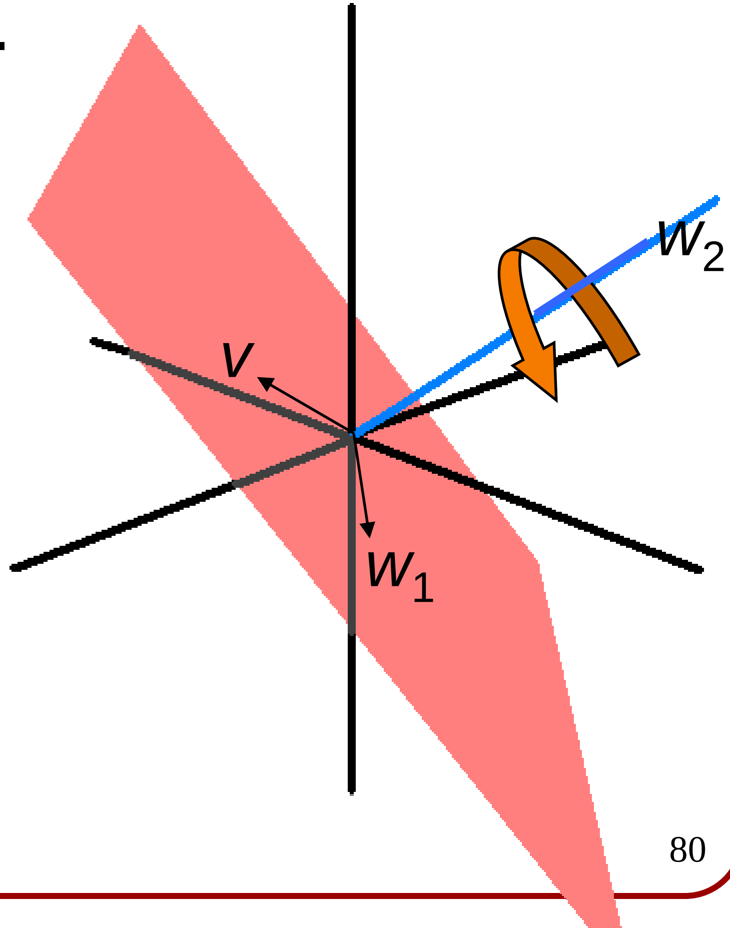
$$R \to \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Representing 3D Rotations (Quaternions)

What happens in the case that *R* is a rotation?

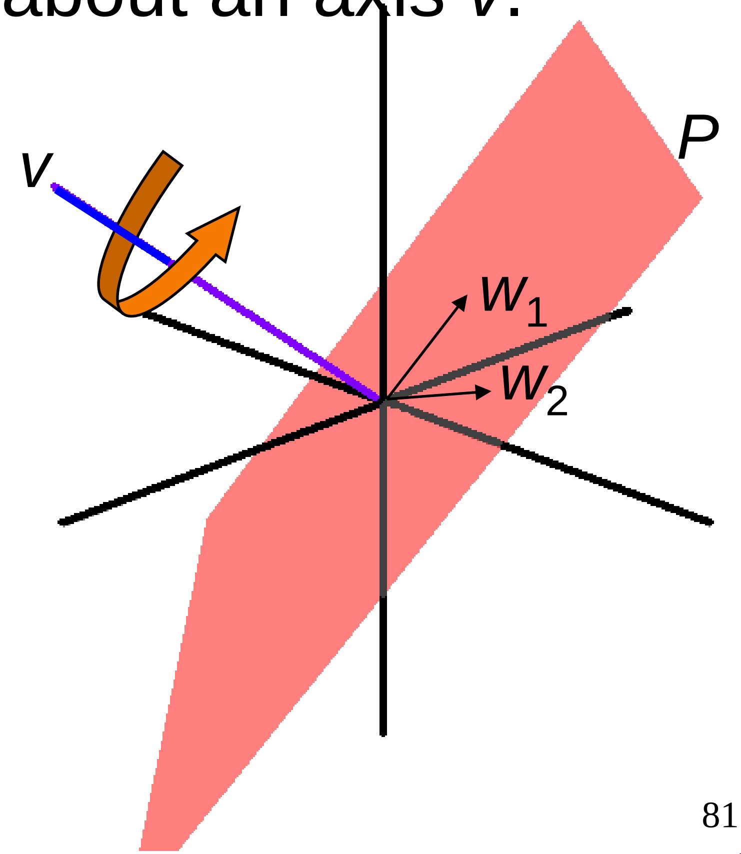
This implies that *R* must be a rotation by 180º in the plane spanned by *v* and w_1.



Representing 3D Rotations (Quaternions)

What happens in the case that *R* is a rotation?

So, in both cases, the rotation *R* can be realized as a rotation by some angle θ about an axis *v*.

