FFTs in Graphics and Vision

The Spherical Laplacian

Ouline

- Stokes' Theorem
- Tangent Spaces
- Gradients
- The Spherical Laplacian
- Applications

Stokes' Theorem

Stokes' Theorem equates the integral of the divergence of a vector field over a region to the surface integral of the vector field over the boundary:

$$\int_V \left(\nabla \cdot F\right) dV = \int_{\partial V} F \cdot dA$$

where *F·dA* is defined by:

$$F \cdot dA = \langle F, \vec{n} \rangle \left| dA \right|$$

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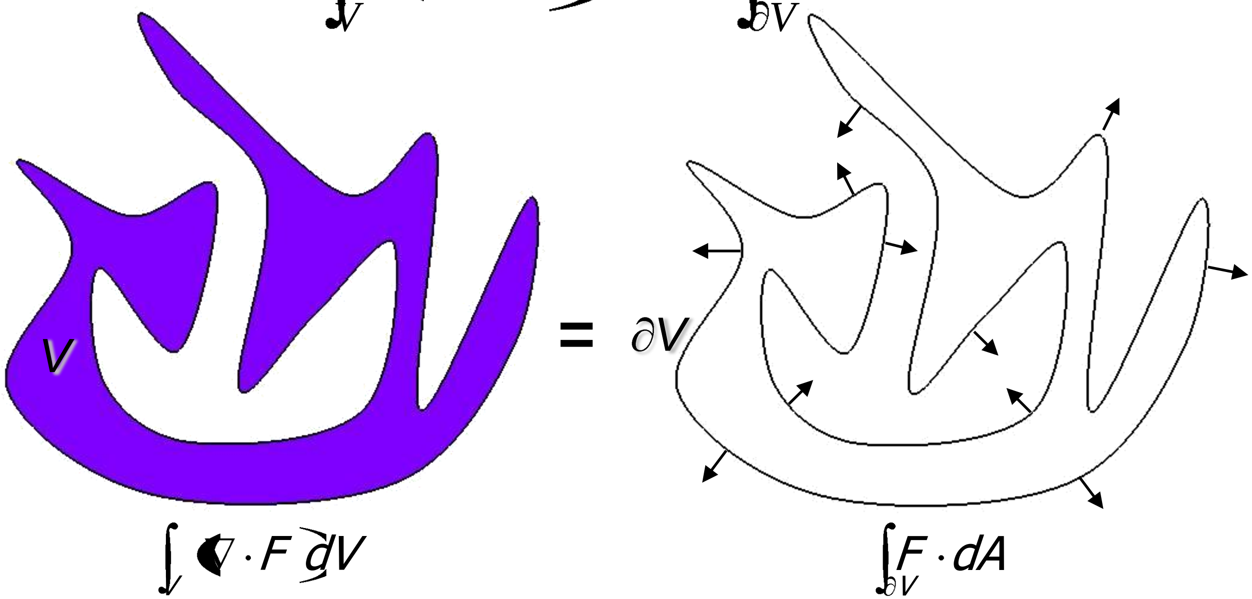

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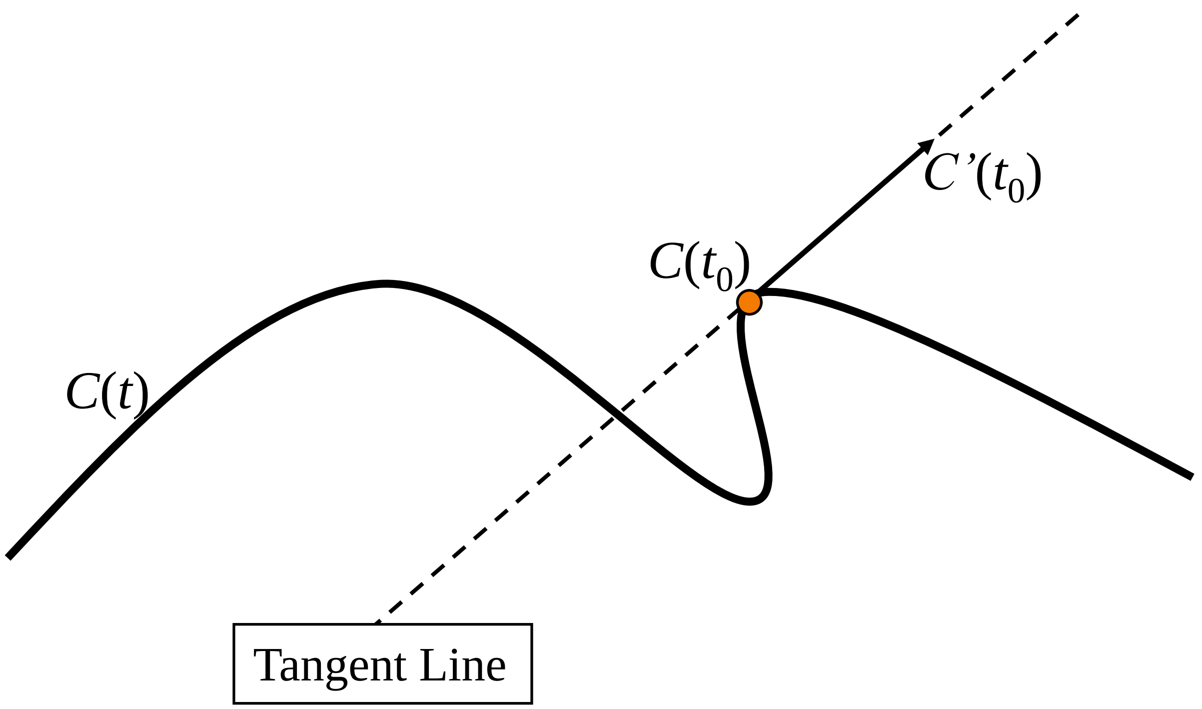
Tangent Spaces

Given a curve $C(t)=(x(t),y(t))$, the <u>tangent line</u> to the curve at a point $p_0=C(t_0)$ is the line passing through p_0 with direction $C'(t_0)=(x'(t_0),y'(t_0))$.



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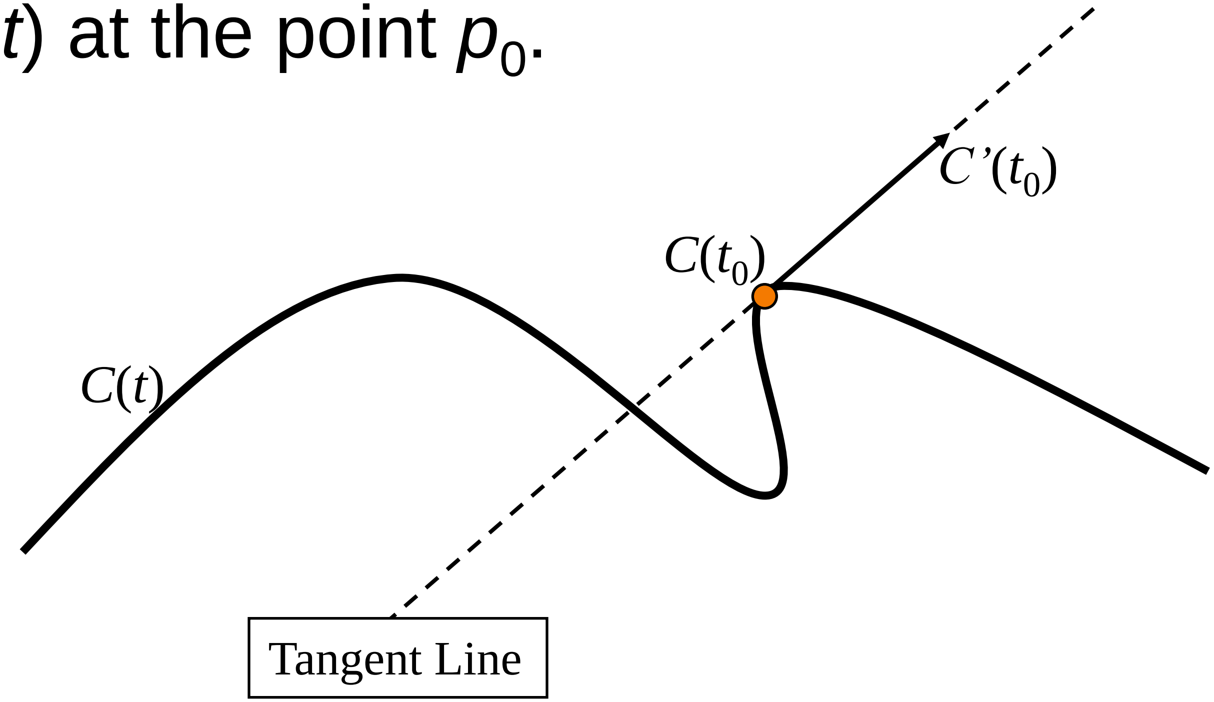
This is the line that most closely approximates the curve $C(t)$ at the point p_0.

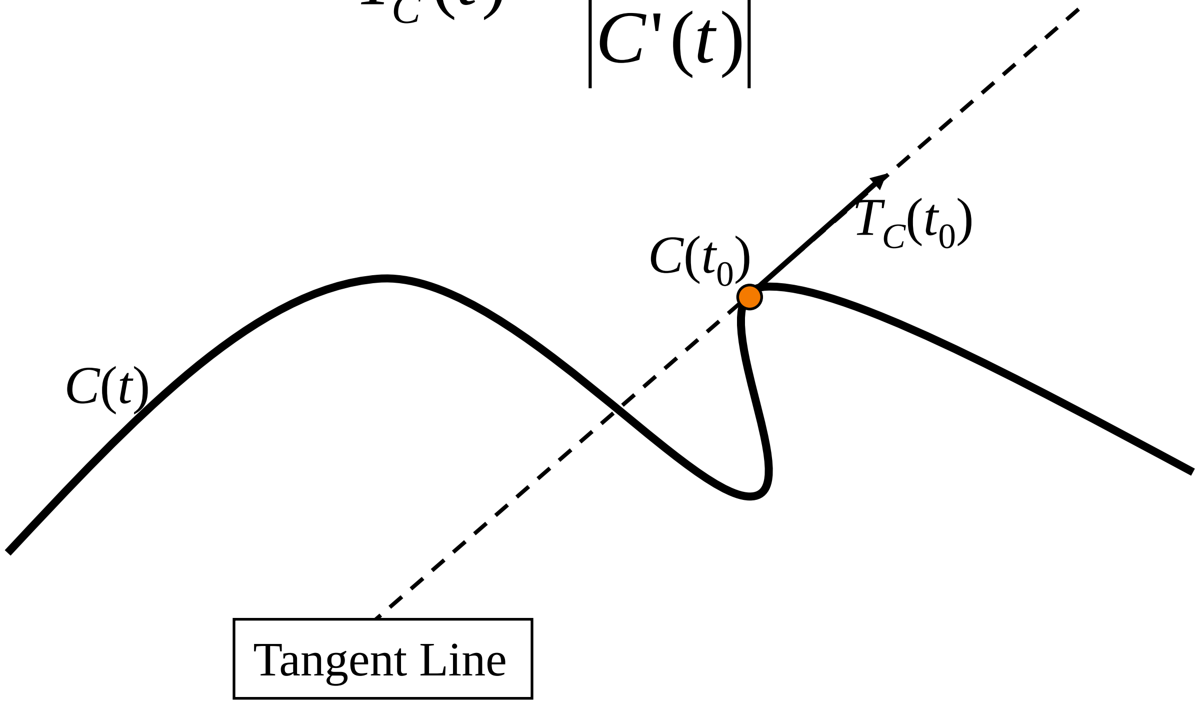


Tangent Spaces

Often, what we want is a unit vector specifying the tangent direction.

