FFTs in Graphics and Vision

Spherical Harmonics

Outline

Math Stuff

Review

Finding the Spherical Harmonics

Homogenous Polynomials

A homogenous polynomial of degree *d* in *n* variables can be expressed in summation notation as:

$$p_d(x_1,\ldots,x_n)=\sum_{j_1+\cdots+j_n=d} a_{j_1\cdots j_n} x_1^{j_1}\cdot\cdots\cdot x_n^{j_n}$$

Homogenous Polynomials

If we fix the value of the first coefficient at $x_1=\zeta$, we get a new polynomial in n-1 variables:

$$q_d(x_2,\ldots,x_n) = p_d(\zeta, x_2 \ldots, x_n)$$

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Since in the summation notation we have:

$$p_d(x_1,\ldots,x_n) = \sum_{j_1+\cdots+j_n=d} a_{j_1\cdots j_n} x_1^{j_1} \cdot\cdots\cdot x_n^{j_n}$$

$$= \sum_{j_1=0}^{d} x_1^{j_1} \left(\sum_{j_2+\cdots+j_n=d-j_1} a_{j_1\cdots j_n} x_2^{j_2} \cdot\cdots\cdot x_n^{j_n} \right)$$

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the summation notation for the new polynomial is:

$$q_d(x_2,\ldots,x_n) = \sum_{j_1=0}^{d} \zeta^{j_1}\left(\sum_{j_2+\cdots+j_n=d-j_1} a_{j_1\cdots j_n} x_2^{j_2}\cdot\cdots\cdot x_n^{j_n}\right)$$

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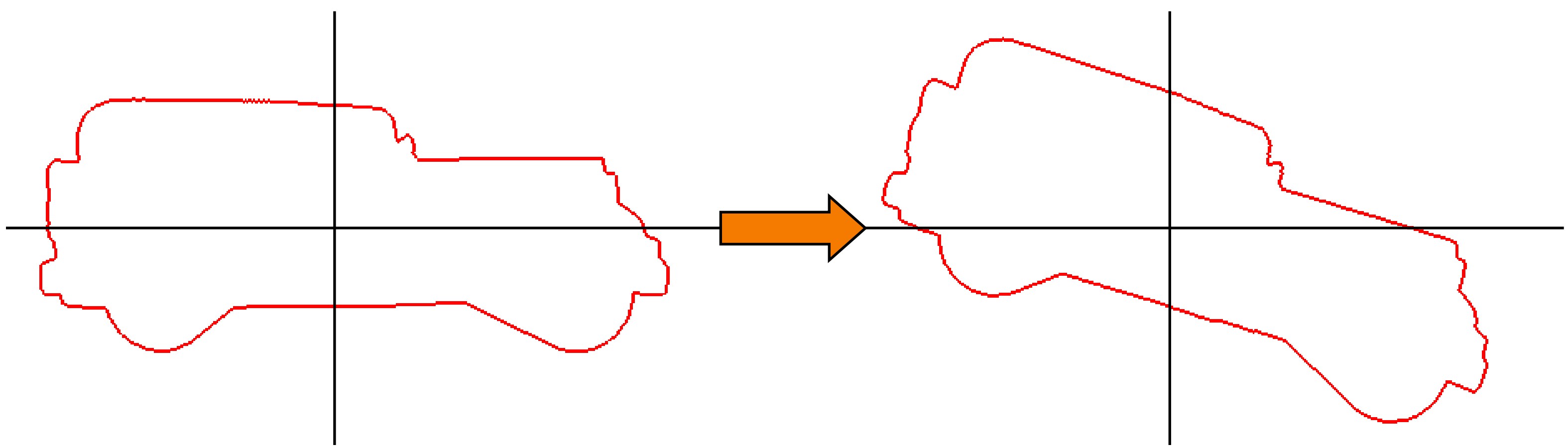
Thus, the new polynomial, obtained by fixing the value of the first variable must be a polynomial of degree at most *d* in *n*-1 variables.

Review

So far, we have considered the representation of the 2D group of rotations, acting on the space of (complex-valued) functions on the unit circle:

$$\left(\rho_R f\right)(p) = f\left(R^{-1} p\right)$$



Review

Since the group of 2D rotations is commutative, Schur's lemma tells us that the space of functions can be expressed as the sum of irreducible representations:

$$F = \sum F_l$$

where each F_l is a one-dimensional space of functions.

Review

In the 2D case, we know that the F_l are spanned by the complex exponentials of degree l:

$$F_l = \{ a e^{il\theta} \}$$

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In the 2D case, we know that the F_l are spanned by the complex exponentials of degree l:

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Thus, the ability to compute the Fourier transform of an arbitrary function $f(\theta)$:

$$f(\theta) = \sum_{l=-\infty}^{\infty} \hat{f}(l) e^{il\theta}$$

has important applications to operations such as smoothing and correlation that are tied to the action of the group of rotation on the space of functions.

Functions on the Sphere

What happens when we consider the space of functions on the unit sphere?

Functions on the Sphere

What happens when we consider the space of functions on the unit sphere?

Since the group of 3D rotations is no longer commutative, we cannot expect to express the space of functions as the sum of irreducible representations:

$$F = \sum F_l$$

where each of the F_l is one-dimensional.

Functions on the Sphere

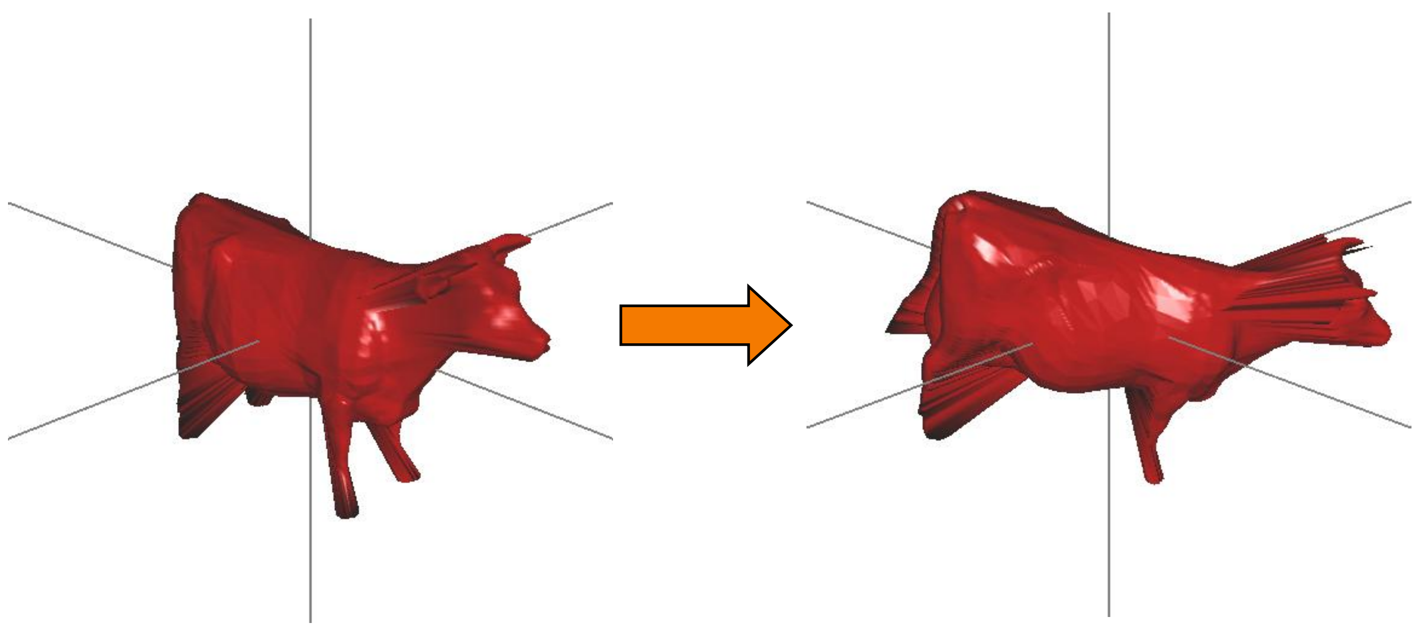
What happens when we consider the space of functions on the unit sphere?

