FFTs in Graphics and Vision

Spherical Harmonics
and
Legendre Polynomials

Outline

Math Stuff

- Gram-Schmidt Orthogonalization
- Height Functions
- Completing Homogenous Polynomials

Review

Defining the Harmonics

Gram–Schmidt Orthogonalization

Given an inner product space V, and given a basis $\{v_1,\ldots,v_n\}$ we can define an orthonormal basis $\{w_1,\ldots,w_n\}$ for V:

$$\left\langle w_i, w_j \right\rangle = \delta_{ij}$$

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<u>Algorithm</u>:

Start by making v_1 a unit vector:

$$w_1 = \frac{v_1}{\|v_1\|}$$

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$$\langle w_i, w_j \rangle = \delta_{ij}$$

Algorithm:

To get the 2nd basis element, subtract off from v_2 the w_1 component and then normalize:

$$w_2 = \frac{v_2 - \langle v_2, w_1 \rangle w_1}{\left\| v_2 - \langle v_2, w_1 \rangle w_1 \right\|}$$

Gram–Schmidt Orthogonalization

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$$\left\langle w_i, w_j \right\rangle = \delta_{ij}$$

<u>Algorithm</u>:

To get the *i*-th basis element, subtract off all the earlier components from v_i and then normalize:

$$w_i = \frac{v_i - \left\langle v_i, w_{i-1} \right\rangle w_{i-1} - \cdots - \left\langle v_i, w_1 \right\rangle w_1}{\left\| v_i - \left\langle v_i, w_{i-1} \right\rangle w_{i-1} - \cdots - \left\langle v_i, w_1 \right\rangle w_1 \right\|}$$

Gram–Schmidt Orthogonalization

Example:

Consider the space of polynomial functions of degree *N* on the interval [-1,1], with the standard inner-product:

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x) g(x) dx$$

We would like to obtain an orthogonal basis:

$$\{ p_0(x), \cdots, p_N(x) \}$$

Gram–Schmidt Orthogonalization

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We would like to obtain an orthogonal basis:

$$\{p_0(x), \cdots, p_N(x)\}$$

An easy basis to start with is the monomials:

$$\{1, x, x^2, \ldots, x^N\}$$

Gram–Schmidt Orthogonalization

Example:

Starting with the constant term, we get:

$$p_0(x) = \frac{1}{\|1\|}$$

Gram–Schmidt Orthogonalization

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Starting with the constant term, we get:

$$p_0(x) = \frac{1}{\|1\|} = \frac{1}{\sqrt{\int_{-1}^{1} dx}}$$

Gram–Schmidt Orthogonalization

Example:

Starting with the constant term, we get:

$$p_0(x) = \frac{1}{\|1\|} = \frac{1}{\sqrt{\int_{-1}^{1} dx}}$$

$$= \frac{1}{\sqrt{2}}$$

Gram–Schmidt Orthogonalization

Example:

Moving on to the linear term, we get:

$$p_1(x) = \frac{x - \langle x, p_0(x) \rangle p_0(x)}{\left\| x - \langle x, p_0(x) \rangle p_0(x) \right\|}$$

Gram–Schmidt Orthogonalization

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$$p_1(x) = \frac{x - \langle x, p_0(x) \rangle p_0(x)}{\left\| x - \langle x, p_0(x) \rangle p_0(x) \right\|}$$

$$= \frac{x - \left(\int_{-1}^{1} x \frac{1}{\sqrt{2}} dx \right) \frac{1}{\sqrt{2}}}{\left\| x - \langle x, p_0(x) \rangle p_0(x) \right\|}$$

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$$= \frac{x - \left(\int_{-1}^{1} x \frac{1}{\sqrt{2}} dx \right) \frac{1}{\sqrt{2}}}{\left\| x - \langle x, p_0(x)\rangle p_0(x) \right\|}$$

$$= \frac{x}{\sqrt{\int_{-1}^{1} x^2 dx}} = \sqrt{\frac{3}{2}} x$$

Gram–Schmidt Orthogonalization

Example:

And the quadratic term:

$$p_2(x) = \frac{x^2 - \langle x^2, p_1(x)\rangle p_1(x) - \langle x^2, p_0(x)\rangle p_0(x)}{\left\| x^2 - \langle x^2, p_1(x)\rangle p_1(x) - \langle x^2, p_0(x)\rangle p_0(x) \right\|}$$

Gram–Schmidt Orthogonalization

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These polynomials are called the *Legendre Polynomials*.

Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials have the same parity as *k*:

$$p_k(x) = a_k x^k + a_{k-2} x^{k-2} + \ldots$$

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Proof by Induction:

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$$p_k(x) = a_k x^k + a_{k-2} x^{k-2} + \ldots$$

Proof by Induction (*k*=0):

$$p_0(x) = \frac{1}{\sqrt{2}}$$

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$$p_k(x) = a_k x^k + a_{k-2} x^{k-2} + \ldots$$

Proof by Induction (assume true for *k*=*n*):

$$p_{n+1}(x) = \frac{x^{n+1} - \left\langle x^{n+1}, p_n(x) \right\rangle p_n(x) - \cdots - \left\langle x^{n+1}, p_0(x) \right\rangle p_0(x)}{\left\| x^{n+1} - \left\langle x^{n+1}, p_n(x) \right\rangle p_n(x) - \cdots - \left\langle x^{n+1}, p_0(x) \right\rangle p_0(x) \right\|}$$

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Recall that:

$$\langle x^{n+1}, p_m(x) \rangle = \int_{-1}^{1} x^{n+1} p_m(x) dx$$

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Proof by Induction:

$$\left\langle x^{n+1}, p_m(x) \right\rangle = \int_{-1}^{1} x^{n+1} p_m(x) dx$$

Since $m \leq n$ we can assume that the monomials comprising $p_m(x)$ are all even if *m* is even and all odd if *m* is odd.

Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials have the same parity as *k*:

$$p_k(x) = a_k x^k + a_{k-2} x^{k-2} + \ldots$$

Proof by Induction:

So if *n* and *m* are both either odd or both even, the polynomial $x^{n+1} p_m(x)$ is comprised of strictly odd-powered monomials:

$$\left\langle x^{n+1}, p_m(x) \right\rangle = \int_{-1}^{1} x^{n+1} p_m(x) dx = 0$$

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Proof by Induction:

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$$p_{n+1}(x) = \frac{x^{n+1} - \left\langle x^{n+1}, p_{n-1}(x) \right\rangle p_{n-1}(x) - \left\langle x^{n+1}, p_{n-3}(x) \right\rangle p_{n-3}(x) - \cdots}{\left\| x^{n+1} - \left\langle x^{n+1}, p_{n-1}(x) \right\rangle p_{n-1}(x) - \left\langle x^{n+1}, p_{n-3}(x) \right\rangle p_{n-3}(x) - \cdots \right\|}$$

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So $p_{n+1}(x)$ is obtained by starting with the monomial x^{n+1} and subtracting off monomials with the same parity.

Gram–Schmidt Orthogonalization

Example:

Consider the space of polynomials of degree *N* on the interval [-1,1], with an inner-product giving more weight to points near the origin:

$$\langle f(x), g(x)\rangle_m = \int_{-1}^{1} \left(1 - x^2\right)^m f(x) g(x)\, dx$$

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We would like to obtain an orthogonal basis:

$$\left\{p_0^m(x), \cdots, p_N^m(x)\right\}$$

Gram–Schmidt Orthogonalization

<u>Example</u>:

$$\left\langle f(x), g(x) \right\rangle_m = \int_{-1}^{1} \left(1 - x^2\right)^m f(x) g(x) dx$$

We proceed exactly as before but now using the new inner-product.

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We proceed exactly as before but now using the new inner-product.

Since the weighting function is even, if *f* is an even function and *g* is an odd function (or vice-versa), the inner product must be zero:

$$\left\langle f(x), g(x) \right\rangle_m = 0$$

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Since the weighting function is even, if *f* is an even function and *g* is an odd function (or vice-versa), the inner product must be zero:

$$\left\langle f(x), g(x) \right\rangle_m = 0$$

Thus, as before, the degree of the monomials comprising $p_l^m(x)$ must all have the same parity.

Height Functions

Given a function $f(x)$ on the interval [-1,1], we can turn the function into a function on the sphere:

$$F(x,y,z)=\frac{f(y)}{\sqrt{2\pi}}$$

Height Functions

Given a function *f*(*x*) on the interval [-1,1], we can turn the function into a function on the sphere:

$$F(x,y,z)=\frac{f(y)}{\sqrt{2\pi}}$$

Given two such functions *f*(*x*) and *g*(*y*), the mapping from 1D functions to spherical functions preserves inner products:

$$\int_{|p|=1} F(p)\cdot G(p)dp=\int_{-1}^{-1} f(y)\cdot g(y)dy$$

Height Functions

To see this, we can parameterize points on the sphere by spherical angle:

$$\Phi(\theta,\phi)=\left(\cos\theta\sin\phi,\cos\phi,\sin\theta\sin\phi\right)$$