In this case, we need to normalize:

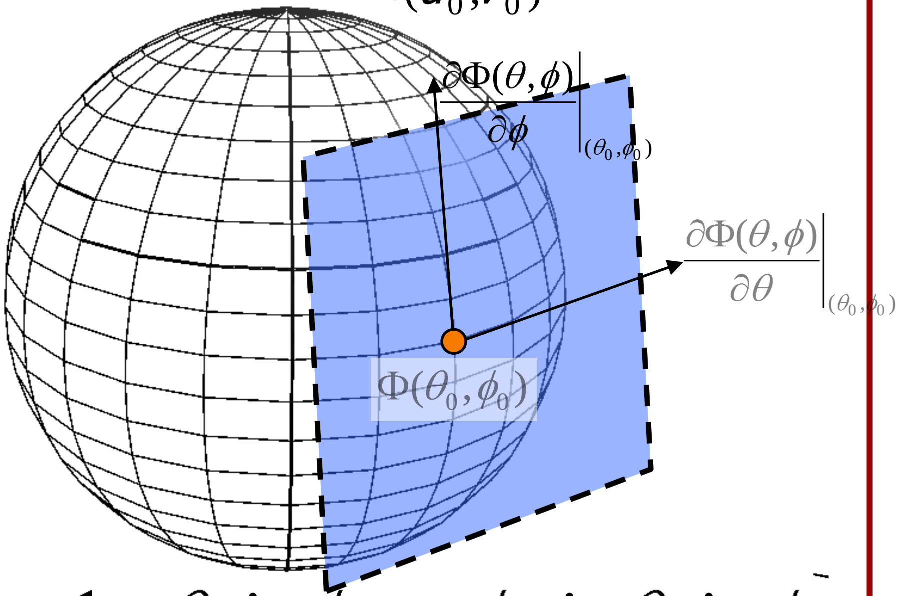
$$T_C(t) = \frac{C'(t)}{\left|C'(t)\right|}$$



Tangent Spaces

Given a surface $S(u,v)$ the <u>tangent plane</u> to the curve at a point $p_0=S(u_0,v_0)$ is the plane passing through p_0, parallel to the plane spanned by:

$$\left.\frac{\partial S(u,v)}{\partial u}\right|_{(u_0,v_0)} \quad \text{and} \quad \left.\frac{\partial S(u,v)}{\partial v}\right|_{(u_0,v_0)}$$



$$\Phi(\theta,\phi) = (\cos\theta\sin\phi, \cos\phi, \sin\theta\sin\phi)$$

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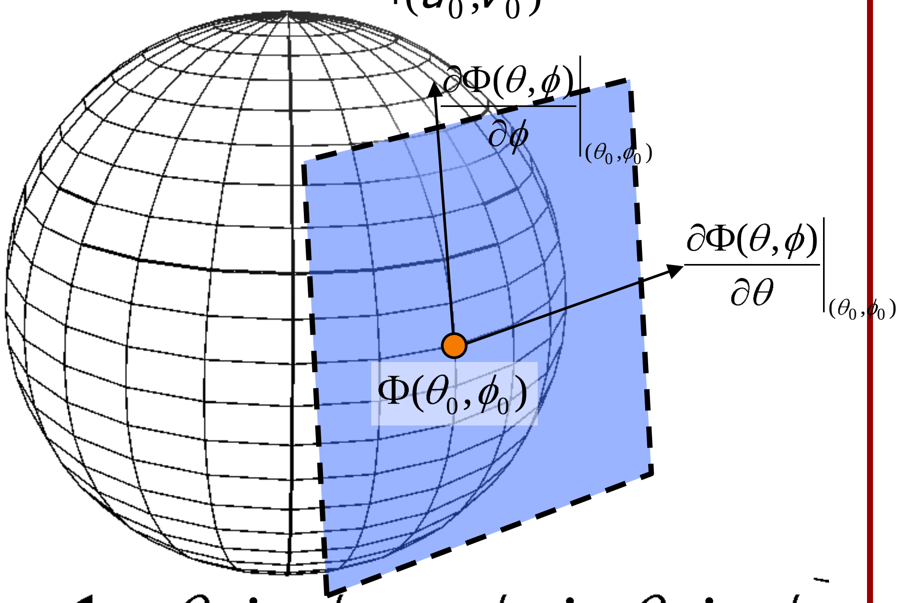
This is the plane that most closely approximates $S(u,v)$ at the point p_0.



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Tangent Spaces

In the case of the sphere, points are parameterized by the equation:

$$\Phi(\theta,\phi) = \left(\cos\theta\sin\phi, \cos\phi, \sin\theta\sin\phi\right)$$

and the two tangent directions are:

$$\frac{\partial\Phi}{\partial\theta} = \left(-\sin\theta\sin\phi, 0, \cos\theta\sin\phi\right)$$

$$\frac{\partial\Phi}{\partial\phi} = \left(\cos\theta\cos\phi, -\sin\phi, \sin\theta\cos\phi\right)$$

Tangent Spaces

If we look at the dot-product of the two vectors:

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we get:

$$\left\langle \frac{\partial \Phi}{\partial \theta}, \frac{\partial \Phi}{\partial \theta} \right\rangle = \sin^2\theta \sin^2\phi + \cos^2\theta \sin^2\phi$$

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$$\left\langle \frac{\partial \Phi}{\partial \theta}, \frac{\partial \Phi}{\partial \phi} \right\rangle = 0$$

Tangent Spaces

So, the vectors:

$$\Phi_\theta(\theta,\phi) = \frac{1}{\sin\phi}\frac{\partial\Phi}{\partial\theta}$$

$$\Phi_\phi(\theta,\phi) = \frac{\partial\Phi}{\partial\phi}$$

form an orthonormal basis for the tangent plane to the sphere at the point $\Phi(\theta,\phi)$.

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Function Gradients

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The gradient of a function is a vector which tells us how the function changes as we move in different directions.

Given a function *f* and given a direction *v*:

$$f(p+v) \approx f(p) + \left\langle \nabla f(p), v \right\rangle$$

Function Gradients

To compute the gradient, we choose two orthogonal unit vectors *u* and *v*, and we set:

$$\nabla f(p) = \frac{d}{dt} f(p+tu)u + \frac{d}{dt} f(p+tv)v$$

Curve Gradients

Given a curve *C*(*t*), and given a function *f*(*t*) the gradient of the function is a vector field on the curve telling us how the function changes as we move along the curve.

Curve Gradients

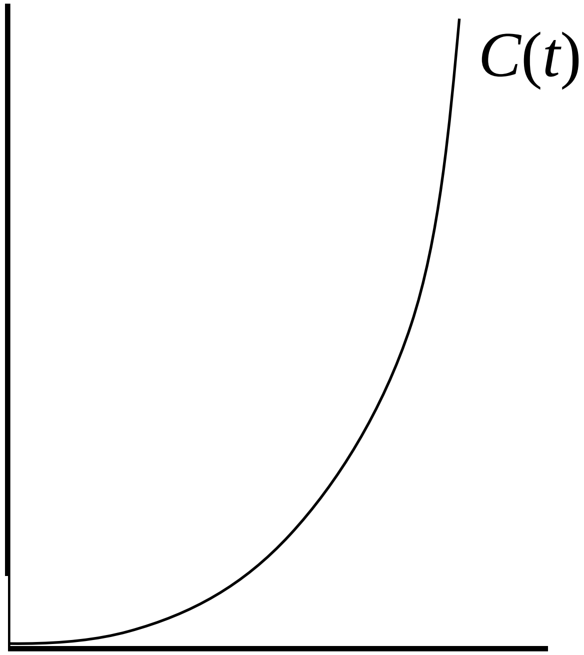
<u>Example</u>:

Let *C* be the curve defined by:

$$C(t) = \left(t, t^2\right)$$

and let *f*(*t*) be the function on the curve defined by:

$$f(t) = t$$



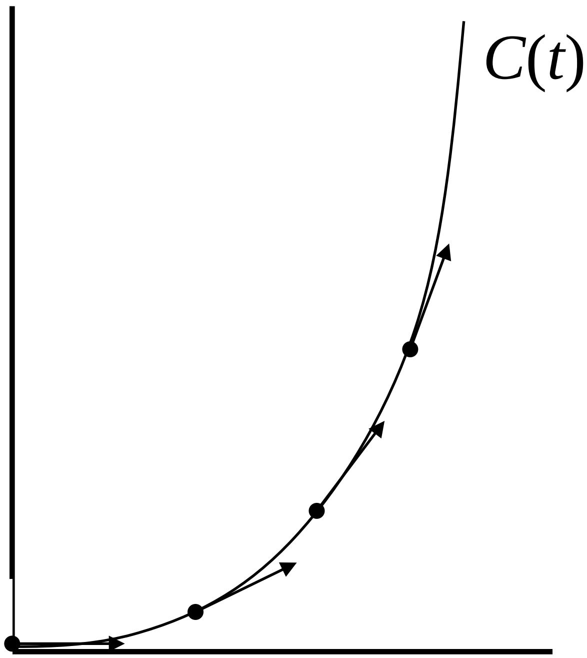
What is the gradient of *f*(*t*)?

Curve Gradients

Example:

The gradient is not the function $\nabla_C f=1$!

This would imply that at any point on the curve moving a unit forward would change the value by a constant amount.