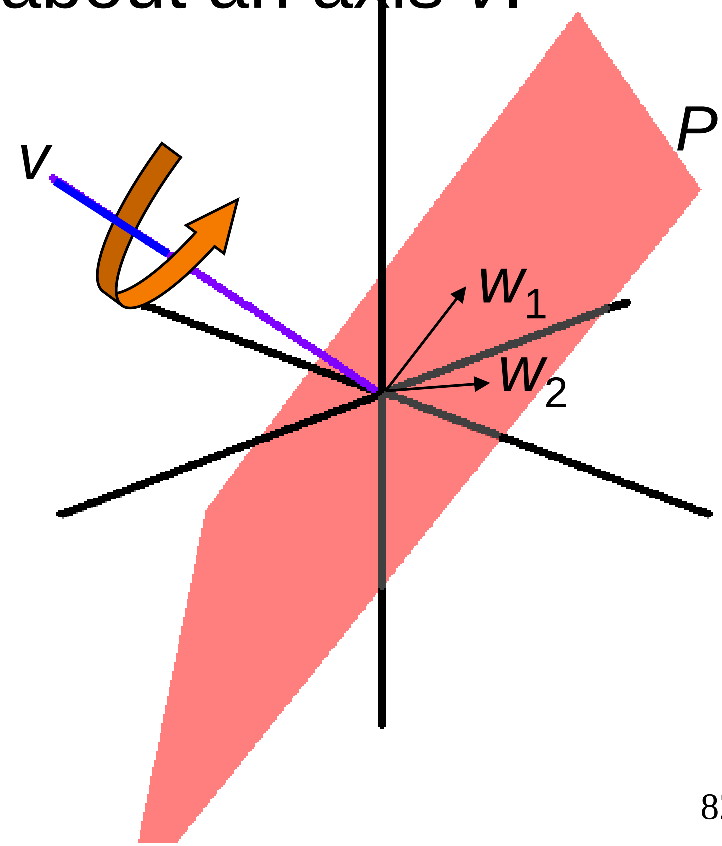


Representing 3D Rotations (Quaternions)

What happens in the case that *R* is a rotation?

So, in both cases, the rotation *R* can be realized as a rotation by some angle θ about an axis *v*.

That is, *R* sends the vector *v* back into itself and rotates vectors in the plane that is perpendicular to *v* by the angle θ.



Representing 3D Rotations (Quaternions)

What happens in the case that *R* is a rotation?

This motivates a representation of rotations by specifying the axis about which the rotation occurs and the angle of the 2D rotation.

Representing 3D Rotations (Quaternions)

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This motivates a representation of rotations by specifying the axis about which the rotation occurs and the angle of the 2D rotation.

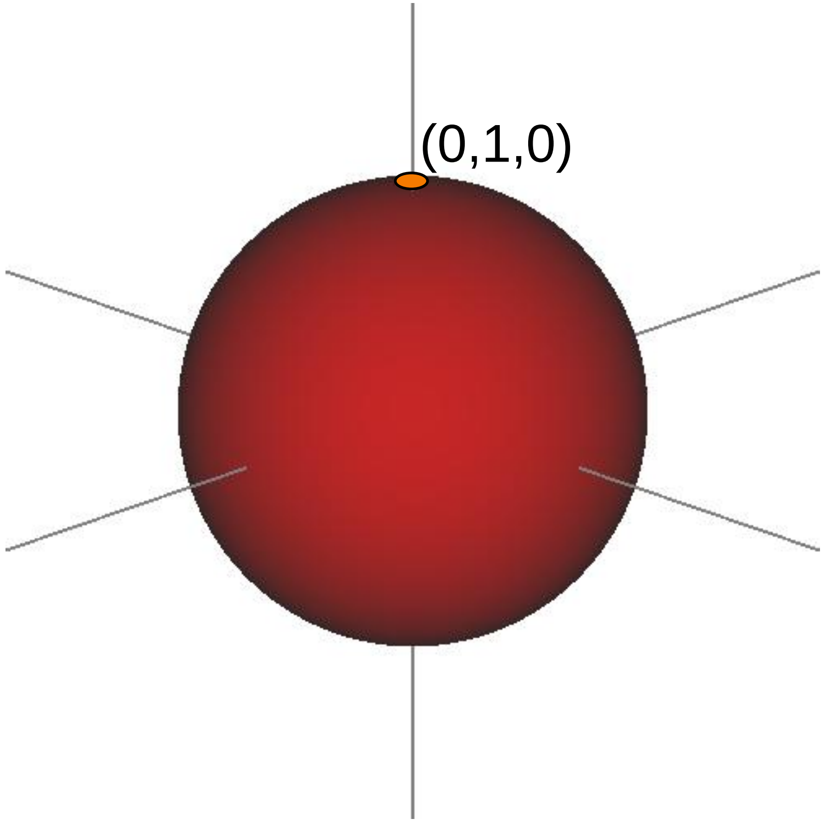
This is precisely the information represented by the quaternion representation of rotation.