Then the integral on the left-hand side becomes:

$$\int_{|p|=1} F(p)\cdot G(p)dp=\int_0^{\pi}\int_0^{2\pi} F\left(\Phi(\theta,\phi)\right)\cdot G\left(\Phi(\theta,\phi)\right)d\theta\sin(\phi)d\phi$$

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$$=\frac{1}{2\pi}\int_0^{\pi}\int_0^{2\pi}f\left(\cos\phi\right)\cdot g\left(\cos\phi\right)d\theta\sin(\phi)d\phi$$

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$$=\int_0^{\pi} f\left(\cos\phi\right)\cdot g\left(\cos\phi\right)\sin(\phi)d\phi$$

Height Functions

On the other hand, setting y=cosϕ, the left-hand side can be expressed as:

$$\int_{-1}^{1} f(y)\cdot g(y)dy = \int_{\pi}^{0} f(\cos\phi)\cdot g(\cos\phi)\frac{dy}{d\phi}d\phi$$

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$$= \int_{\pi}^{0} f(\cos\phi)\cdot g(\cos\phi)\big(-\sin(\phi)\big)d\phi$$

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$$= \int_{0}^{\pi} f(\cos\phi)\cdot g(\cos\phi)\sin(\phi)d\phi$$

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$$= \int_{0}^{\pi} f(\cos\phi)\cdot g(\cos\phi)\sin(\phi)d\phi$$

$$= \int_{|p|=1} F(p)\cdot G(p)dp$$

Completing Homogenous Polynomials

Given a polynomial $p(x,y,z)$ of degree d, consisting of monomials of powers d,d-2,d,…:

$$p(x,y,z)=\sum_{k=0}^{\lfloor d/2\rfloor}\left(\sum_{l+m+n=d-2k} a_{lmn}x^l y^m z^n\right)$$

Completing Homogenous Polynomials

Given a polynomial $p(x,y,z)$ of degree d, consisting of monomials of powers $d,d\text{-}2,d,\ldots$:

$$p(x,y,z)=\sum_{k=0}^{\lfloor d/2\rfloor}\left(\sum_{l+m+n=d-2k}a_{lmn}x^l y^m z^n\right)$$

This is *not* a homogenous polynomial.

Completing Homogenous Polynomials

Given a polynomial *p*(*x*,*y*,*z*) of degree *d*, consisting of monomials of powers *d*,*d*-2,*d*,…:

$$p(x,y,z)=\sum_{k=0}^{\lfloor d/2\rfloor}\left(\sum_{l+m+n=d-2k}a_{lmn}x^l y^m z^n\right)$$

This is *not* a homogenous polynomial.

However, if we restrict it to the sphere, we *can* think of it as homogenous:

$$p(x,y,z)\rightarrow\sum_{k=0}^{\lfloor d/2\rfloor}\left(x^2+y^2+z^2\right)^k\left(\sum_{l+m+n=d-2k}a_{lmn}x^l y^m z^n\right)$$

Completing Homogenous Polynomials

Example:

$$p(x, y, z) = x^2 y + y + z$$

Is not a homogenous polynomial.

Completing Homogenous Polynomials

<u>Example</u>:

$$p(x,y,z) = x^2 y + y + z$$

Is not a homogenous polynomial.

But on the unit-sphere, the polynomial:

$$q(x,y,z) = x^2 y + (y+z)\left(x^2 + y^2 + z^2\right)$$

has identical values and is homogenous of degree 3.

Outline

Math Stuff

Review

- Spherical Harmonics

Defining the Harmonics

Spherical Harmonics

For each non-negative integer l, there are $2l+1$ spherical harmonics of degree l satisfying:

1. Each spherical harmonic of degree l can be expressed as the restriction of a homogenous polynomial of degree l to the unit-sphere.
2. The different spherical harmonics are orthogonal to each other.

Spherical Harmonics

We had seen that by considering just the rotations about the *y*-axis, we could factor the spherical harmonics as:

$$Y_l^m(\theta,\phi) = e^{im\theta} P_l^m(\phi)$$

$$= (\cos\theta + i\sin\theta)^m P_l^m(\phi)$$

where $|m| \leq l$.

Outline

Math Stuff

Review

Defining the Harmonics

Defining the Harmonics (m ≥0)

To define the spherical harmonics, we would like to express the function:

$$Y_l^m(\theta,\phi) = (\cos\theta + i\sin\theta)^m P_l^m(\phi)$$

as the restriction of a homogenous polynomial of degree *l* to the unit sphere.