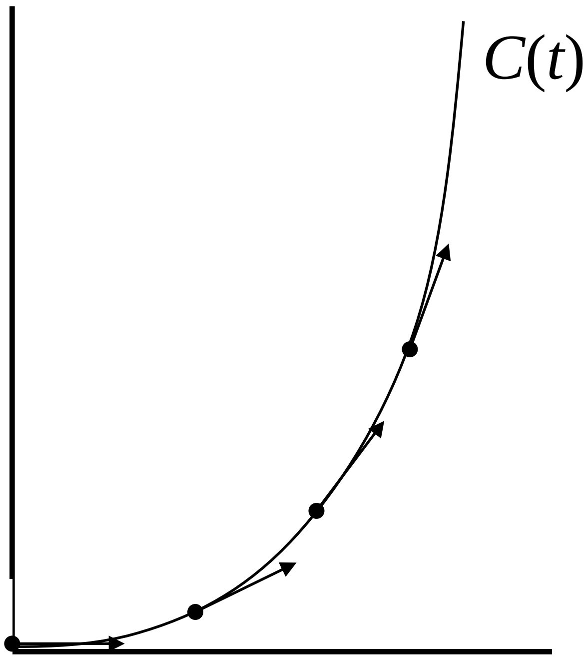


Curve Gradients

Example:

The gradient is not the function $\nabla_C f = 1$!

As we move from t=1 to t=2, the function changes by a value of 1. Similarly, as we move from t=10 to t=11, the function changes by a value of 1.



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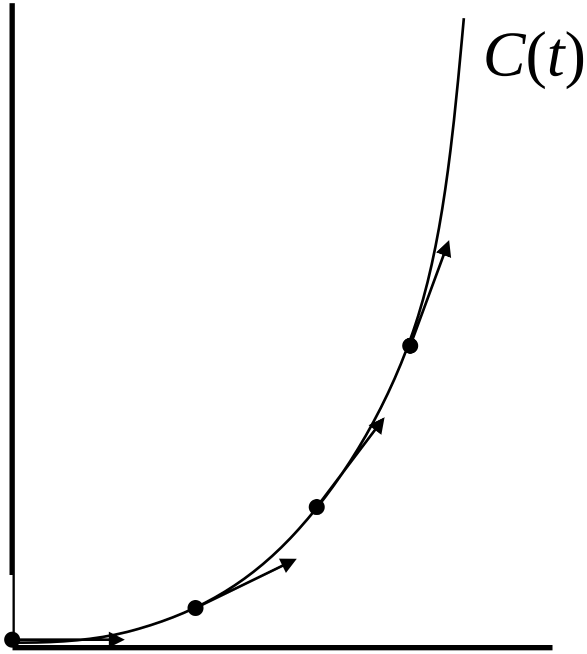

But in the first case, we have moved a distance of:

$$d_1 \approx \left\| C(2) - C(1) \right\| = \sqrt{1^2 + 3^2}$$

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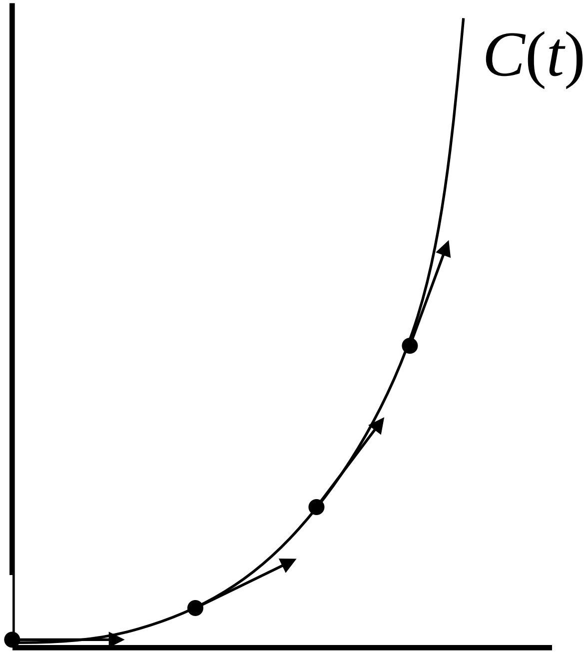


In the second case, we have moved a distance of:

$$d_2 \approx \left\| C(11) - C(10) \right\| = \sqrt{1^2 + 21^2}$$

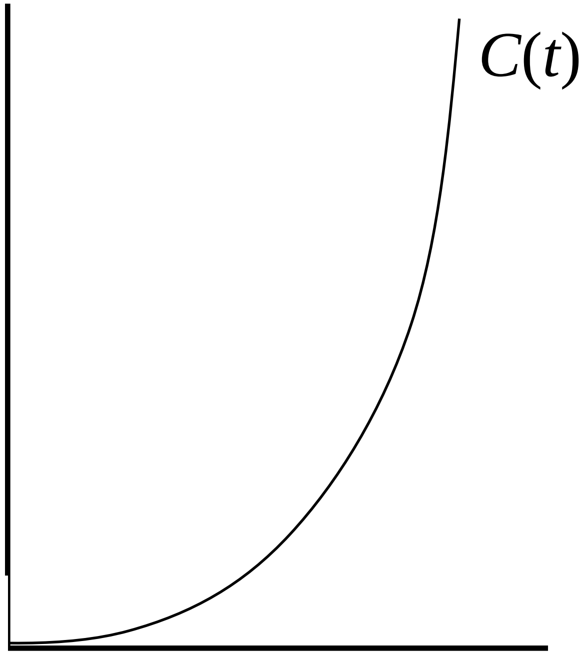
Curve Gradients

Example:

We need to measure the ratio of the change in *f* over the distance traveled:

$$\nabla_C f(t) \approx \frac{f(t+\varepsilon)-f(t)}{\left|C(t+\varepsilon)-C(t)\right|}$$

$$\nabla_C f(t) = \frac{f'(t)}{\left|C'(t)\right|}$$



Curve Gradients

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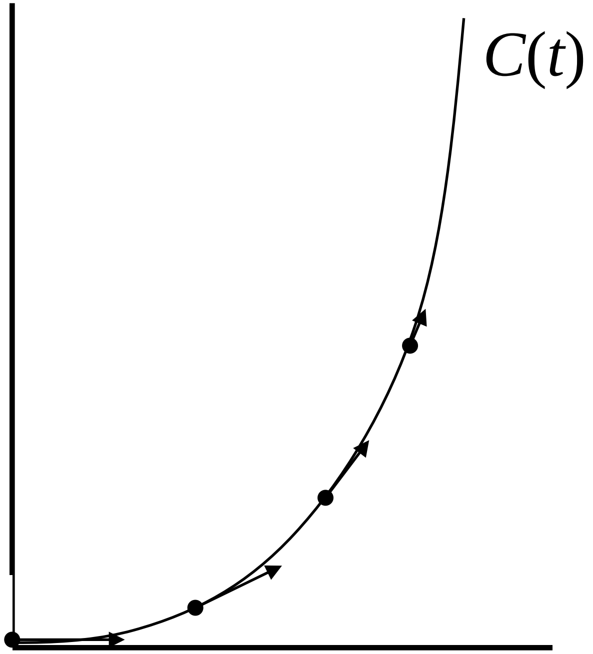
We need to measure the ratio of the change in *f* over the distance traveled:

$$\nabla_C f(t) = \frac{1}{\sqrt{1+2t}}$$



Spherical Gradients

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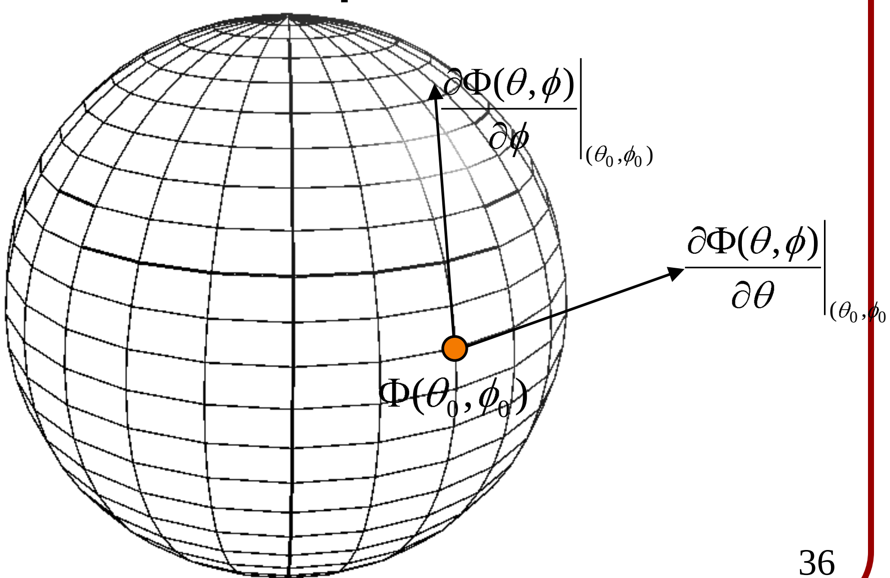
The directions θ and ϕ are two such directions:



Spherical Gradients

We could try taking the partial derivatives in the θ and ϕ directions:

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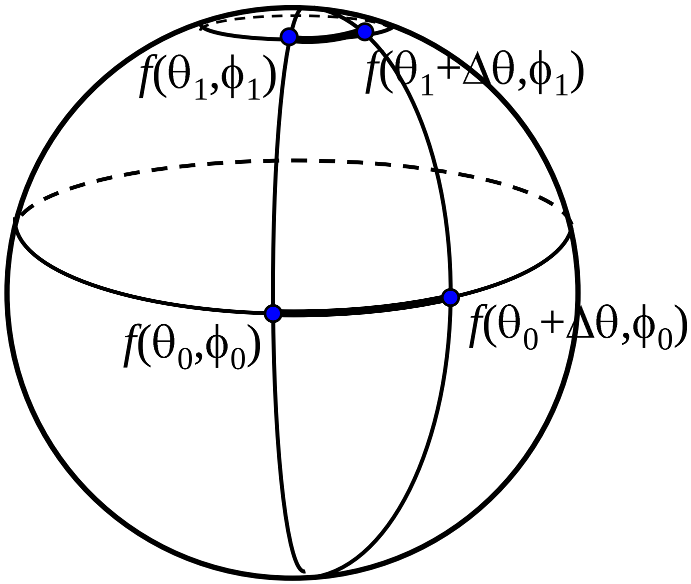
But this introduces bias!



Shifting by a constant $\Delta\theta$ will move us different distances depending on where we are on the sphere.

Spherical Gradients

How does the scale change as we change θ or ϕ by a value of ε?