Outline

- Math Review
- Representing 3D Rotations
    - Quaternions
    - Euler Angles

Representing 3D Rotations (Euler)

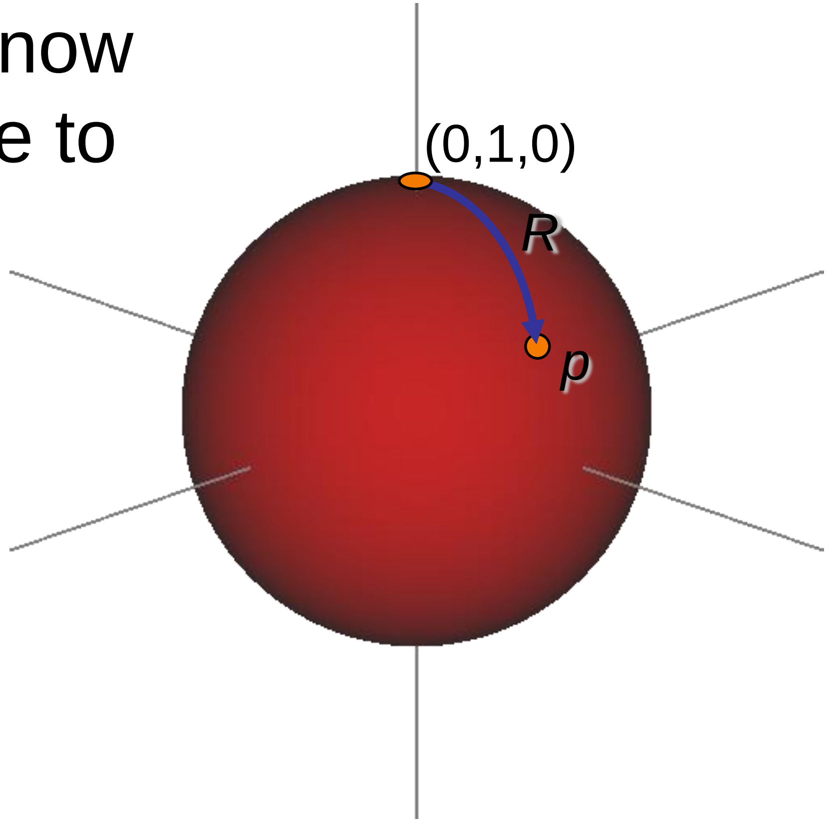
We will consider a representation of rotations that describe a rotation in terms of what it does to the point (0,1,0) on the North pole of the sphere.



Representing 3D Rotations (Euler)

We will consider a representation of rotations that describe a rotation in terms of what it does to the point (0,1,0) on the North pole of the sphere.

Given a rotation *R*, if we know that *R* maps the North pole to the point *p*, is that enough information to define *R*?



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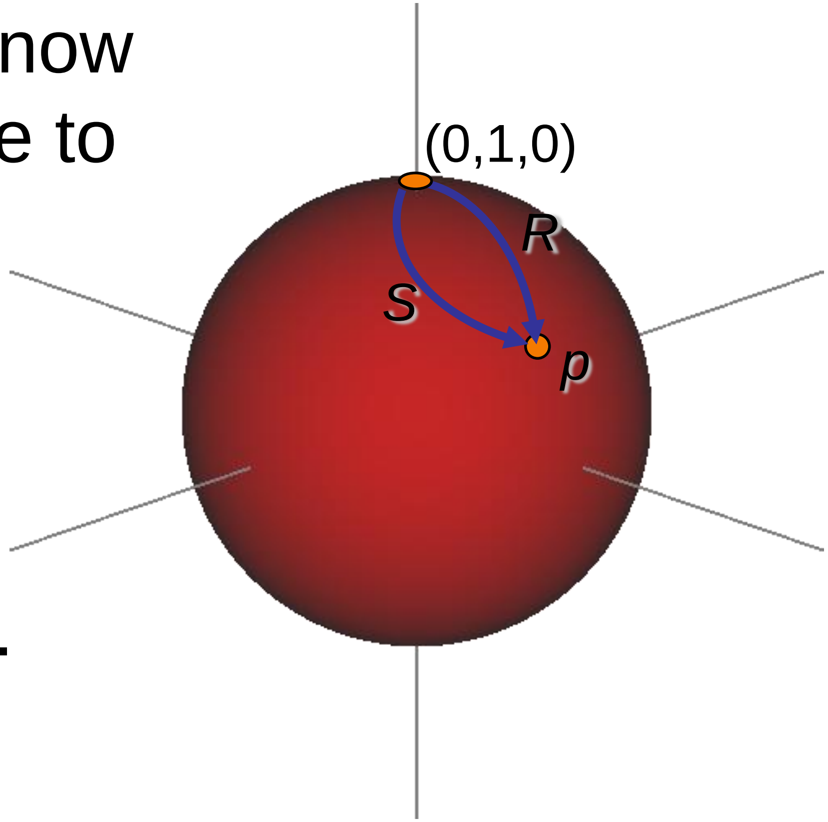
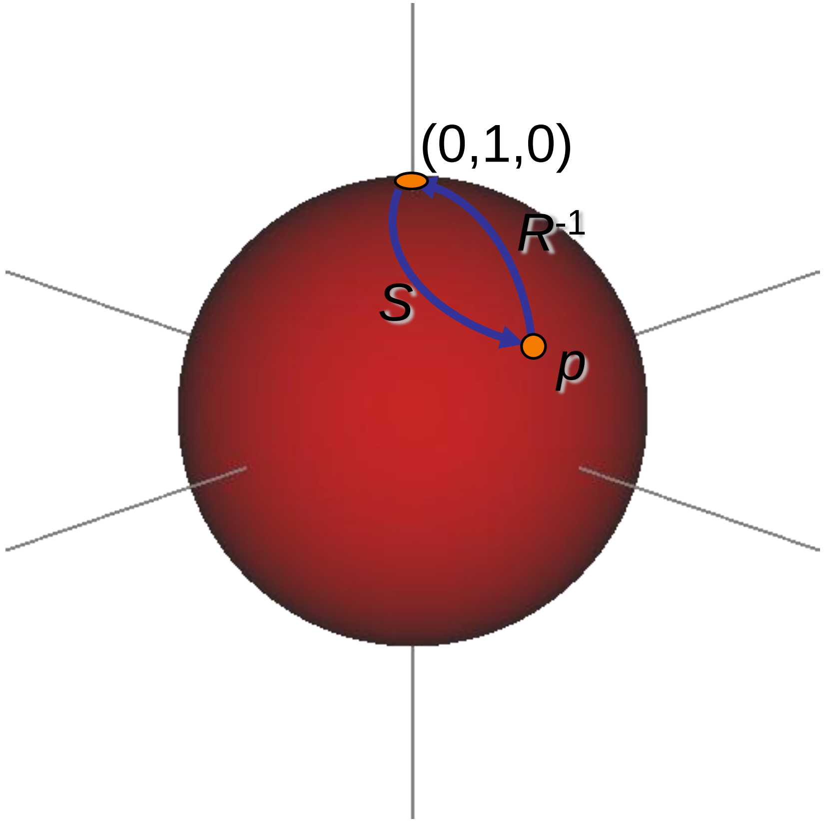