Defining the Harmonics (m ≥0)

Using the parameterization of the unit-sphere

$$\Phi(\theta,\phi)=\left(\cos\theta\sin\phi,\cos\phi,\sin\theta\sin\phi\right)$$

we get:

$$Y_l^m\left(\theta,\phi\right)=\left(\frac{x}{\sin\phi}+i\frac{z}{\sin\phi}\right)^m P_l^m(\phi)$$

Defining the Harmonics (m ≥0)

Using the parameterization of the unit-sphere

$$\Phi(\theta,\phi)=\left(\cos\theta\sin\phi,\cos\phi,\sin\theta\sin\phi\right)$$

we get:

$$Y_l^m(\theta,\phi)=\left(\frac{x}{\sin\phi}+i\frac{z}{\sin\phi}\right)^m P_l^m(\phi)$$

$$=\left(x+iz\right)^m\frac{P_l^m(\phi)}{\sin^m\phi}$$

Defining the Harmonics (m ≥0)

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$$\Phi(\theta,\phi)=\left(\cos\theta\sin\phi,\cos\phi,\sin\theta\sin\phi\right)$$

we get:

$$Y_l^m(\theta,\phi)=\left(\frac{x}{\sin\phi}+i\frac{z}{\sin\phi}\right)^m P_l^m(\phi)$$

$$=\left(x+iz\right)^m\frac{P_l^m(\phi)}{\sin^m\phi}$$

$$=\left(x+iz\right)^m\frac{P_l^m(\cos^{-1}y)}{\left(\sqrt{1-y^2}\right)^m}$$

Defining the Harmonics (m ≥0)

$$Y_l^m(\theta,\phi) = \boxed{(x+iz)^m} \frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1-y^2}\right)^m}$$

This • is a homogenous polynomial of degree *m*.

Defining the Harmonics (m $\geq$0)

$$Y_l^m(\theta,\phi) = \boxed{(x+iz)^m} \boxed{\frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1-y^2}\right)^m}}$$

This • is a homogenous polynomial of degree *m*.

So we want:

1. This • to complete to a homogenous polynomial of degree *l*-*m*.

Defining the Harmonics (m ≥0)

$$Y_l^m(\theta,\phi) = \boxed{(x+iz)^m} \boxed{\frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1-y^2}\right)^m}}$$

This (first box) is a homogenous polynomial of degree *m*.

So we want:

1. This (second box) to complete to a homogenous polynomial of degree *l*-*m*.

2. The different Y_l^m to be orthogonal.

Defining the Harmonics (m ≥0)

$$Y_l^m(\theta,\phi) = (x+iz)^m \frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1-y^2}\right)^m}$$

Homogeneity:

To satisfy the homogeneity constraint, we need:

$$\frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1-y^2}\right)^m} = a_{l-m} y^{l-m} + a_{l-m-2} y^{l-m-2} + \ldots$$

Defining the Harmonics (m ≥0)

$$Y_l^m(\theta,\phi)=(x+iz)^m\frac{P_l^m(\cos^{-1}y)}{\left(\sqrt{1-y^2}\right)^m}$$

Homogeneity:

To satisfy the homogeneity constraint, we need:

$$\frac{P_l^m(\cos^{-1}y)}{\left(\sqrt{1-y^2}\right)^m}=a_{l-m}y^{l-m}+a_{l-m-2}y^{l-m-2}+\ldots$$

Or equivalently:

$$P_l^m\left(\cos^{-1}y\right)=q_l^m(y)\left(\sqrt{1-y^2}\right)^m$$

for a polynomial:

$$q_l^m(y)=a_{l-m}y^{l-m}+a_{l-m-2}y^{l-m-2}+\ldots$$

Defining the Harmonics (m ≥0)

$$Y_l^m(\theta,\phi) = (x + iz)^m \frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1-y^2}\right)^m}$$

<u>Orthogonality</u>:

To satisfy the orthogonality constraint, we need:

$$\left\langle Y_l^m(\theta,\varphi), Y_{l'}^{m'}(\theta,\varphi) \right\rangle = 0 \qquad \forall l \neq l' \text{ or } m \neq m'$$

Defining the Harmonics (m ≥0)

$$Y_l^m(\theta,\phi) = (x+iz)^m \frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1-y^2}\right)^m}$$

Orthogonality ($m \neq m'$):