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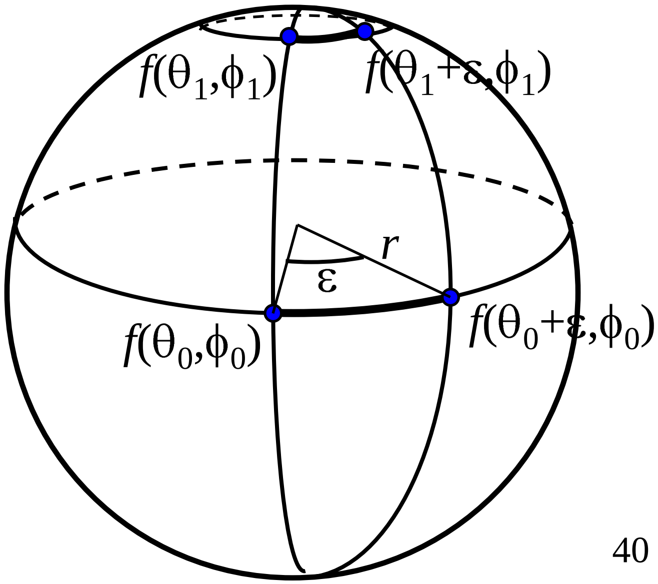
At the point $p=\Phi(\theta,\phi)$, changing the value of θ by ε, moves us a distance of εr along the circle about the y-axis, where r is the radius of the circle:



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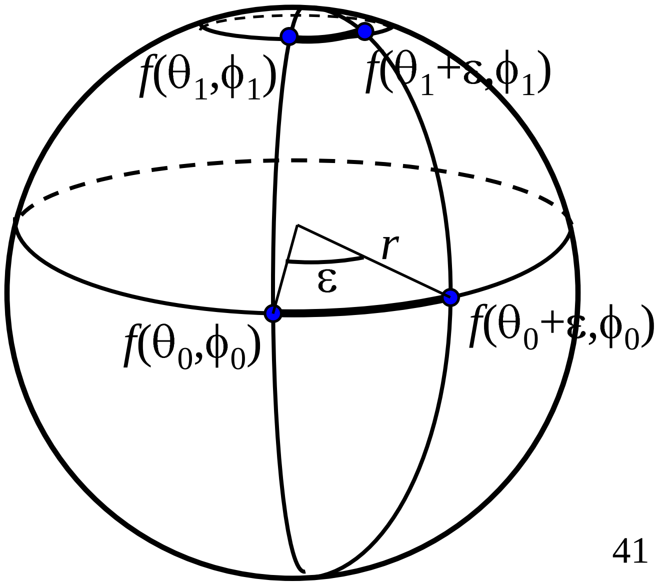
On the sphere, the radius is defined by:

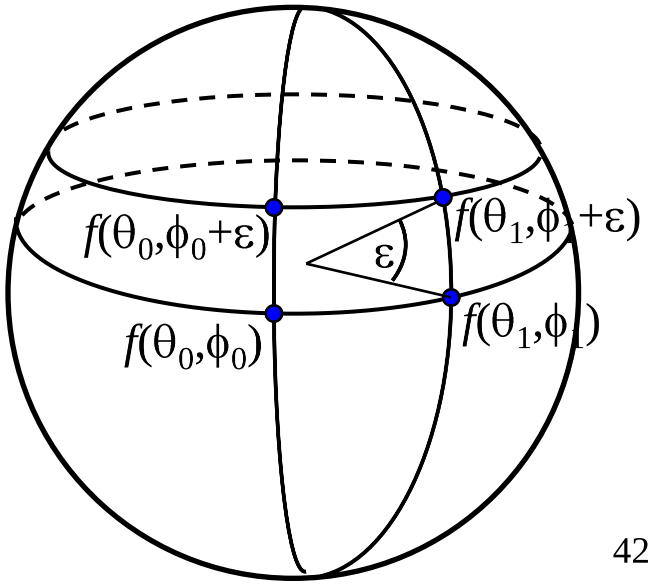
$$r(\phi) = \sin\phi$$



Spherical Gradients

How does the scale change as we change θ or ϕ by a value of ε?

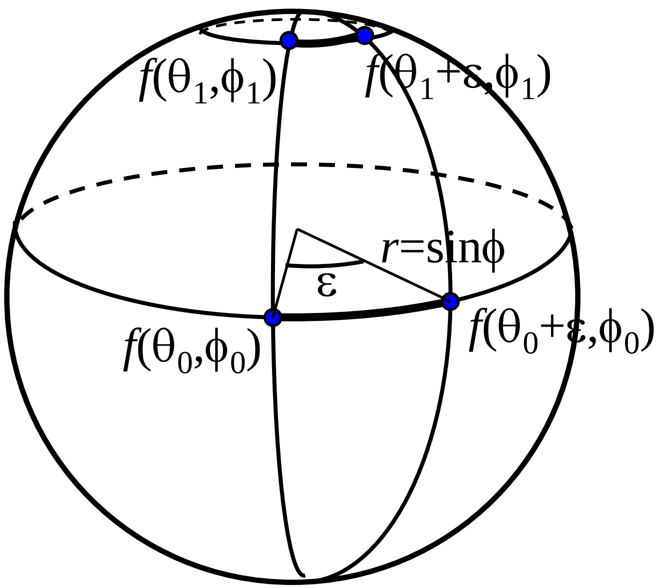
At the point $p=\Phi(\theta,\phi)$, changing the value of ϕ by ε, moves us a distance of ε along a great circle, regardless of where on the sphere we are:

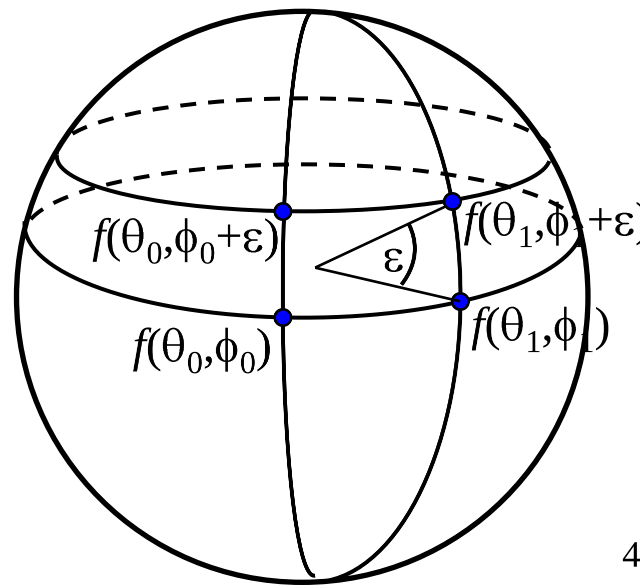


Spherical Gradients

Taking the scaling into account, we get:

$$\nabla f(\theta,\phi) \approx \left(\frac{f(\theta+\varepsilon,\phi) - f(\theta,\phi)}{\varepsilon \sin\phi}, \frac{f(\theta,\phi+\varepsilon) - f(\theta,\phi)}{\varepsilon} \right)$$





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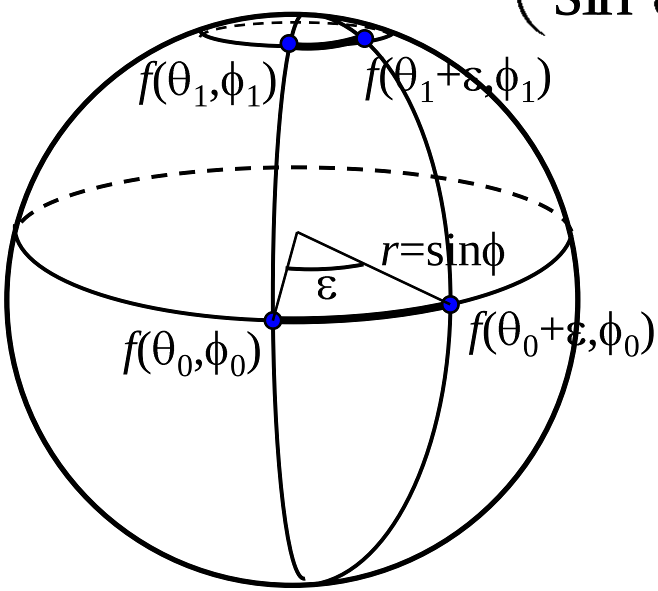
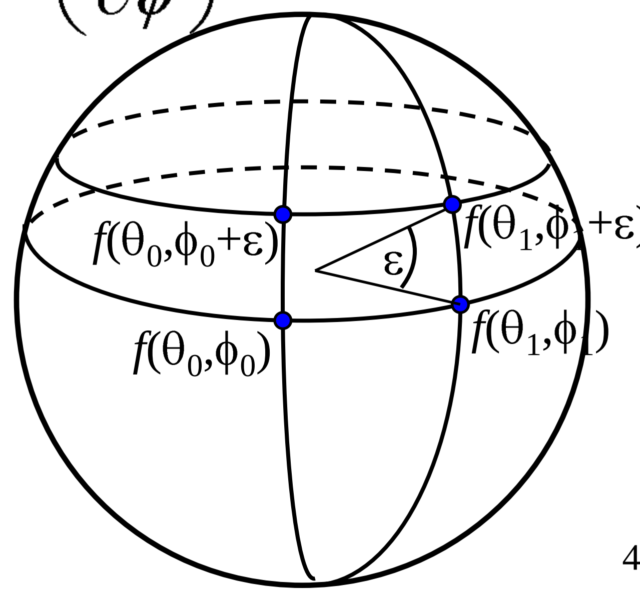
$$\nabla f(\theta,\phi) \approx \left(\frac{f(\theta+\varepsilon,\phi) - f(\theta,\phi)}{\varepsilon \sin\phi}, \frac{f(\theta,\phi+\varepsilon) - f(\theta,\phi)}{\varepsilon} \right)$$

$$\nabla f(\theta,\phi) = \left(\frac{1}{\sin\phi} \frac{\partial f}{\partial \theta} \right) \Phi_\theta + \left(\frac{\partial f}{\partial \phi} \right) \Phi_\phi$$





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Recall:

The Laplacian operator is self-adjoint (symmetric)

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$\Rightarrow$ If F_λ are the eigenfunctions of the Laplacian with eigenvalue λ, rotations fix F_λ.

$\Rightarrow$ The irreducible representations are subspaces of the F_λ.

The Spherical Laplacian

All this implies that for a fixed degree *l*, the spherical harmonics of degree *l*:

$$Y_l^k(\theta,\phi) = e^{ik\theta} P_l^k(\cos\phi)$$

($-l \le k \le l$) must be eigenvectors of the Laplacian with the same eigenvalue.