However, we would still like to compute the irreducible representations. And in particular…

Goal

Let *F* be the space of (complex-value) functions on the unit sphere and let ρ be the representation of the group of 3D rotations, acting on the space of functions by rotation:

$$(\rho_R f)(p) = f(R^{-1}p)$$



Goal

Let F be the space of (complex-value) functions on the unit sphere and let ρ be the representation of the group of 3D rotations, acting on the space of functions by rotation:

$$\left(\rho_R f\right)(p) = f\left(R^{-1}p\right)$$

We would like to know what the irreducible representations are.

What We Know

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- Given a homogenous polynomial of degree *l*, a rotation of the polynomial will still be a homogenous polynomial of degree *l*:

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$$\rho_R\left(HP^l(x,y,z)\right) \subseteq HP^l(x,y,z)$$

- If we "throw out" the homogenous polynomials whose restriction to the unit sphere can be expressed as the restriction of a homogenous polynomial of smaller degree, we get a (2*l*+1)-dimensional representation.

What We Know

So, if we let l index the degree of the homogenous polynomial, we should be able to express the space of spherical functions as:

$$F = \sum_{l=0}^{\infty} F_l$$

where:

$$\dim(F_l) = 2l + 1$$

and any function $f \in F_l$ can be expressed as the restriction of a homogenous polynomial in three variables, of degree l.

What We Want to Know

What are the functions in F_l?

That is, what are the functions forming a basis for each F_l:

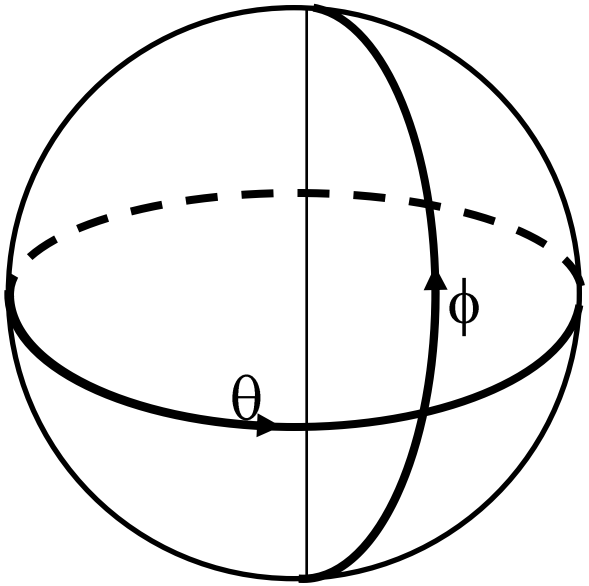
$$F_l = \text{Span}\{ f_l^0, f_l^1, \ldots, f_l^{2l-1}, f_l^{2l} \}$$

Parameterizing Spherical Functions

Every point on the unit sphere can be parameterized by its angles of longitude and latitude:

$$\Phi(\theta,\phi) = (\cos\theta\sin\phi, \cos\phi, \sin\theta\sin\phi)$$

$$\theta \in [0,2\pi) \qquad \phi \in [0,\pi]$$

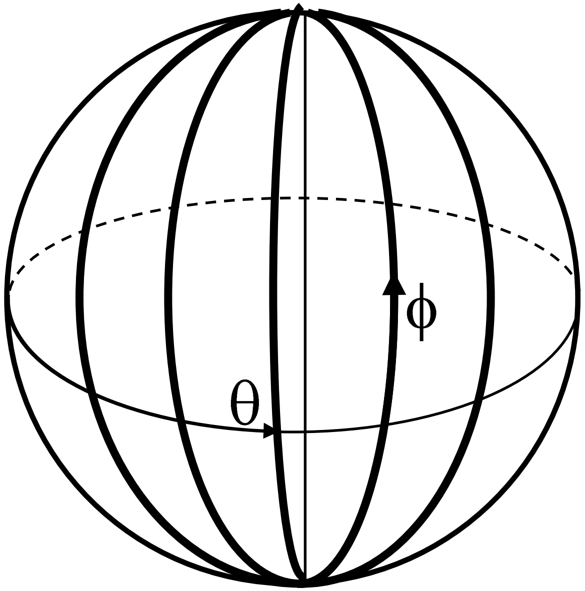


Parameterizing Spherical Functions

$$\Phi(\theta,\phi) = \left(\cos\theta\sin\phi, \cos\phi, \sin\theta\sin\phi\right)$$

$$\theta \in [0,2\pi) \qquad \phi \in [0,\pi]$$

Holding θ constant, we get great semi-circles through the North and South poles:



Parameterizing Spherical Functions

$$\Phi(\theta,\phi) = \left(\cos\theta\sin\phi, \cos\phi, \sin\theta\sin\phi\right)$$

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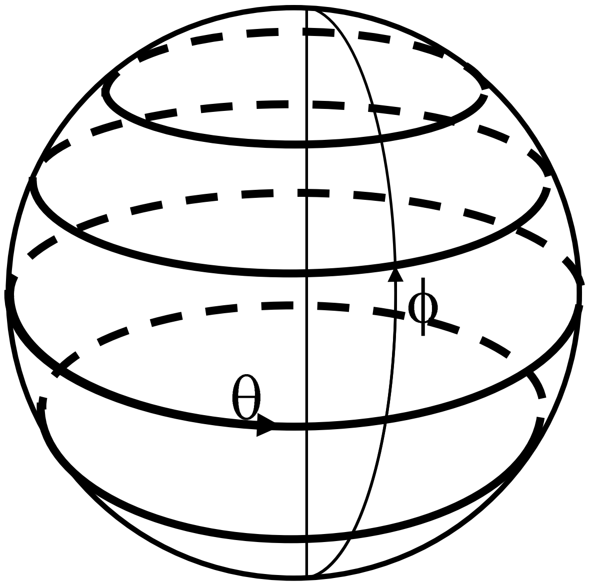
Holding ϕ constant, we get great circles about the *y*-axis:

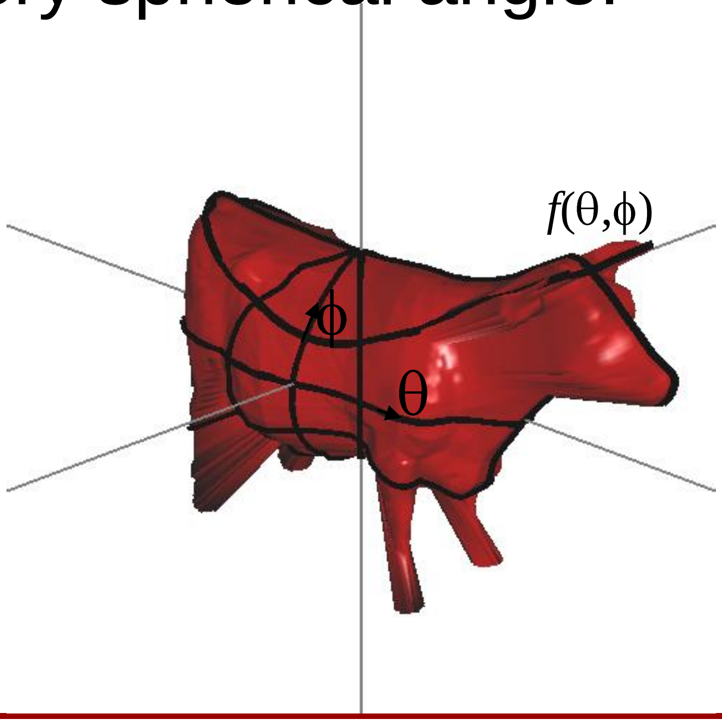


Parameterizing Spherical Functions

$$\Phi(\theta,\phi) = (\cos\theta\sin\phi, \cos\phi, \sin\theta\sin\phi)$$

$$\theta \in [0,2\pi) \qquad \phi \in [0,\pi]$$

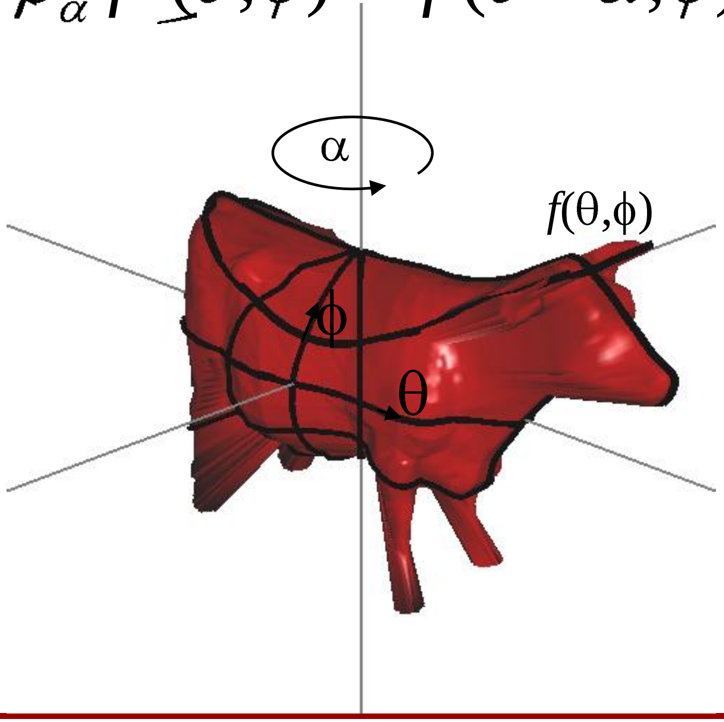
A spherical function can be represented by its values at every spherical angle:



Rotations About the *y*-Axis

Instead of considering the action of the entire group of rotations on the space of spherical functions, we can consider the subset of rotations that rotate about the *y*-axis:

$$(\rho_\alpha f)(\theta,\phi) = f(\theta-\alpha,\phi)$$



Rotations About the *y*-Axis

Instead of considering the action of the entire group of rotations on the space of spherical functions, we can consider the subset of rotations that rotate about the *y*-axis.

- This set of rotations is a group:
  - » The product of two rotations about the *y*-axis is still a rotation about the *y*-axis.
  - » The rotation by $-\alpha$ degrees about the *y*-axis is the inverse of the rotation by α degrees.

Rotations About the *y*-Axis

Instead of considering the action of the entire group of rotations on the space of spherical functions, we can consider the subset of rotations that rotate about the *y*-axis.

- This set of rotations is a group.
- This sub-group of rotations is commutative.

Rotations About the *y*-Axis

Since we know that:

- Rotations map the sub-spaces F_l back into themselves, and
- Rotations about the *y*-axis are a sub-group of the group of 3D rotations

$\Rightarrow$ The sub-spaces F_l are representations for the sub-group of rotations about the *y*-axis.

Rotations About the *y*-Axis

Moreover, since the group of rotations about the *y*-axis is commutative:

$\Rightarrow$ Each F_l can be expressed as the sum of one-dimensional representations that are fixed by rotations about the *y*-axis.

Rotations About the *y*-Axis

Thus, for each l, there must exist a basis of orthogonal functions $\{f_l^0(\theta,\phi),\ldots, f_l^{2l}(\theta,\phi)\}$ such that any rotation by α degrees about the *y*-axis acts on the basis functions by multiplication by a complex number:

$$\rho_R(f_l^k) = \lambda_l^k(\alpha) f_l^k$$

Rotations About the *y*-Axis

$$\rho_R(f_l^k) = \lambda_l^k(\alpha) f_l^k$$

Since the representation is unitary, we know that for any angle of rotation α, we must have:

$$\left\| \lambda_l^k(\alpha) \right\| = 1$$

Rotations About the *y*-Axis

$$\rho_R(f_l^k) = \lambda_l^k(\alpha) f_l^k$$

Since the representation is unitary, we know that for any angle of rotation α, we must have:

$$\left\| \lambda_l^k(\alpha) \right\| = 1$$

Since we know that representations preserve the group structure, and since rotating by α degree and then by β degrees is equivalent to rotating by $(\alpha+\beta)$ degrees, we must have:

$$\lambda_l^k(\alpha+\beta) = \lambda_l^k(\alpha) \cdot \lambda_l^k(\beta)$$

Rotations About the *y*-Axis

$$\rho_R(f_l^k) = \lambda_l^k(\alpha) f_l^k$$

The only functions that satisfy these properties are of the form:

$$\lambda_l^k(\alpha) = e^{ik_l\alpha}$$

Moreover, since we know that rotations by $\alpha=2\pi$ degrees about the y-axis do not change a function, the powers k_l must be integers.

Rotations About the *y*-Axis

$$\boxed{\rho_R(f_l^k) = \lambda_l^k(\alpha) f_l^k}$$

Thus, we can consider the function:

$$\tilde{f}_l^k(\theta,\phi) = \frac{f_l^k(\theta,\phi)}{e^{ik_l\theta}}$$

Rotations About the *y*-Axis

$$\tilde{f}_l^k(\theta,\phi) = \frac{f_l^k(\theta,\phi)}{e^{ik_l\theta}}$$

What happens when we rotate these functions by α degrees about the *y*-axis?

$$\rho_\alpha\left(\tilde{f}_l^k\right)(\theta,\phi) = \tilde{f}_l^k(\theta-\alpha,\phi)$$

Rotations About the *y*-Axis

$$\tilde{f}_l^k(\theta,\phi) = \frac{f_l^k(\theta,\phi)}{e^{ik_l\theta}}$$

What happens when we rotate these functions by α degrees about the *y*-axis?

$$\left(\rho_\alpha \tilde{f}_l^k\right)(\theta,\phi) = \tilde{f}_l^k(\theta-\alpha,\phi)$$

$$= \frac{f_l^k(\theta-\alpha,\phi)}{e^{ik_l(\theta-\alpha)}}$$

Rotations About the *y*-Axis

$$\tilde{f}_l^k(\theta,\phi) = \frac{f_l^k(\theta,\phi)}{e^{ik_l\theta}}$$

What happens when we rotate these functions by α degrees about the *y*-axis?

$$\rho_\alpha\left[\tilde{f}_l^k\right](\theta,\phi) = \tilde{f}_l^k(\theta-\alpha,\phi)$$

$$= \frac{f_l^k(\theta-\alpha,\phi)}{e^{ik_l(\theta-\alpha)}}$$

$$= \frac{e^{-ik_l\alpha} f_l^k(\theta,\phi)}{e^{-ik_l\alpha}e^{ik_l\theta}}$$

Rotations About the *y*-Axis

$$\tilde{f}_l^k(\theta,\phi) = \frac{f_l^k(\theta,\phi)}{e^{ik_l\theta}}$$

What happens when we rotate these functions by α degrees about the *y*-axis?

$$\begin{aligned}
\rho_\alpha\left(\tilde{f}_l^k\right)(\theta,\phi) &= \tilde{f}_l^k(\theta-\alpha,\phi) \\
&= \frac{f_l^k(\theta-\alpha,\phi)}{e^{ik_l(\theta-\alpha)}} \\
&= \frac{e^{-ik_l\alpha} f_l^k(\theta,\phi)}{e^{-ik_l\alpha}e^{ik_l\theta}} \\
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\end{aligned}$$

Rotations About the *y*-Axis

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What happens when we rotate these functions by α degrees about the *y*-axis?