No. There can be many different rotations that all send (0,1,0) to the point *p*.

Representing 3D Rotations (Euler)

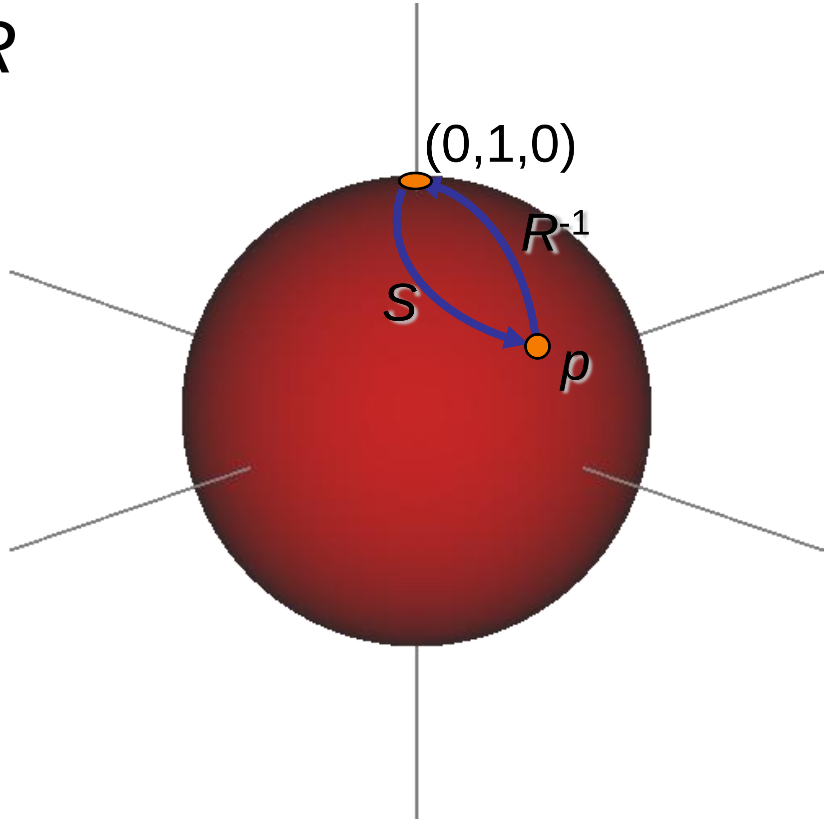
In particular, if R and S are two rotations mapping the North pole to the point p, we know that $R^{-1}S$ must map the North pole back to itself.



Representing 3D Rotations (Euler)

In particular, if *R* and *S* are two rotations mapping the North pole to the point *p*, we know that $R^{-1}S$ must map the North pole back to itself.

That is, for any rotations *R* and *S* mapping the North pole to the point *p*, $R^{-1}S$ must be a rotation about the North pole.