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1. What is the Laplacian?

2. What are the eigenvalues?

The Spherical Laplacian

How do we compute the Laplacian of a spherical function $f(\theta,\phi)$?

The Spherical Laplacian

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Recall:

The Laplacian of a function is the divergence of its gradient:

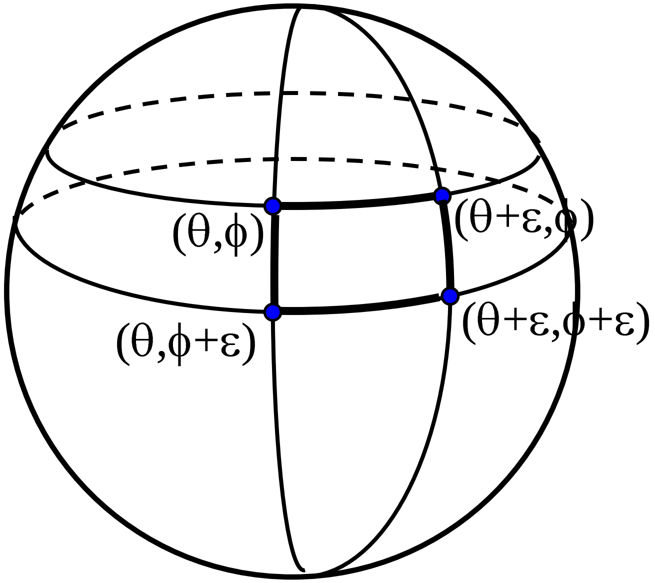
$$\nabla^2 f = \nabla \cdot (\nabla f)$$

The Spherical Laplacian

By Stokes' Theorem, we can compute the integral of the Laplacian (the divergence of the gradient) over a region by computing the surface integral of the gradient field over the boundary:

The Spherical Laplacian

Consider the "square" on the sphere with end-points (θ,ϕ), $(\theta+\varepsilon,\phi)$, $(\theta+\varepsilon,\phi+\varepsilon)$ and $(\theta,\phi+\varepsilon)$:

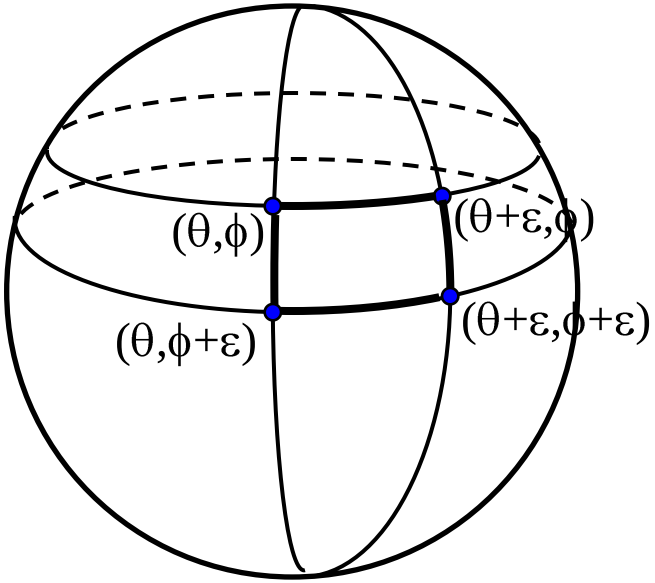


The Spherical Laplacian

Consider the "square" on the sphere with end-points (θ,ϕ), $(\theta+\varepsilon,\phi)$, $(\theta+\varepsilon,\phi+\varepsilon)$ and $(\theta,\phi+\varepsilon)$:

The integral of the Laplacian is approximately:

$$\int_R \left(\nabla^2 f\right) dR \approx \text{Area}(R)\nabla^2 f(\theta,\phi)$$



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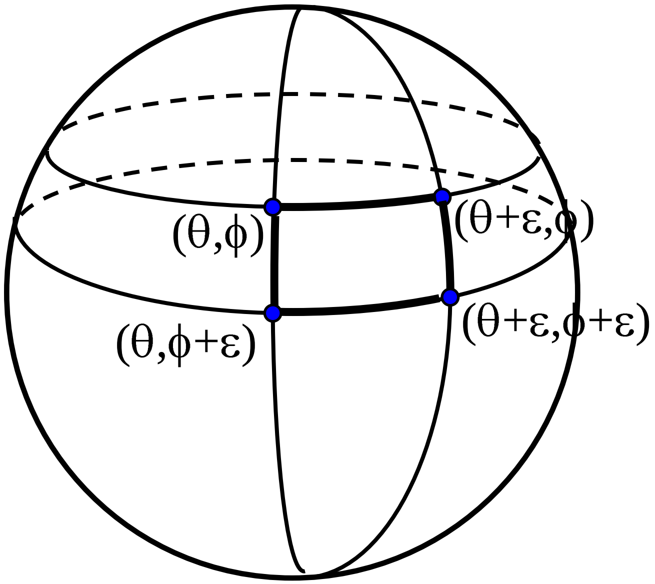
$$= \varepsilon^2 \sin\phi \, \nabla^2 f(\theta,\phi)$$

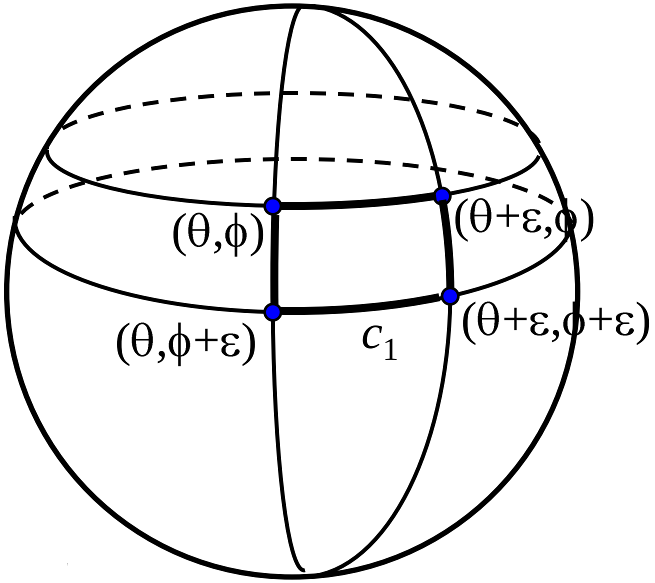


The Spherical Laplacian

Consider the "square" on the sphere with end-points (θ,ϕ), $(\theta+\varepsilon,\phi)$, $(\theta+\varepsilon,\phi+\varepsilon)$ and $(\theta,\phi+\varepsilon)$:

On the curve c_1, the surface integral of the gradient is approximately:

$$\int_{c_1} \langle \nabla f, \cdot \rangle \, dA \approx \text{Length}(c_1)\langle \nabla f, \Phi_\phi \rangle$$



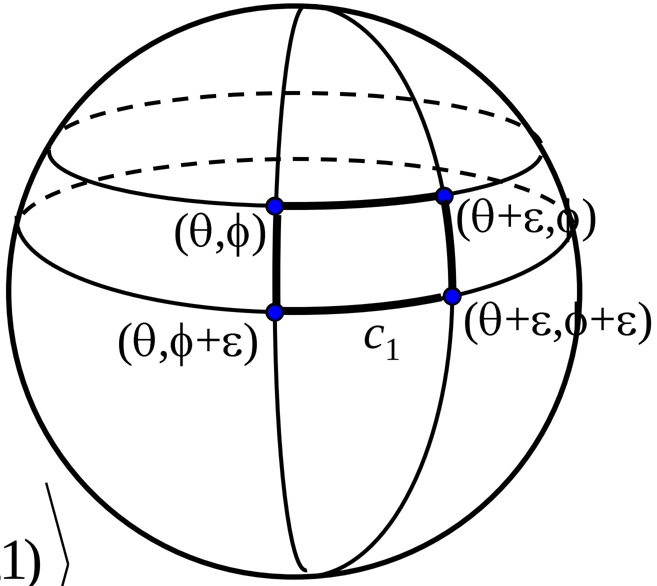
The Spherical Laplacian

Consider the “square” on the sphere with end-points (θ,ϕ), (θ+ ε,ϕ), (θ+ ε,ϕ + ε) and (θ,ϕ + ε):

On the curve c_1, the surface integral of the gradient is approximately:

$$\int_{c_1} (\nabla f)\cdot dA \approx \mathrm{Length}\,(c_1)\langle \nabla f,\Phi_\phi\rangle$$

$$=\varepsilon\sin(\phi+\varepsilon)\left\langle\left(\frac{1}{\sin(\phi+\varepsilon)}\frac{\partial f}{\partial\theta},\frac{\partial f}{\partial\phi}\right),(0,1)\right\rangle$$



The Spherical Laplacian

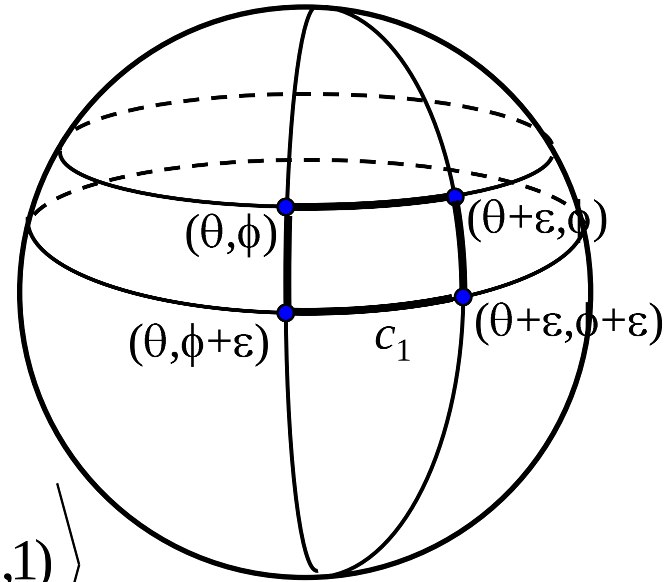
Consider the "square" on the sphere with end-points (θ,ϕ), (θ+ ε,ϕ), (θ+ ε,ϕ + ε) and (θ,ϕ + ε):

On the curve c_1, the surface integral of the gradient is approximately:

$$\int_{c_1} \left(\nabla f \right) \cdot dA \approx \text{Length}\,(c_1) \left\langle \nabla f , \Phi_{\phi} \right\rangle$$

$$= \varepsilon \sin(\phi + \varepsilon) \left\langle \left(\frac{1}{\sin(\phi + \varepsilon)} \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi} \right), (0,1) \right\rangle$$

$$= \varepsilon \sin(\phi + \varepsilon) \frac{\partial f}{\partial \phi} (\theta, \phi + \varepsilon)$$