$$\begin{aligned}
\rho_\alpha\left(\tilde{f}_l^k\right)(\theta,\phi) &= \tilde{f}_l^k(\theta-\alpha,\phi) \\
&= \frac{f_l^k(\theta-\alpha,\phi)}{e^{ik_l(\theta-\alpha)}} \\
&= \frac{e^{-ik_l\alpha}\, f_l^k(\theta,\phi)}{e^{-ik_l\alpha}e^{ik_l\theta}} \\
&= \frac{f_l^k(\theta,\phi)}{e^{ik_l\theta}} \\
&= \tilde{f}_l^k(\theta,\phi)
\end{aligned}$$

Rotations About the *y*-Axis

$$\tilde{f}_l^k(\theta,\phi) = \frac{f_l^k(\theta,\phi)}{e^{ik_l\theta}}$$

What happens when we rotate these functions by α degrees about the *y*-axis?

$$\rho_\alpha\left(\tilde{f}_l^k\right)(\theta,\phi) = \tilde{f}_l^k(\theta,\phi)$$

Since these functions are unchanged by rotations about the *y*-axis, this must imply that they are only functions of ϕ:

$$\tilde{f}_l^k(\theta,\phi) = p_l^k(\phi)$$

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Thus, we get:

$$f_l^k(\theta,\phi) = e^{ik_l\theta}\, p_l^k(\phi)$$

Factoring the Spherical Harmonics

$$f_l^k(\theta,\phi) = e^{ik_l\theta} p_l^k(\phi)$$

What can we say about the integers k_l?

Using the fact that the (*x*,*y*,*z*) coordinates of a point on the sphere are defined by:

$$\Phi(\theta,\phi) = (\cos\theta\sin\phi, \cos\phi, \sin\theta\sin\phi)$$

$$\cos\theta = \frac{x}{\sin\phi} \qquad \cos\phi = y \qquad \sin\theta = \frac{z}{\sin\phi}$$

Factoring the Spherical Harmonics

$$f_l^k(\theta,\phi) = e^{ik_l\theta} p_l^k(\phi)$$

What can we say about the integers k_l?

Fixing the value of the angle of latitude, $\phi=\phi_0$, we get an expression for $f_l^k(\theta,\phi_0)$ as:

$$f_l^k(\theta,\phi_0) = (\cos\theta + i\sin\theta)^{k_l} p_l^k(\phi_0)$$

Factoring the Spherical Harmonics

$$f_l^k(\theta,\phi) = e^{ik_l\theta} p_l^k(\phi)$$

What can we say about the integers k_l?

We can express this as a polynomial of degree k_l in x and z:

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Factoring the Spherical Harmonics

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$$f_l^k(\theta,\phi_0) = (\cos\theta + i\sin\theta)^{k_l} p_l^k(\phi_0)$$

$$= \left(\frac{x}{\sin\phi_0} + i\frac{z}{\sin\phi_0}\right)^{k_l} p_l^k(\phi_0)$$

Factoring the Spherical Harmonics

$$\boxed{f_l^k(\theta,\phi) = e^{ik_l\theta}\, p_l^k(\phi)}$$

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$$= \left(\frac{x}{\sin\phi_0} + i\frac{z}{\sin\phi_0}\right)^{k_l} p_l^k(\phi_0)$$

$$= (x+iz)^{k_l}\,\frac{p_l^k(\phi_0)}{\sin^{k_l}\phi_0}$$

Factoring the Spherical Harmonics

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But we also now that $f_l^k(\theta,\phi)$ is the restriction of a homogenous polynomial of degree l to the unit sphere.

Factoring the Spherical Harmonics

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But we also now that $f_l^k(\theta,\phi)$ is the restriction of a homogenous polynomial of degree l to the unit sphere.

So fixing $y=\cos(\phi_0)$, we must get a polynomial of degree at most l:

$$-l \leq k_l \leq l$$

The Spherical Harmonics

In sum, we know that the space of spherical functions F can be expressed as the sum of sub-representations:

$$F = \sum F_l$$

where the functions in F_l are obtained by considering the restrictions of homogenous polynomials of degree l to the unit sphere.

The Spherical Harmonics

Each F_l is a $(2l+1)$-dimensional space of functions, spanned by an orthogonal basis of functions $\{f_l^0(\theta,\phi),\ldots, f_l^{2l}(\theta,\phi)\}$ where the k-th basis function can be expressed as:

$$f_l^k(\theta,\phi) = e^{ik_l\theta} p_l^k(\phi)$$

where k_l is an integer in the range $[-l,l]$.

The Spherical Harmonics

It turns out that for every value of -*l*≤*k*≤*l* there is exactly one basis function:

$$Y_l^k(\theta,\phi) = \frac{e^{ik\theta}\,p_l^k(\phi)}{\left\| e^{ik\theta}\,p_l^k(\phi) \right\|}$$

The Spherical Harmonics

It turns out that for every value of $-l \leq k \leq l$ there is exactly one basis function:

$$Y_l^k(\theta,\phi) = \frac{e^{ik\theta} p_l^k(\phi)}{\left\| e^{ik\theta} p_l^k(\phi) \right\|}$$

These are the <u>spherical harmonics</u> of degree *l*.

Aside

To evaluate the spherical harmonics, we need to know what the functions $p_l^k(\phi)$ are.

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To evaluate the spherical harmonics, we need to know what the functions $p_l^k(\phi)$ are.

These are defined by setting:

$$p_l^k(\phi) = P_l^k(\cos\phi)$$

where the P_l^k are the associated Legendre polynomials, defined by:

$$P_l^k(x) = \frac{(-1)^k}{2^l l!}\left(1-x^2\right)^{k/2}\frac{d^{l+k}}{dx^{l+k}}\left(x^2-1\right)^l$$

$$P_l^{-k}(x) = (-1)^k\frac{(l-k)!}{(l+k)!}P_l^k(x)$$

for $k \geq 0$.

The Spherical Harmonics

Examples (l=0):

$$Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$

The Spherical Harmonics

Examples (l=1):

$$Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin(\phi)e^{-i\theta}$$

$$Y_1^{0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos(\phi)$$

$$Y_1^{1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin(\phi)e^{i\theta}$$

The Spherical Harmonics

Examples (*l*=2):

$$Y_2^{-2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2(\phi)e^{-2i\theta}$$

$$Y_2^{-1}(\theta,\phi) = \sqrt{\frac{15}{8\pi}}\sin(\phi)\cos(\phi)e^{-i\theta}$$

$$Y_2^{0}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2(\phi)-1\right)$$

$$Y_2^{1}(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}\sin(\phi)\cos(\phi)e^{i\theta}$$

$$Y_2^{2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2(\phi)e^{2i\theta}$$

The Spherical Harmonics

Using the fact that we can write out the *x*, *y*, and *z* coordinates as:

$$\cos\theta = \frac{x}{\sin\phi} \qquad \cos\phi = y \qquad \sin\theta = \frac{z}{\sin\phi}$$

gives:

$$Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$

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The Spherical Harmonics

Using the fact that we can write out the *x*, *y*, and *z* coordinates as:

$$\cos\theta = \frac{x}{\sin\phi} \qquad \cos\phi = \mathrm{y} \qquad \sin\theta = \frac{z}{\sin\phi}$$

gives:

$$Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin(\phi)e^{-i\theta}$$

$$Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}(x - iz)$$

The Spherical Harmonics

Using the fact that we can write out the *x*, *y*, and *z* coordinates as:

$$\cos\theta = \frac{x}{\sin\phi} \qquad \cos\phi = y \qquad \sin\theta = \frac{z}{\sin\phi}$$

gives:

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos(\phi)$$

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\,y$$

The Spherical Harmonics

Using the fact that we can write out the *x*, *y*, and *z* coordinates as:

$$\cos\theta = \frac{x}{\sin\phi} \qquad \cos\phi = y \qquad \sin\theta = \frac{z}{\sin\phi}$$

gives:

$$Y_1^1(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin(\phi)e^{i\theta}$$

$$Y_1^1(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}(x+iz)$$

The Spherical Harmonics

Using the fact that we can write out the *x*, *y*, and *z* coordinates as:

$$\cos\theta = \frac{x}{\sin\phi} \qquad \cos\phi = y \qquad \sin\theta = \frac{z}{\sin\phi}$$

gives:

$$Y_2^{-2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2(\phi)e^{-2i\theta}$$

$$Y_2^{-2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}(x-iz)^2$$

The Spherical Harmonics

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$$\cos\theta = \frac{x}{\sin\phi} \qquad \cos\phi = y \qquad \sin\theta = \frac{z}{\sin\phi}$$

gives:

$$Y_2^{-1}(\theta,\phi) = \sqrt{\frac{15}{8\pi}}\sin(\phi)\cos(\phi)e^{-i\theta}$$

$$Y_2^{-1}(\theta,\phi) = \sqrt{\frac{15}{8\pi}}y(x-iz)$$

The Spherical Harmonics

Using the fact that we can write out the *x*, *y*, and *z* coordinates as:

$$\cos\theta = \frac{x}{\sin\phi} \qquad \cos\phi = y \qquad \sin\theta = \frac{z}{\sin\phi}$$

gives:

$$Y_2^0(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2(\phi) - 1\right)$$

$$Y_2^0(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\left(2z^2 - x^2 - y^2\right)$$

The Spherical Harmonics

Using the fact that we can write out the *x*, *y*, and *z* coordinates as:

$$\cos\theta = \frac{x}{\sin\phi} \qquad \cos\phi = \mathrm{y} \qquad \sin\theta = \frac{z}{\sin\phi}$$

gives:

$$Y_2^1(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}\sin(\phi)\cos(\phi)e^{i\theta}$$

$$Y_2^1(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}y(x+iz)$$

The Spherical Harmonics

Using the fact that we can write out the *x*, *y*, and *z* coordinates as:

$$\cos\theta = \frac{x}{\sin\phi} \qquad \cos\phi = y \qquad \sin\theta = \frac{z}{\sin\phi}$$

gives:

$$Y_2^2(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2(\phi)e^{2i\theta}$$

$$Y_2^2(\theta,\phi) = \sqrt{\frac{15}{32\pi}}(x+iz)^2$$

Implications

For any spherical function $f(\theta,\phi)$, we can express f as the sum of functions in F_k:

$$f(\theta,\phi) = \sum_{l=0}^{\infty} f_l(\theta,\phi)$$

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Giving an expression for the function $f(\theta,\phi)$ as:

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In the case that the function *f* is real-valued, we may want to express it as the sum of real-valued functions.

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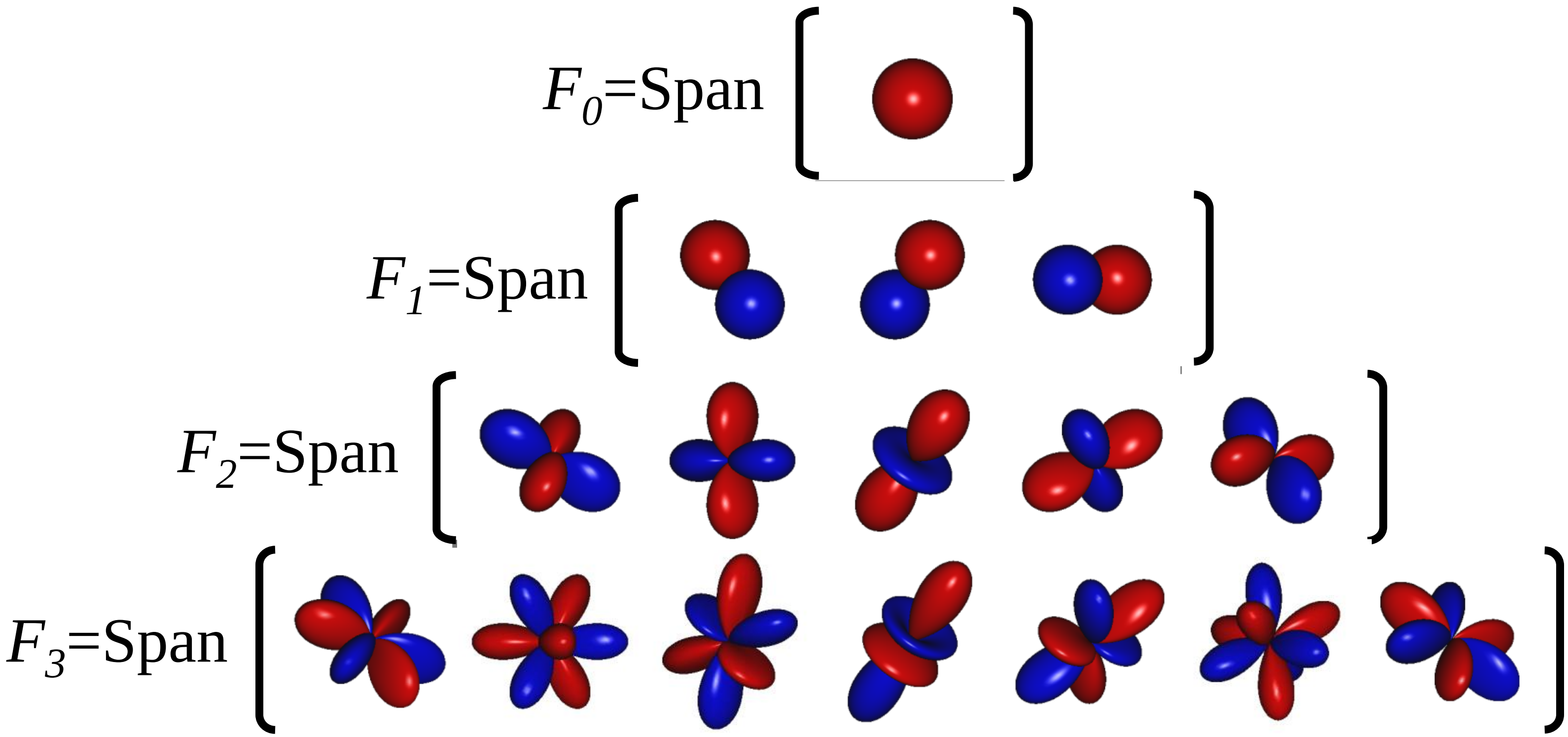
We can do this by considering the real and imaginary parts of the spherical harmonics independently:

$$\mathrm{Re}\left(Y_l^k(\theta,\phi)\right) = (-1)^k\,\mathrm{Re}\left(Y_l^{-k}(\theta,\phi)\right) = \frac{\cos(k\theta)\,p_l^k(\phi)}{\left\|e^{ik\theta}\,p_l^k(\phi)\right\|}$$

$$\mathrm{Im}\left(Y_l^k(\theta,\phi)\right) = (-1)^{k+1}\,\mathrm{Im}\left(Y_l^{-k}(\theta,\phi)\right) = \frac{\sin(k\theta)\,p_l^k(\phi)}{\left\|e^{ik\theta}\,p_l^k(\phi)\right\|}$$