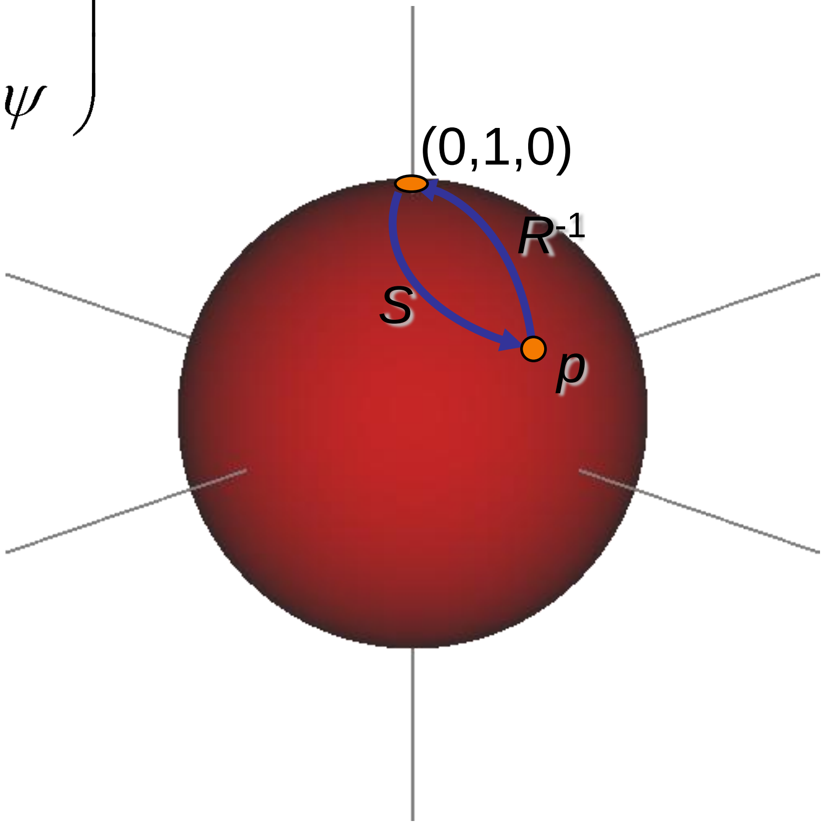
Representing 3D Rotations (Euler)

Thus, if we denote by $R_y(\psi)$ a rotation about the *y*-axis (North pole) by ψ degrees:

$$R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}$$

then if *R* and *S* are two rotations sending the North pole to *p*, then:

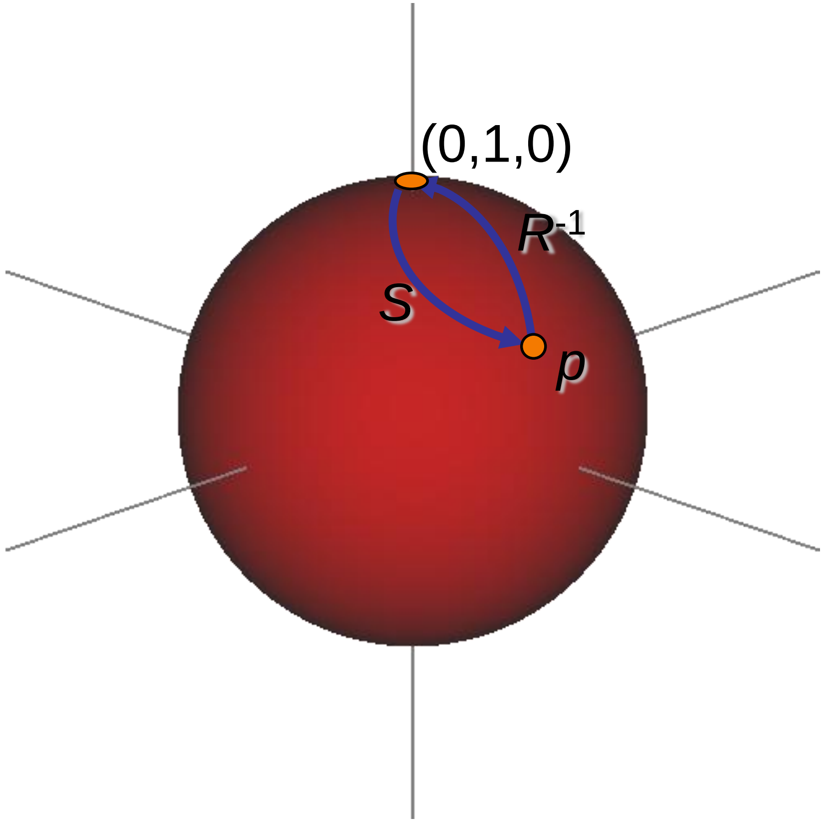
$$R^{-1}S = R_y(\psi)$$



Representing 3D Rotations (Euler)

In particular, this implies that if *R* is a rotation sending the North pole to *p*, then any other rotation *S* that sends the North pole to *p* must be of the form:

$$S = R \cdot R_y(\psi)$$



Representing 3D Rotations (Euler)

In order to represent all rotations, we need to find an expression for the map that sends the North pole to the point p.