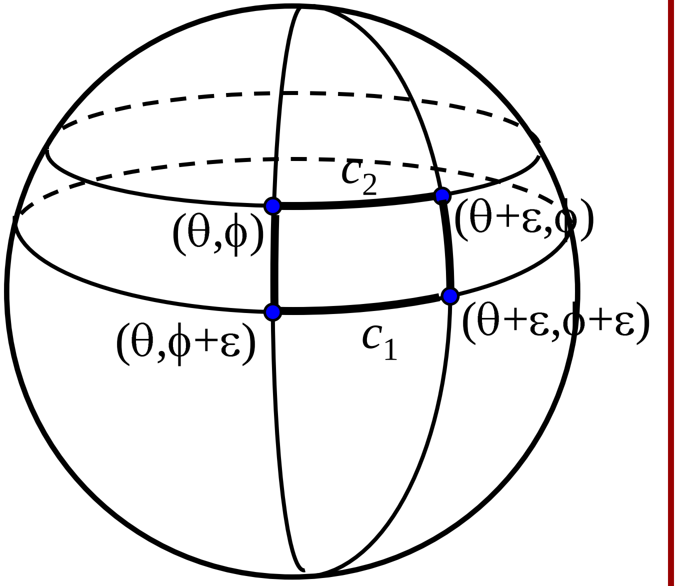


The Spherical Laplacian

Consider the “square” on the sphere with end-points (θ,ϕ), $(\theta+\varepsilon,\phi)$, $(\theta+\varepsilon,\phi+\varepsilon)$ and $(\theta,\phi+\varepsilon)$:

Similarly, on the curve c_2, the surface integral of the gradient is approximately:

$$\int_{c_2} \langle \nabla f \cdot \rangle \, dA \approx -\varepsilon \sin(\phi) \frac{\partial f}{\partial \phi}(\theta,\phi)$$

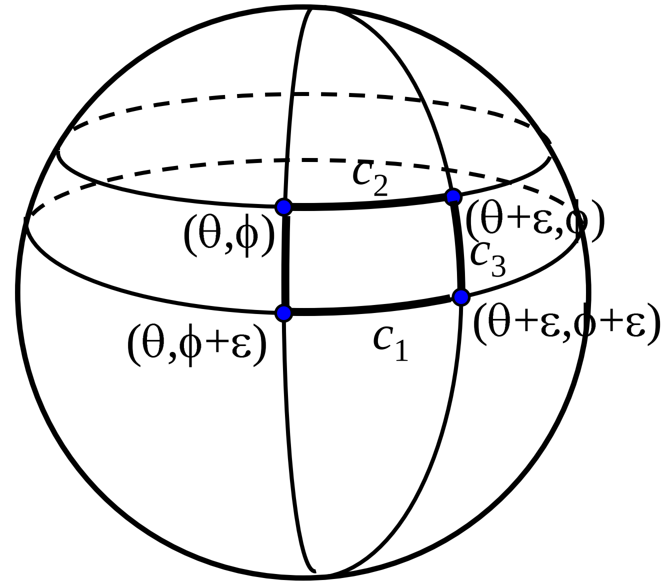


The Spherical Laplacian

Consider the "square" on the sphere with end-points (θ,ϕ), $(\theta+\varepsilon,\phi)$, $(\theta+\varepsilon,\phi+\varepsilon)$ and $(\theta,\phi+\varepsilon)$:

On the curve c_3, the surface integral of the gradient is approximately:

$$\int_{c_3} \langle \nabla f, \cdot \rangle \, dA \approx \text{Length}\,(c_3)\langle \nabla f, \Phi_\theta \rangle$$



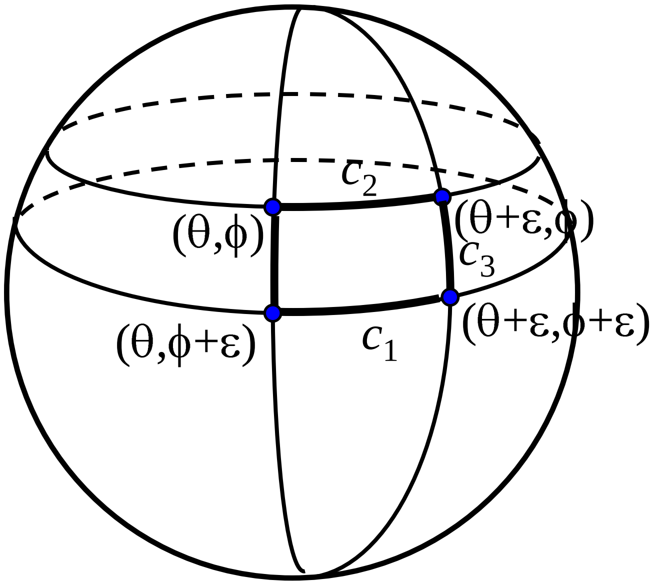
The Spherical Laplacian

Consider the “square” on the sphere with end-points (θ,ϕ), (θ+ ε,ϕ), (θ+ ε,ϕ + ε) and (θ,ϕ + ε):

On the curve c_3, the surface integral of the gradient is approximately:

$$\int_{c_3} \left\langle \nabla f, \cdot \right\rangle dA \approx \text{Length}\,(c_3)\left\langle \nabla f, \Phi_\theta \right\rangle$$

$$= \varepsilon \left\langle \left(\frac{1}{\sin(\phi)} \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi} \right), (1,0) \right\rangle$$



The Spherical Laplacian

Consider the “square” on the sphere with end-points (θ,ϕ), $(\theta+\varepsilon,\phi)$, $(\theta+\varepsilon,\phi+\varepsilon)$ and $(\theta,\phi+\varepsilon)$:

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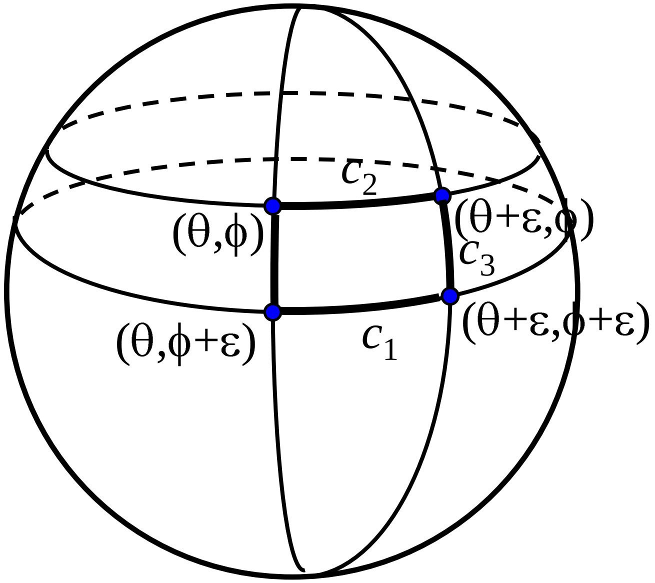
$$\int_{c_3} (\nabla f)\cdot dA \approx \text{Length}\,(c_3)\langle \nabla f, \Phi_\theta \rangle$$

$$= \varepsilon \left\langle \left(\frac{1}{\sin(\phi)}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi} \right), (1,0) \right\rangle$$

$$= \varepsilon \frac{1}{\sin(\phi)}\frac{\partial f}{\partial \theta}(\theta+\varepsilon,\phi)$$

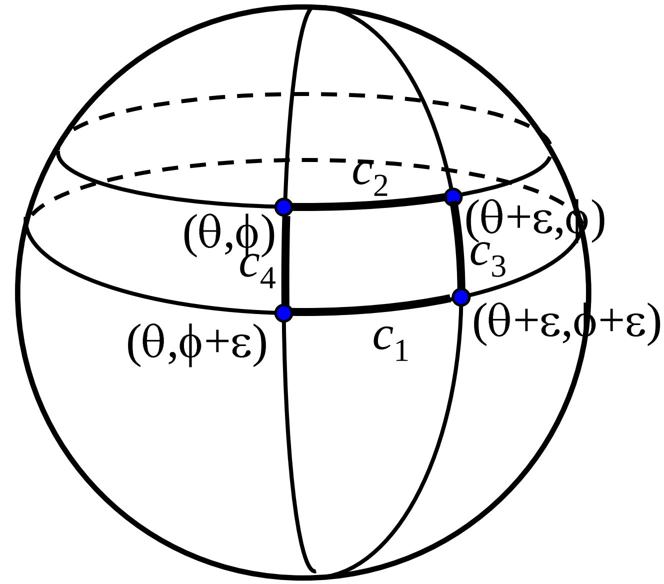


The Spherical Laplacian

Consider the “square” on the sphere with end-points (θ,ϕ), $(\theta+\varepsilon,\phi)$, $(\theta+\varepsilon,\phi+\varepsilon)$ and $(\theta,\phi+\varepsilon)$:

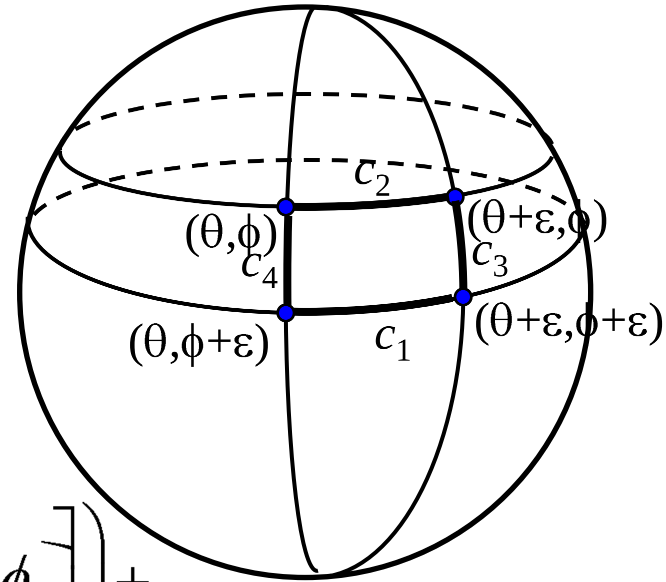
Similarly, on the curve c_4, the surface integral of the gradient is approximately:

$$\int_{c_4} \left(\nabla f\right) \cdot dA \approx -\varepsilon \frac{1}{\sin(\phi)} \frac{\partial f}{\partial \theta}\left(\theta,\phi\right)$$