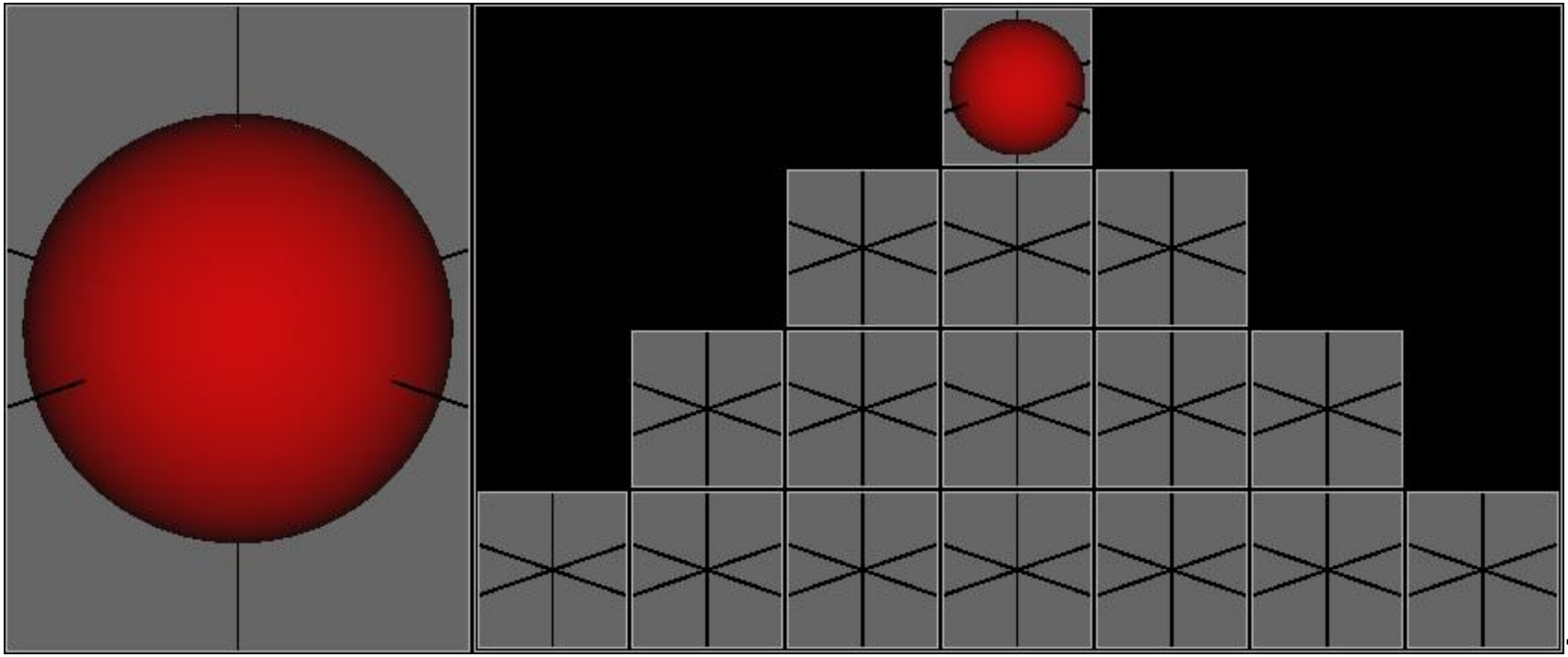
The Spherical Harmonics

In the case that the function *f* is real-valued, we may want to express it as the sum of real-valued functions.



Implications

Since the spaces F_l are sub-representations, rotating a function that is the sum of the l-th spherical harmonics, will give a function that is the sum of the l-th spherical harmonics.



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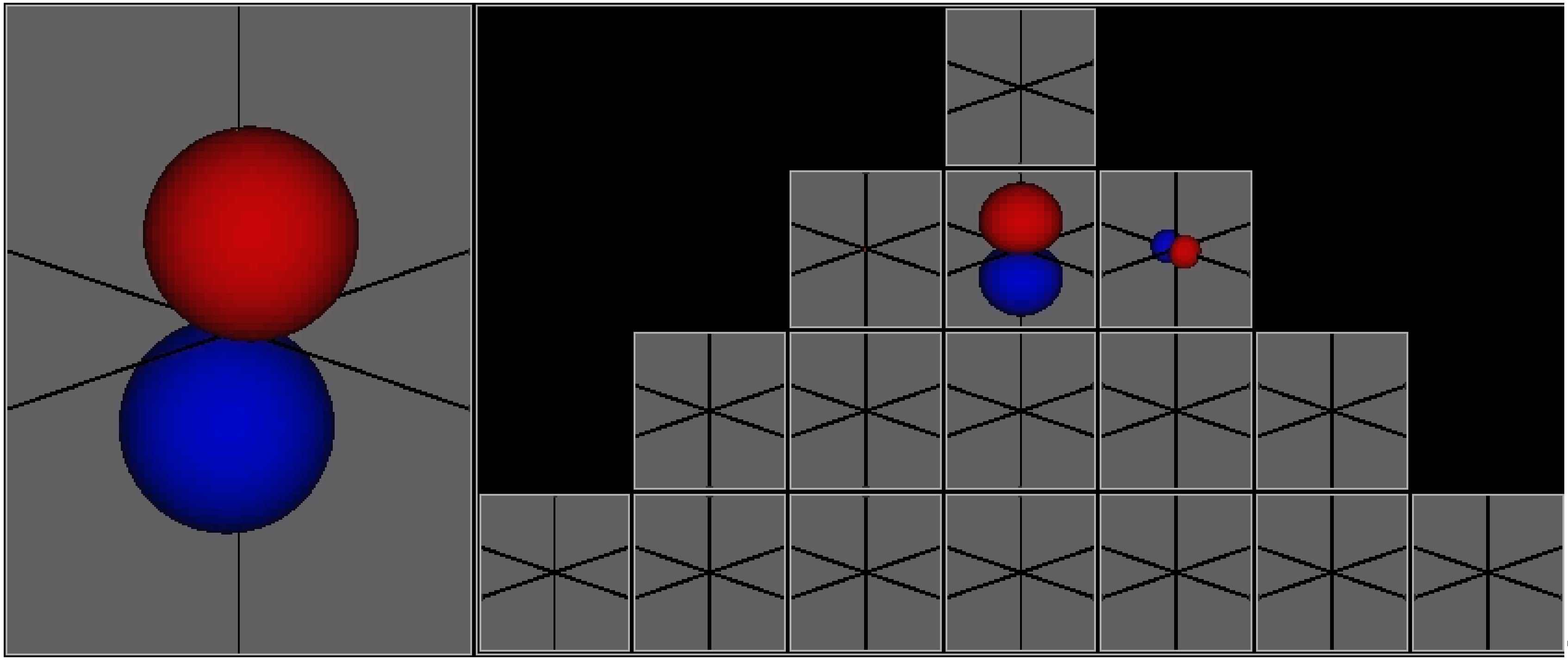

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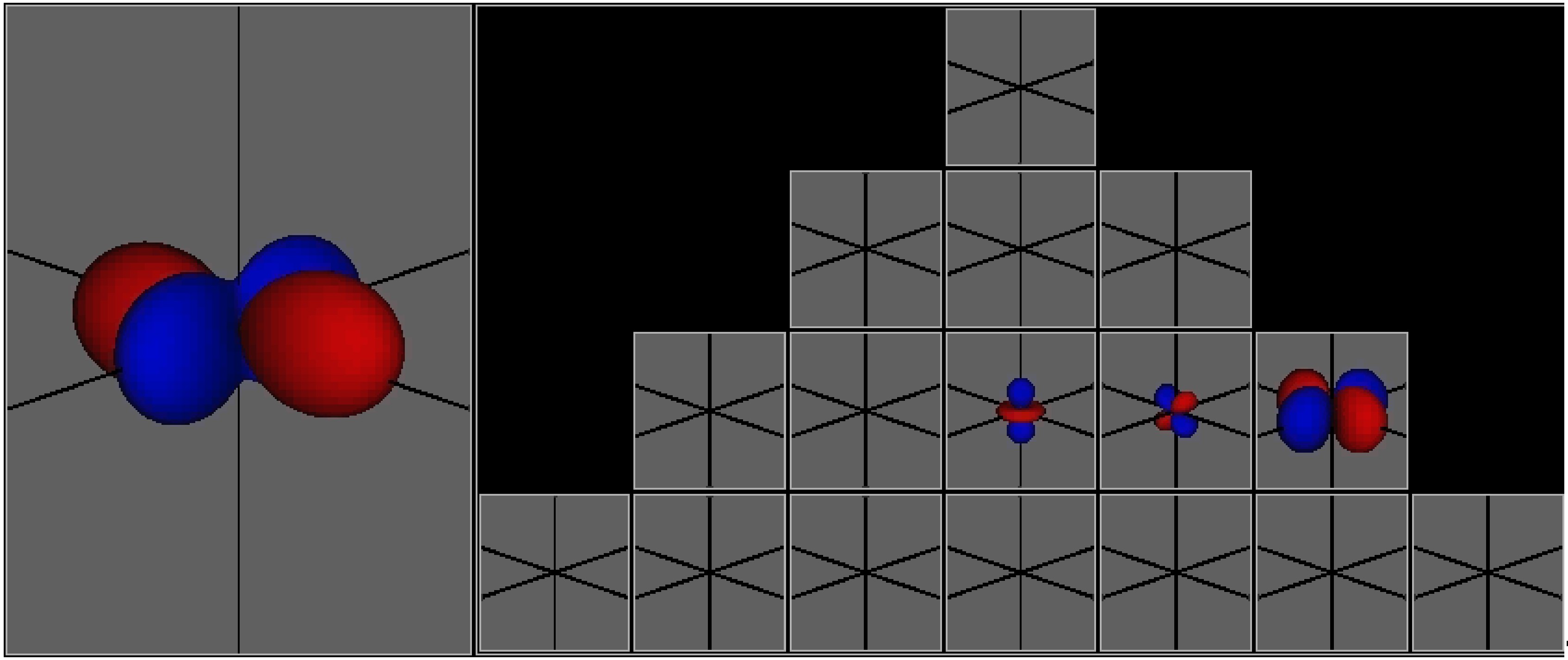