Representing 3D Rotations (Euler)

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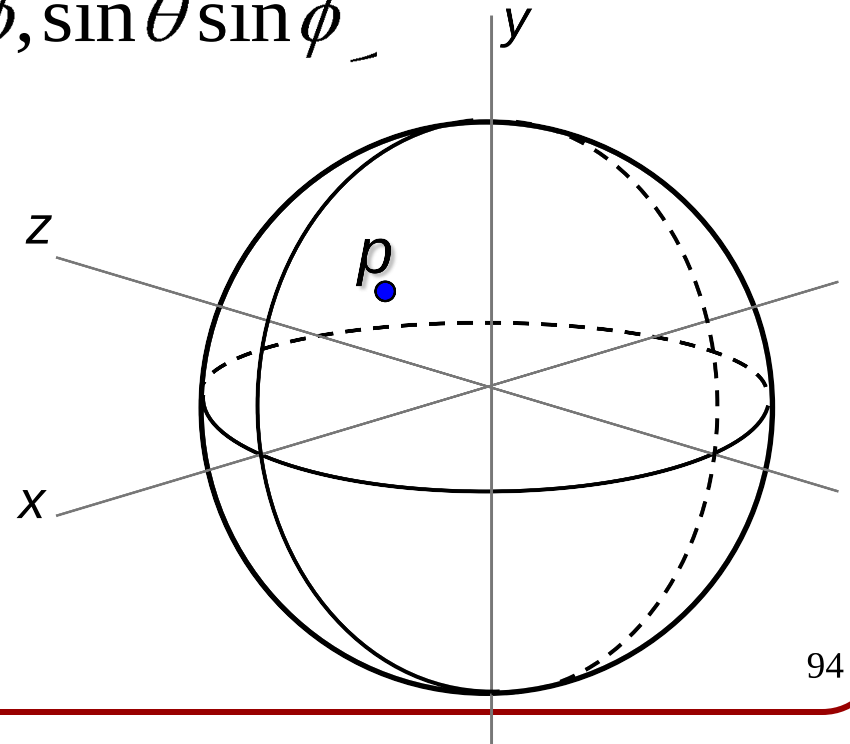
Let (θ,ϕ) be the spherical coordinates of *p*:

$$p = (\cos\theta\sin\phi, \cos\phi, \sin\theta\sin\phi)$$



Representing 3D Rotations (Euler)

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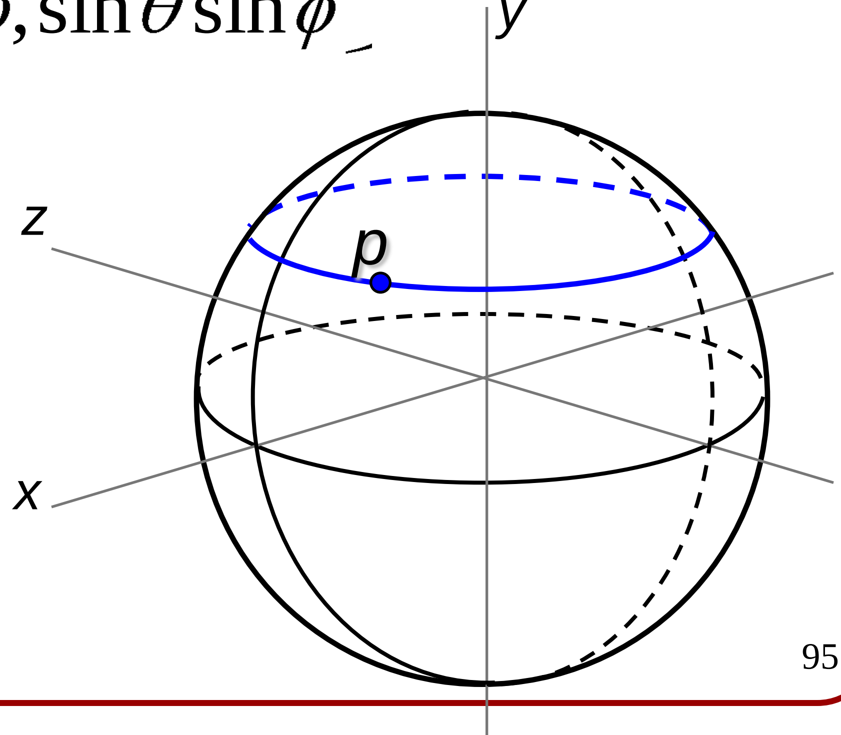
Let (θ,ϕ) be the spherical coordinates of *p*:

$$p = \left(\cos\theta\sin\phi, \cos\phi, \sin\theta\sin\phi\right)$$

The point *p* must lie on the circle about the *y*-axis with height $\cos\phi$.