The Spherical Laplacian

Consider the "square" on the sphere with end-points (θ,ϕ), (θ+ ε,ϕ), (θ+ ε,ϕ + ε) and (θ,ϕ + ε):



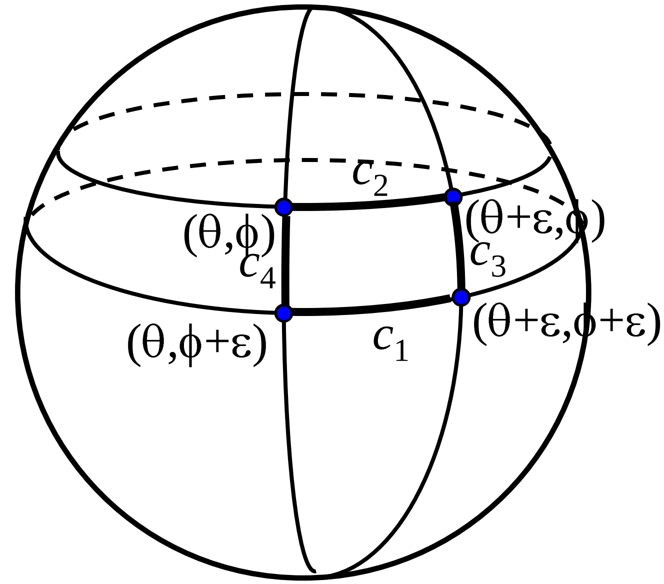
Summing these together, we can approximate the boundary integral by:

$$\int_{\partial R} \langle \nabla f, \cdot \rangle \, dA \approx \varepsilon\left(\frac{1}{\sin(\phi)}\left[\frac{\partial f}{\partial \theta}(\theta+\varepsilon,\phi)-\frac{\partial f}{\partial \theta}(\theta,\phi)\right]\right)+$$

$$\varepsilon\left(\sin(\phi+\varepsilon)\frac{\partial f}{\partial \phi}(\theta,\phi+\varepsilon)-\sin(\phi)\frac{\partial f}{\partial \phi}(\theta,\phi)\right)$$

The Spherical Laplacian

Consider the "square" on the sphere with end-points (θ,ϕ), $(\theta+\varepsilon,\phi)$, $(\theta+\varepsilon,\phi+\varepsilon)$ and $(\theta,\phi+\varepsilon)$:

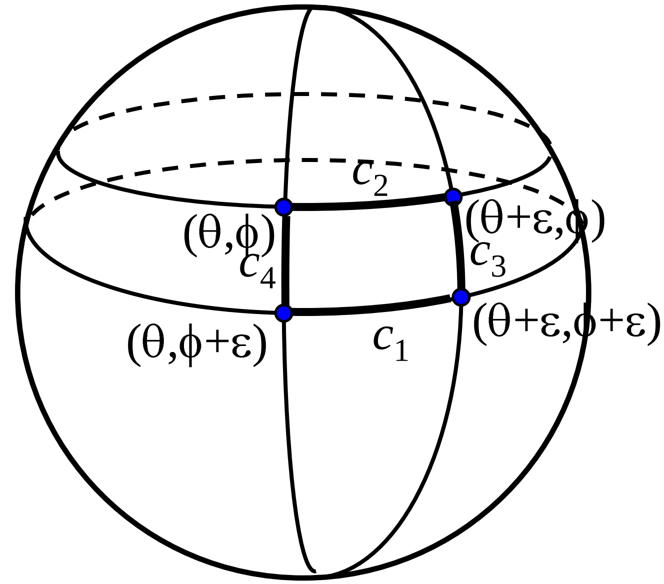


Summing these together, we can approximate the boundary integral by:

$$\int_{\partial R} \langle \nabla f \rangle \cdot dA \approx \varepsilon\left(\frac{1}{\sin(\phi)}\varepsilon\frac{\partial}{\partial\theta}\left[\frac{\partial f}{\partial\theta}(\theta,\phi)\right]\right)+\varepsilon\left(\varepsilon\frac{\partial}{\partial\phi}\left[\sin(\phi)\frac{\partial f}{\partial\phi}(\theta,\phi)\right]\right)$$

The Spherical Laplacian

Consider the "square" on the sphere with end-points (θ,ϕ), (θ+ ε,ϕ), (θ+ ε,ϕ + ε) and (θ,ϕ + ε):



Summing these together, we can approximate the boundary integral by:

$$\int_{\partial R} \langle \nabla f, \cdot \rangle \, dA \approx \varepsilon^2 \left(\frac{1}{\sin\phi} \frac{\partial^2 f}{\partial \theta^2}(\theta,\phi) + \frac{\partial}{\partial \phi}\left[\sin\phi \frac{\partial f}{\partial \phi}(\theta,\phi) \right] \right)$$

The Spherical Laplacian

Using the fact that the boundary integral can be approximated by:

$$\int_{\partial R} \left(\nabla f \cdot\right) dA \approx \varepsilon^2 \left(\frac{1}{\sin\phi} \frac{\partial^2 f}{\partial \theta^2}(\theta,\phi) + \frac{\partial}{\partial \phi}\left[\sin\phi \frac{\partial f}{\partial \phi}(\theta,\phi) \right] \right)$$

and the surface integral can be approximated by:

$$\int_{R} \left(\nabla^2 f\right) dR \approx \varepsilon^2 \sin\phi \, \nabla^2 f(\theta,\phi)$$

The Spherical Laplacian

Using the fact that the boundary integral can be approximated by:

$$\int_{\partial R} (\nabla f)\cdot dA \approx \varepsilon^2\left(\frac{1}{\sin\phi}\frac{\partial^2 f}{\partial\theta^2}(\theta,\phi)+\frac{\partial}{\partial\phi}\left[\sin\phi\,\frac{\partial f}{\partial\phi}(\theta,\phi)\right]\right)$$

and the surface integral can be approximated by:

$$\int_{R} (\nabla^2 f)\, dR \approx \varepsilon^2 \sin\phi\, \nabla^2 f(\theta,\phi)$$

we can apply Stokes' Theorem to get:

$$\boxed{\nabla^2 f(\theta,\phi)=\frac{1}{\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}(\theta,\phi)+\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left[\sin\phi\,\frac{\partial f}{\partial\phi}(\theta,\phi)\right]}$$

The Spherical Laplacian

$$\nabla^2 f(\theta,\phi) = \frac{1}{\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}(\theta,\phi) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left[\sin\phi\frac{\partial f}{\partial\phi}(\theta,\phi)\right]$$

In order to compute the eigenvalues of the Laplacian associated to the the spherical harmonics, we need to compute:

$$\nabla^2 Y_l^k(\theta,\phi) = \nabla^2\left(e^{ik\theta}P_l^k(\cos\phi)\right)$$

The Spherical Laplacian

$$\nabla^2 f(\theta,\phi) = \frac{1}{\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}(\theta,\phi) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left[\sin\phi\frac{\partial f}{\partial\phi}(\theta,\phi)\right]$$

Taking the derivative with respect to θ is easy:

$$\frac{1}{\sin^2\phi}\frac{\partial^2 Y_l^k(\theta,\phi)}{\partial\theta^2} = \frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2}\left(e^{ik\theta}P_l^k(\cos\phi)\right)$$

The Spherical Laplacian

$$\nabla^2 f(\theta,\phi) = \frac{1}{\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}(\theta,\phi) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left[\sin\phi\frac{\partial f}{\partial\phi}(\theta,\phi)\right]$$

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$$= \frac{-k^2}{\sin^2\phi}e^{ik\theta}P_l^k(\cos\phi)$$

The Spherical Laplacian

$$\nabla^2 f(\theta,\phi) = \frac{1}{\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}(\theta,\phi) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left[\sin\phi\frac{\partial f}{\partial\phi}(\theta,\phi)\right]$$

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$$= \frac{-k^2}{\sin^2\phi}e^{ik\theta}P_l^k(\cos\phi)$$

$$= \frac{-k^2}{\sin^2\phi}Y_l^k(\theta,\phi)$$

The Spherical Laplacian

$$\nabla^2 f(\theta,\phi) = \frac{1}{\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}(\theta,\phi) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left[\sin\phi\frac{\partial f}{\partial\phi}(\theta,\phi)\right]$$

Taking the derivative with respect to ϕ is more complicated:

$$\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial Y_l^k(\theta,\phi)}{\partial\phi}\right) = \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\left[e^{ik\theta}P_l^k(\cos\phi)\right]\right)$$

The Spherical Laplacian

$$\nabla^2 f(\theta,\phi) = \frac{1}{\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}(\theta,\phi) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left[\sin\phi\frac{\partial f}{\partial\phi}(\theta,\phi)\right]$$

Taking the derivative with respect to ϕ is more complicated:

$$\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial Y_l^k(\theta,\phi)}{\partial\phi}\right) = \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\left[e^{ik\theta}P_l^k(\cos\phi)\right]\right)$$

$$= \frac{e^{ik\theta}}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial P_l^k(\cos\phi)}{\partial\phi}\right)$$

as it requires taking the derivatives of the associated Legendre polynomials.