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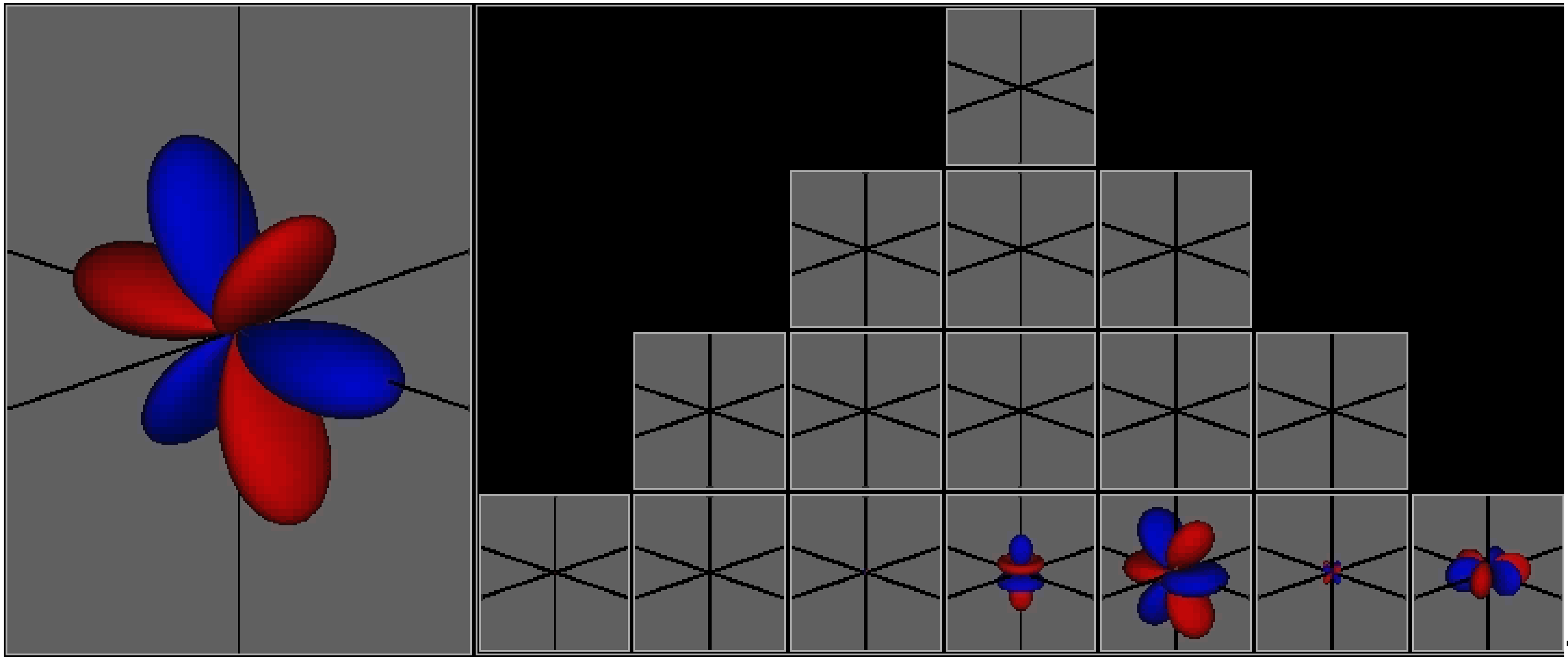
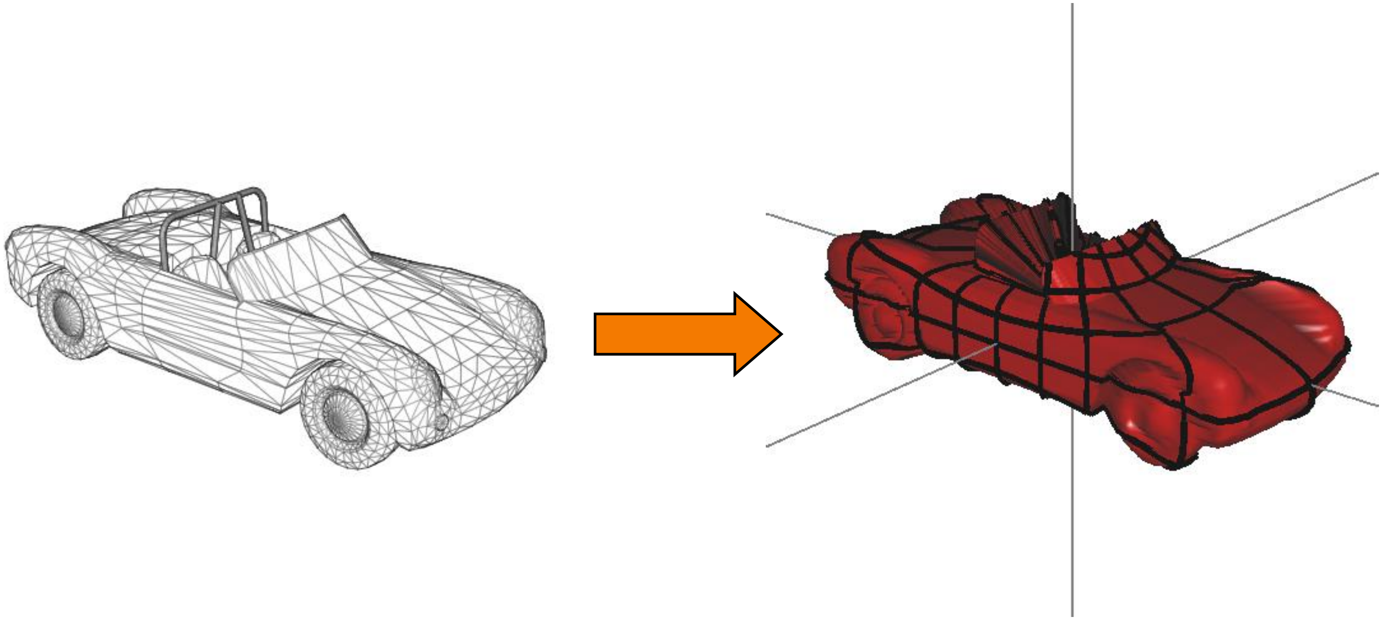


Application (Shape Matching)

<u>Goal</u>:

Given a spherical function representing the surface of a 3D model by a spherical function:



we would like to obtain a rotation invariant representation.