Since we have separation of variables:

$$Y_l^m(\theta,\phi) = e^{im\theta} P_l^m(\phi)$$

we know that:

$$\left\langle Y_l^m(\theta,\varphi), Y_{l'}^{m'}(\theta,\varphi) \right\rangle = \int_0^{\pi} \int_0^{2\pi} e^{im\theta} P_l^m(\phi) \overline{e^{im'\theta} P_{l'}^{m'}(\phi)} d\theta \sin\phi d\phi$$

Defining the Harmonics (m ≥0)

$$Y_l^m(\theta,\phi) = (x+iz)^m \frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1-y^2}\right)^m}$$

<u>Orthogonality (*m*≠*m*')</u>:

Since we have separation of variables:

$$Y_l^m(\theta,\phi) = e^{im\theta} P_l^m(\phi)$$

we know that:

$$\left\langle Y_l^m(\theta,\varphi), Y_{l'}^{m'}(\theta,\varphi)\right\rangle = \int_0^{\pi}\int_0^{2\pi} e^{im\theta} P_l^m(\phi)\overline{e^{im'\theta} P_{l'}^{m'}(\phi)}\, d\theta \sin\phi\, d\phi$$

$$= \left(\int_0^{\pi} P_l^m(\phi) P_{l'}^{m'}(\phi) \sin\phi\, d\phi\right)\left(\int_0^{2\pi} e^{i(m-m')\theta} d\theta\right)$$

Defining the Harmonics (m ≥0)

$$Y_l^m(\theta,\phi)=(x+iz)^m\frac{P_l^m(\cos^{-1}y)}{\left(\sqrt{1-y^2}\right)^m}$$

Orthogonality ($m \neq m'$):

$$\left\langle Y_l^m(\theta,\varphi),Y_{l'}^{m'}(\theta,\varphi)\right\rangle=\left(\int_0^{\pi}P_l^m(\phi)P_{l'}^{m'}(\phi)\sin\phi\, d\phi\right)\boxed{\left(\int_0^{2\pi}e^{i(m-m')\theta}d\theta\right)}$$

But this is zero whenever $m \neq m'$:

$$\int_0^{2\pi}e^{i(m-m')\theta}d\theta=\frac{1}{i(m-m')}e^{i(m-m')\theta}\Big|_0^{2\pi}=0$$

Defining the Harmonics (m ≥0)

$$Y_l^m(\theta,\phi) = (x+iz)^m \frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1-y^2}\right)^m}$$

<u>Orthogonality (*m*=*m*' and *l*≠*l*')</u>:

We have to choose the function:

$$P_l^m\left(\cos^{-1} y\right) = q_l^m(y)\left(\sqrt{1-y^2}\right)^m$$

so that:

$$\left\langle Y_l^m(\theta,\varphi), Y_{l'}^m(\theta,\varphi)\right\rangle = \int_0^{\pi}\int_0^{2\pi} e^{im\theta} P_l^m(\phi)\overline{e^{im'\theta}} P_{l'}^{m'}(\phi)\, d\theta \sin\phi\, d\phi = 0$$

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Since *m*=*m*', this reduces to:

$$\int_0^{\pi} P_l^m(\phi) P_{l'}^m(\phi) \sin\phi\, d\phi = 0$$

Defining the Harmonics (m ≥0)

$$Y_l^m(\theta,\phi) = (x+iz)^m \frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1-y^2}\right)^m}$$

<u>Orthogonality (*m*=*m*’ and *l*≠*l*’)</u>:

We have to choose the function:

$$P_l^m\left(\cos^{-1} y\right) = q_l^m(y)\left(\sqrt{1-y^2}\right)^m$$

Using the change of variable formulation we get:

$$\int_0^\pi P_l^m(\phi)P_{l'}^m(\phi)\sin\phi d\phi = \int_{-1}^1 P_l^m(\cos^{-1} y)P_{l'}^m(\cos^{-1} y)dy$$

Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta,\phi) = (x+iz)^m \frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1-y^2}\right)^m}$$

<u>Orthogonality (*m*=*m*' and *l*≠*l*')</u>:

We have to choose the function:

$$P_l^m(\cos^{-1} y) = q_l^m(y)\left(\sqrt{1-y^2}\right)^m$$

Using the change of variable formulation we get:

$$\int_0^\pi P_l^m(\phi)P_{l'}^m(\phi)\sin\phi\, d\phi = \int_{-1}^1 P_l^m(\cos^{-1} y)P_{l'}^m(\cos^{-1} y)\,dy$$

$$= \int_{-1}^1 q_l^m(y)q_{l'}^m(y)\left(1-y^2\right)^m dy$$

Defining the Harmonics (m ≥0)

$$Y_l^m(\theta,\phi) = (x+iz)^m \frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1-y^2}\right)^m}$$

$$P_l^m\left(\cos^{-1} y\right) = p_l^m(y)\left(\sqrt{1-y^2}\right)^m$$

Thus, we require:

1. The polynomials $q_l^m(y)$ to complete to homogenous polynomials of degree l-m:

$$q_l^m(y) = a_{l-m} y^{l-m} + a_{l-m-2} y^{l-m-2} + \ldots$$

Defining the Harmonics (m ≥0)

$$Y_l^m(\theta,\phi)=(x+iz)^m\frac{P_l^m(\cos^{-1}y)}{\left(\sqrt{1-y^2}\right)^m}$$

$$P_l^m\left(\cos^{-1}y\right)=p_l^m(y)\left(\sqrt{1-y^2}\right)^m$$

Thus, we require:

1. The polynomials $q_l^m(y)$ to complete to homogenous polynomials of degree *l-m*:

$$q_l^m(y)=a_{l-m}y^{l-m}+a_{l-m-2}y^{l-m-2}+\ldots$$

2. And they satisfy the orthogonality condition:

$$\int_{-1}^{1}q_l^m(y)q_{l'}^m(y)\left(1-y^2\right)^m dy=0$$

Defining the Harmonics (m ≥0)

$$Y_l^m(\theta,\phi)=(x+iz)^m\,\frac{P_l^m(\cos^{-1}y)}{\left(\sqrt{1-y^2}\right)^m}$$

$$P_l^m\left(\cos^{-1}y\right)=p_l^m(y)\left(\sqrt{1-y^2}\right)^m$$

This is precisely what we get when we compute the G.S. orthogonalization of $\{1,y,y^2,\ldots\}$ relative to the inner-product:

$$\left\langle f(y),g(y)\right\rangle_m=\int_{-1}^{1}f(y)g(y)\left(1-y^2\right)^m dy$$

and set:

$$q_l^m(y)=p_{l-m}^m(y)$$

Defining the Harmonics (m ≥0)

$$Y_l^m(\theta,\phi) = e^{im\theta} P_l^m(\phi)$$

$$P_l^m\left(\cos^{-1} y\right) = p_l^m(y)\left(\sqrt{1-y^2}\right)^m$$

In sum, we get an expression for the spherical harmonics as:

$$Y_l^m(\theta,\phi) = e^{im\theta} p_{l-m}^m(\cos\phi)\left(\sqrt{1-\cos^2\phi}\right)^m$$

Defining the Harmonics (m ≥0)

$$Y_l^m(\theta,\phi) = e^{im\theta} P_l^m(\phi)$$

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In sum, we get an expression for the spherical harmonics as:

$$Y_l^m(\theta,\phi) = e^{im\theta} p_{l-m}^m(\cos\phi)\left(\sqrt{1-\cos^2\phi}\right)^m$$

$$= e^{im\theta} p_{l-m}^m(\cos\phi)\sin^m\phi$$

where $p_{l-m}^m(y)$ is a polynomial of degree *l-m*.

The Spherical Harmonics

$$Y_l^m(\theta,\phi) = e^{im\theta} p_{l-m}^m(\cos\phi)\sin^m\phi$$

Examples (*l*=0):

$$Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$

The Spherical Harmonics

$$Y_l^m(\theta,\phi) = e^{im\theta} p_{l-m}^m(\cos\phi)\sin^m\phi$$

<u>Examples (*l*=1)</u>:

$$Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin(\phi)e^{-i\theta}$$

$$Y_1^{0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos(\phi)$$

$$Y_1^{1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin(\phi)e^{i\theta}$$

The Spherical Harmonics

$$Y_l^m(\theta,\phi) = e^{im\theta} p_{l-m}^m(\cos\phi)\sin^m\phi$$

Examples (l=2):

$$Y_2^{-2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2(\phi)e^{-2i\theta}$$

$$Y_2^{-1}(\theta,\phi) = \sqrt{\frac{15}{8\pi}}\sin(\phi)\cos(\phi)e^{-i\theta}$$

$$Y_2^{0}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2(\phi)-1\right)$$

$$Y_2^{1}(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}\sin(\phi)\cos(\phi)e^{i\theta}$$

$$Y_2^{2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2(\phi)e^{2i\theta}$$