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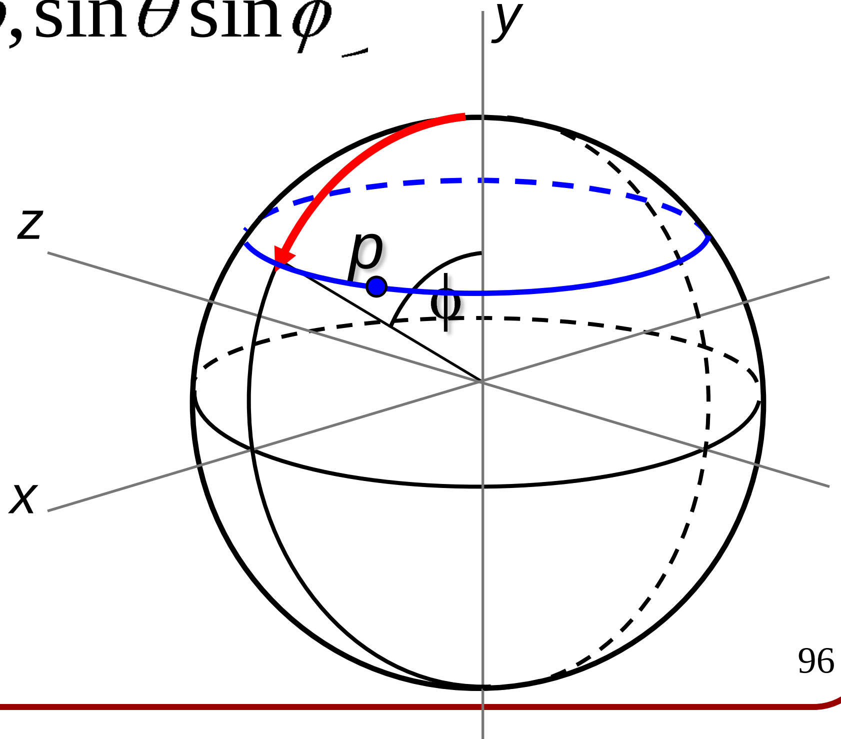
The point *p* must lie on the circle about the *y*-axis with height $\cos\phi$.

We can get (0,1,0) to this circle with a rotation by an angle of ϕ about the *z*-axis.



Representing 3D Rotations (Euler)

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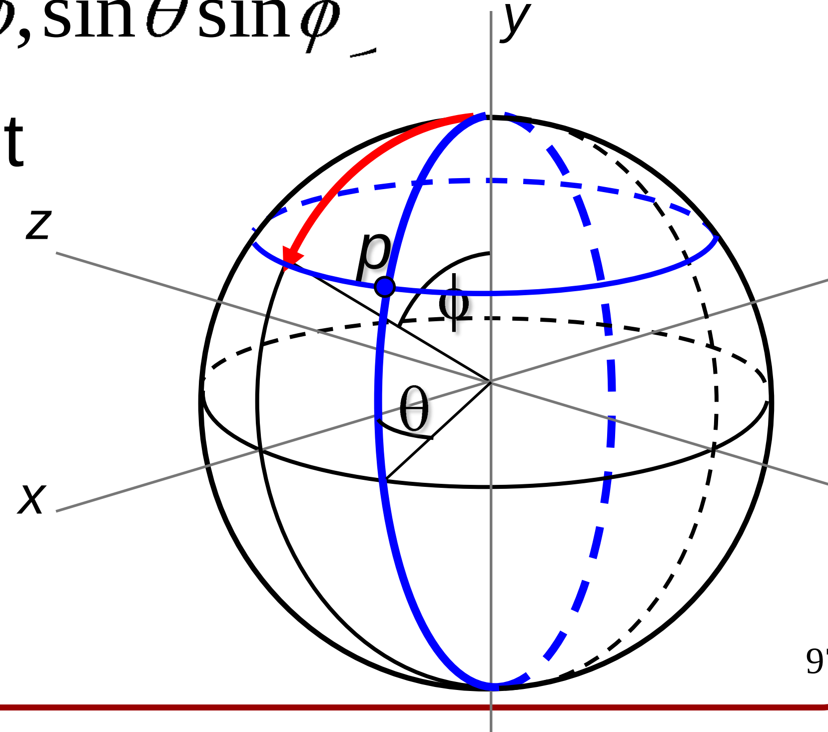
Let (θ,φ) be the spherical coordinates of *p*:

$$p = \left(\cos\theta \sin\phi, \cos\phi, \sin\theta \sin\phi\right)$$

We also know that the point *p* makes an angle of θ with the *xy*-plane.



Representing 3D Rotations (Euler)