Associated Legendre Polynomials

Recall:

The associated Legendre polynomials are defined by the (somewhat hairy) formula:

$$P_l^k(x) = \frac{(-1)^k}{2^l\, l!}\left(1-x^2\right)^{k/2}\frac{d^{\,l+k}}{dx^{l+k}}\left(x^2-1\right)^l$$

Associated Legendre Polynomials

One can show, (but we won't) that the associated Legendre polynomials satisfy the following two identities:

$$\frac{dP_l^k(\cos\phi)}{d\phi} = \frac{l\cos(\phi)P_l^k(\cos\phi) - (l+k)P_{l-1}^k(\cos\phi)}{\sin\phi}$$

$$0 = (l-k)P_l^k(\cos\phi) - \cos\phi(2l-1)P_{l-1}^k(\cos\phi) + (l+k-1)P_{l-2}^k(\cos\phi)$$

The Spherical Laplacian

Plugging these identities into the equation for the Laplacian, we get:

$$\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial Y_l^k(\theta,\phi)}{\partial\phi}\right)=\frac{k^2 Y_l^m(\theta,\phi)}{\sin^2\phi}-l(l+1)Y_l^m(\theta,\phi)$$

The Spherical Laplacian

Plugging these identities into the equation for the Laplacian, we get:

$$\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial Y_l^k(\theta,\phi)}{\partial\phi}\right)=\frac{-k^2Y_l^m(\theta,\phi)}{\sin^2\phi}-l(l+1)Y_l^m(\theta,\phi)$$

In sum, this gives:

$$\boxed{\nabla^2 Y_l^k(\theta,\phi)=-l(l+1)Y_l^k(\theta,\phi)}$$

Ouline

- Stokes' Theorem
- Tangent Spaces
- Gradients
- The Spherical Laplacian
- Applications

Smoothing

In the case of a functions on a plane, we had Newton's Law of Cooling:

"*The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.*"

Smoothing

In the case of a functions on a plane, we had Newton's Law of Cooling:

"*The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.*"

This can be expresses as a PDE:

$$\frac{\partial F}{\partial t} = \lambda \nabla^2 F$$

Smoothing

Using the fact that the spherical harmonics are eigenvectors of the Laplacian, we get solutions of the form:

$$F_l^k(\theta,\phi,t) = e^{-\lambda l(l+1)t} Y_l^k(\theta,\phi)$$

Smoothing

Using the fact that the spherical harmonics are eigenvectors of the Laplacian, we get solutions of the form:

$$F_l^k(\theta,\phi,t) = e^{-\lambda l(l+1)t} Y_l^k(\theta,\phi)$$

Since the linear sum of solutions is also a solution, solutions to the PDE will have the form:

$$F(\theta,\phi,t) = \sum_{l=0}^{\infty} \sum_{k=-l}^{l} a_l^k e^{-\lambda l(l+1)t} Y_l^k(\theta,\phi)$$

and we have freedom in choosing the linear coefficients.

Smoothing

To satisfy the initial condition:

$$F(\theta,\phi,0) = f(\theta,\phi)$$

we need to compute the spherical harmonic decomposition of *f*:

$$f(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{k=-l}^{l} \hat{f}(l,k) Y_l^k(\theta,\phi)$$

Smoothing

To satisfy the initial condition:

$$F(\theta,\phi,0) = f(\theta,\phi)$$

we need to compute the spherical harmonic decomposition of *f*:

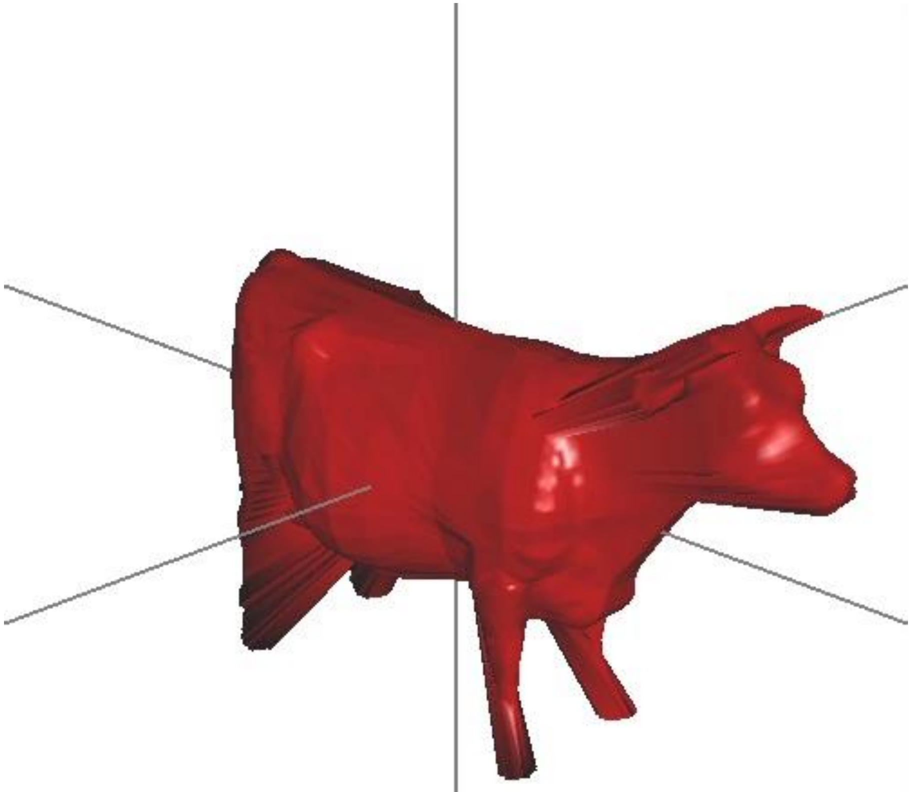
$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{k=-l}^{l}\hat{f}(l,k)Y_l^k(\theta,\phi)$$

and then we set the solution to be:

$$F(\theta,\phi,t) = \sum_{l=0}^{\infty}\sum_{k=-l}^{l}\hat{f}(l,k)e^{-\lambda l(l+1)t}Y_l^k(\theta,\phi)$$

Smoothing

$$F(\theta,\phi,t)=\sum_{l=0}^{\infty}\sum_{k=-l}^{l}\hat{f}(l,k)e^{-\lambda l(l+1)t}Y_l^k(\theta,\phi)$$



Cooling Cow

The Spherical Wave Equation

We can repeat the same type of argument for the wave equation, where the acceleration is proportional to the difference in values:

$$\frac{\partial^2 F}{\partial t^2} = \lambda \nabla^2 F$$

The Spherical Wave Equation

Again, using the fact that the spherical harmonics Y_l^k are eigenvectors of the Laplacian with eigenvalues $l(l+1)$, we get solutions of the form:

$$F_l^{k+}(\theta,\phi,t) = e^{i\sqrt{\lambda l(l+1)}t} \cdot Y_l^m(\theta,\phi)$$

$$F_l^{k-}(\theta,\phi,t) = e^{-i\sqrt{\lambda l(l+1)}t} \cdot Y_l^m(\theta,\phi)$$

The Spherical Wave Equation

Thus, given the initial conditions:

$$F(\theta,\phi,0) = f(\theta,\phi)$$

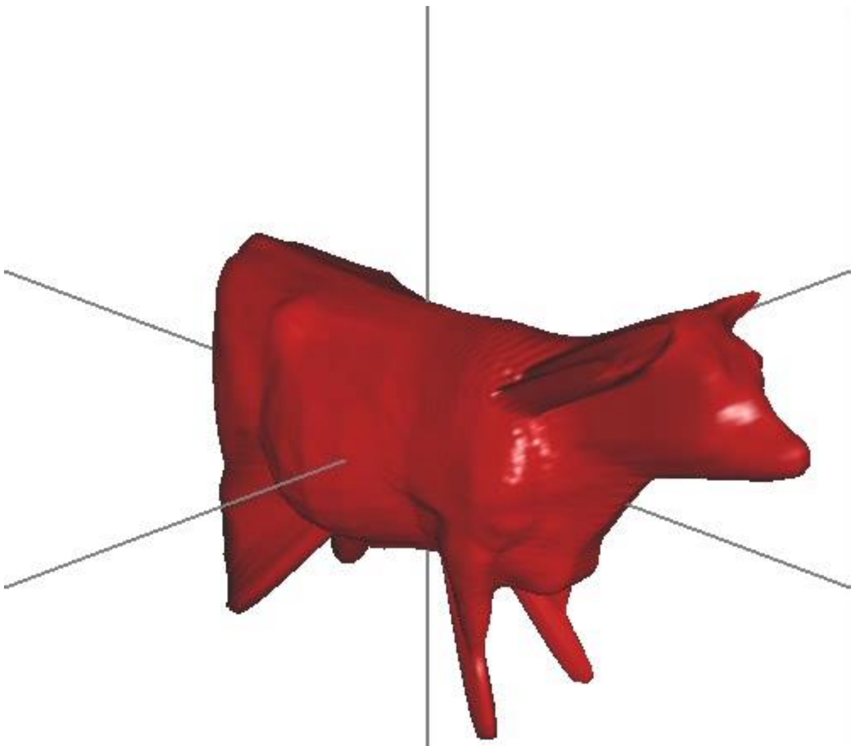
$$\frac{\partial}{\partial t} F(\theta,\phi,0) = 0$$

we get the solution:

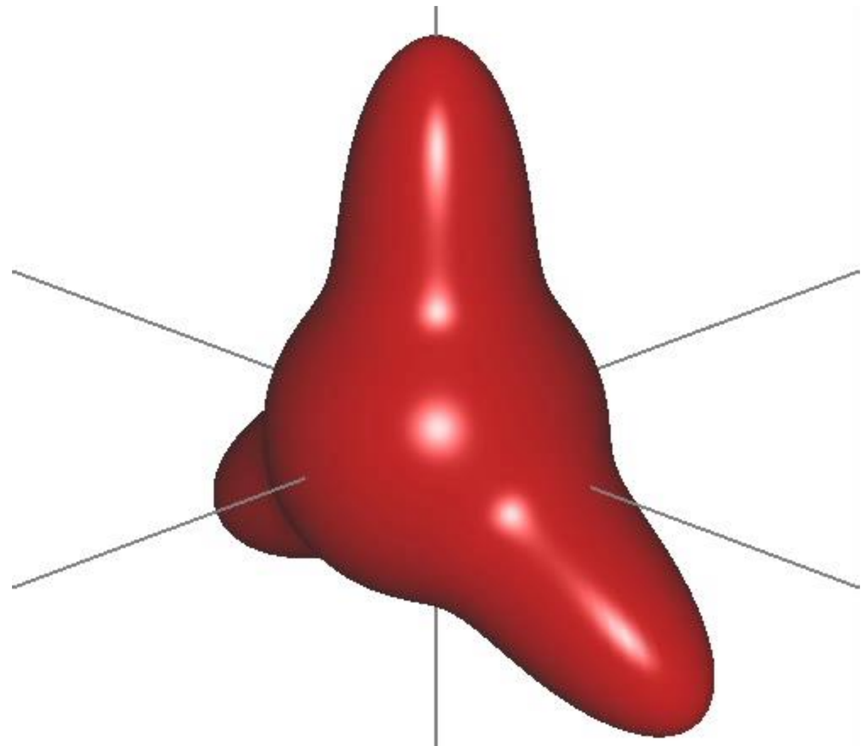
$$F(\theta,\phi,t) = \sum_{l=0}^{\infty} \sum_{k=-l}^{l} \hat{f}(k,l) \cos\left(\sqrt{\lambda l(l+1)}\,t\right) Y_l^k(\theta,\phi)$$

The Spherical Wave Equation

$$F(\theta,\phi,t)=\sum_{l=0}^{\infty}\sum_{k=-l}^{l}\hat{f}(k,l)\cos\left(\sqrt{\lambda l(l+1)}\,t\right)Y_l^k(\theta,\phi)$$



Waving Cow



Waving Gaussians