Application (Shape Matching)

Approach:

We can use the facts that:

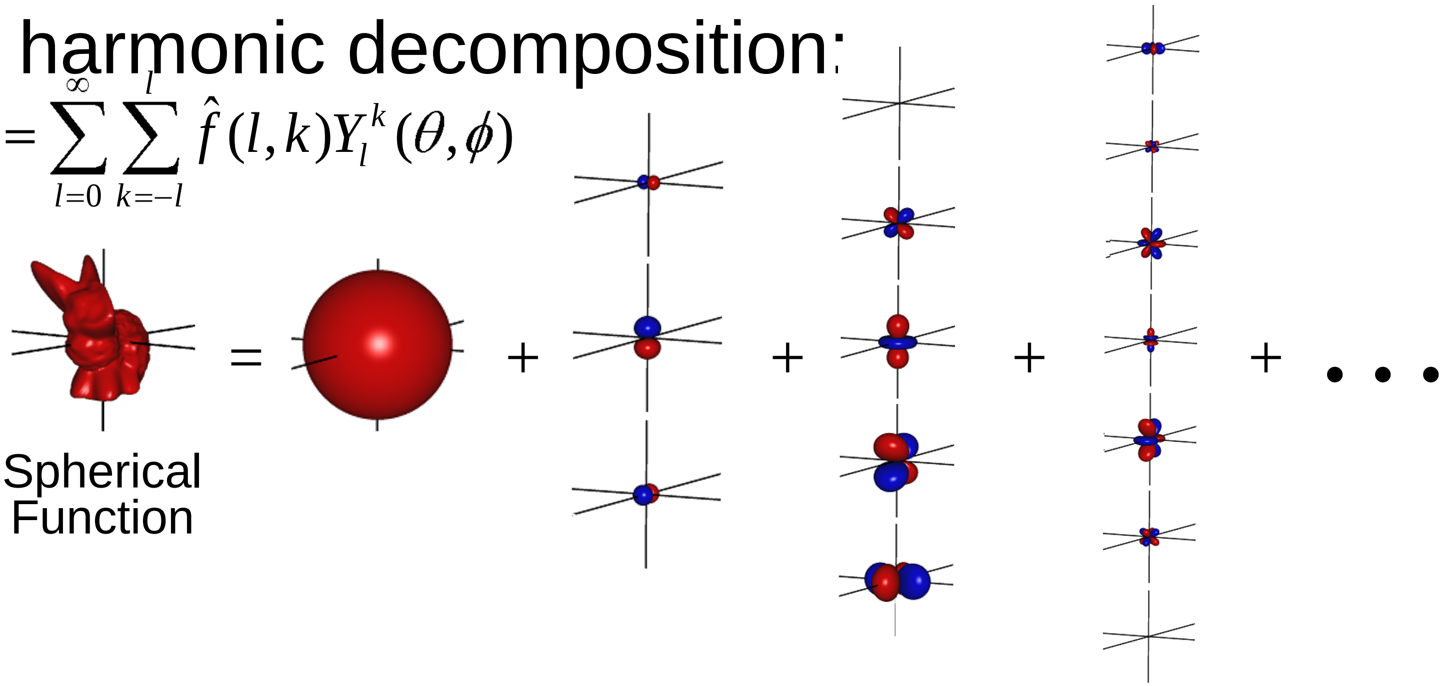
- Rotations are unitary, so they don't change the norms of functions.
- The function spaces spanned by the spherical harmonics of degree l are representations of the group of rotation.

Application (Shape Matching)

Approach:

Specifically, given a spherical function, we obtain its spherical harmonic decomposition:

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{k=-l}^{l}\hat{f}(l,k)Y_l^k(\theta,\phi)$$

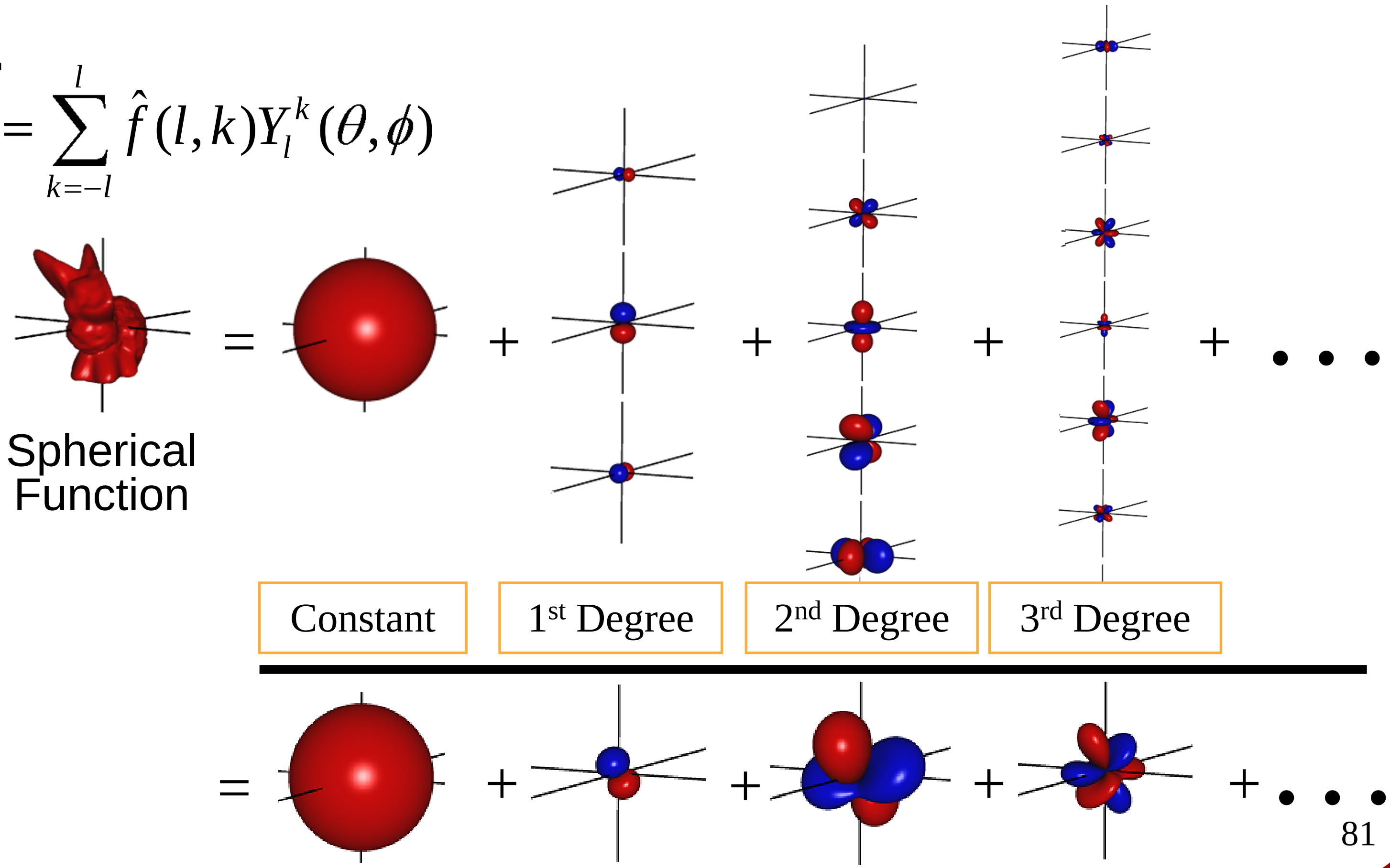


Application (Shape Matching)

Approach:

We combine the spherical harmonics of the same degree…

$$f_l(\theta,\phi) = \sum_{k=-l}^{l} \hat{f}(l,k) Y_l^k(\theta,\phi)$$



Application (Shape Matching)

Approach:

… to get functions whose norm does not change with rotation:

$$\|\rho_R f_l\| = \|f_l\|$$

Storing the norms, we obtain a rotation invariant shape descriptor.