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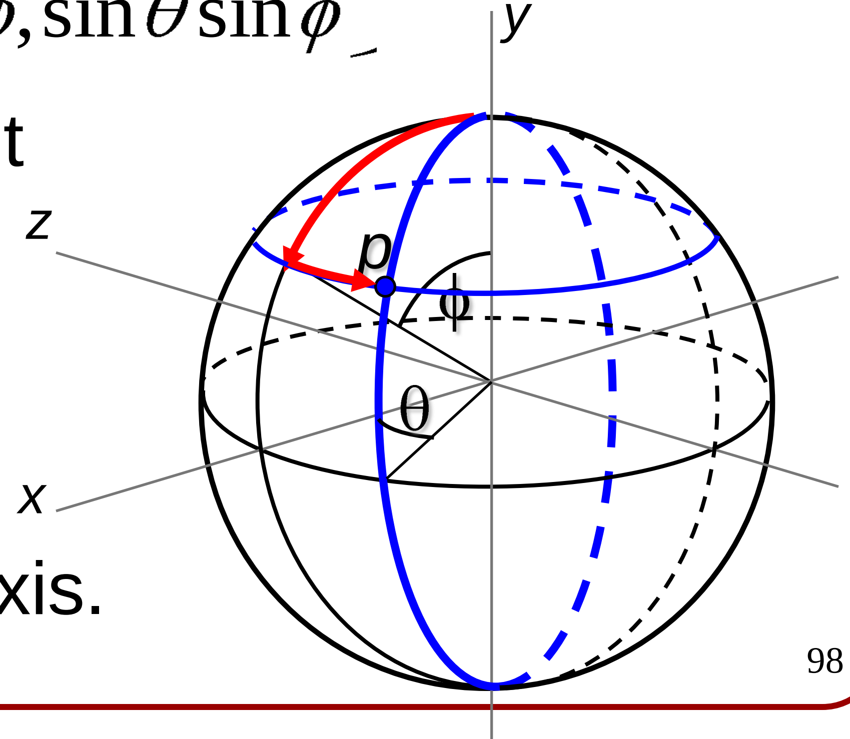
$$p=\left(\cos\theta\sin\phi,\cos\phi,\sin\theta\sin\phi\right)$$

We also know that the point *p* makes an angle of θ with the *xy*-plane.

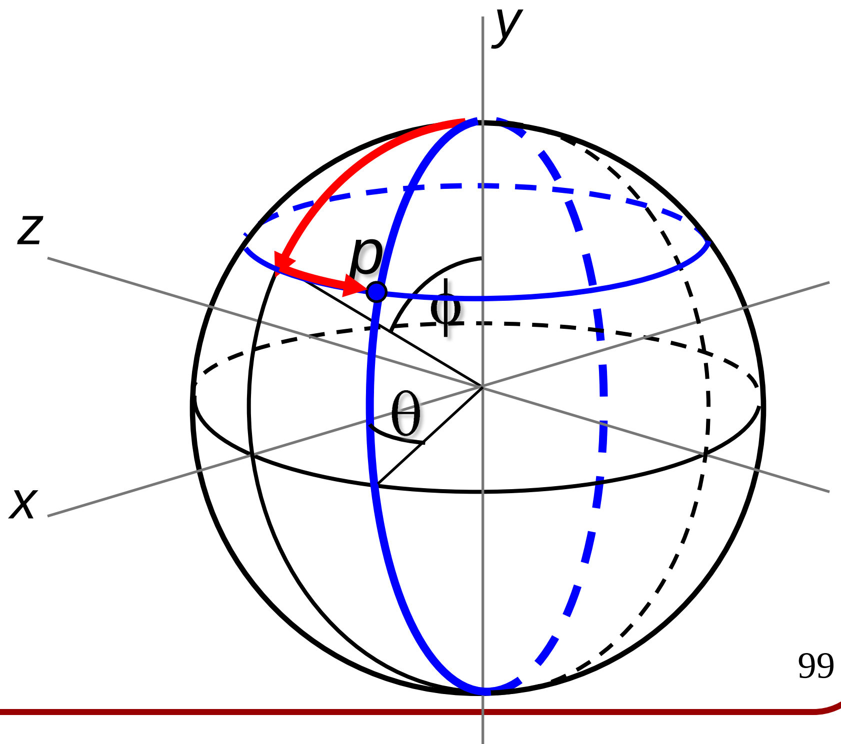
We can get the rotation of (0,1,0) to *p* by rotating by an angle of θ about the *y*-axis.



Representing 3D Rotations (Euler)

Thus, when the spherical coordinates of the point *p* are (θ,ϕ), we can rotate (0,1,0) to *p* by:

- First rotating by ϕ degrees about the *z*-axis, and
- Then rotating by θ degrees about the *y*-axis.



Representing 3D Rotations (Euler)

Since a rotation *R* can be described by a rotation about the *y*-axis, followed by a rotation that maps (0,1,0) to *R*(0,1,0), we can represent rotations by:

$$R = R_y(\theta) \cdot R_z(\phi) \cdot R_y(\psi)$$

where $R_y(\alpha)$ is the rotation about the *y*-axis by α, and $R_z(\beta)$ is the rotation about the *z*-axis by β.

Representing 3D Rotations (Euler)

In matrix form, the triplet of angles (θ,φ,ψ) represents the rotation:

$$R(\theta,\phi,\psi)=\underbrace{\begin{pmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}}_{\text{Rotation sending }(0,1,0)\text{ to }p=\Phi(\theta,\phi)}\underbrace{\begin{pmatrix}\cos\psi & 0 & -\sin\psi\\ 0 & 1 & 0\\ \sin\psi & 0 & \cos\psi\end{pmatrix}}_{\text{Rotation about the }y\text{-axis by }\psi}$$

Rotation sending (0,1,0) to *p*=Φ(θ,φ)

Rotation about the *y*-axis by ψ

Representing 3D Rotations (Euler)

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This is the Euler Angle parameterization of 3D rotations.