FFTs in Graphics and Vision

The Laplacian Operator

Outline

Math

- Symmetric/Hermitian Matrices
- Lagrange Multipliers
- Diagonalizing Symmetric Matrices

The Laplacian Operator

Linear Operators

<u>Definition</u>:

Given a real inner product space $(V,\langle\cdot,\cdot\rangle)$ and given a linear operator $L:V\rightarrow V$, the <u>adjoint</u> of the L is the linear operator M, with the property that:

$$\langle v, Lw\rangle = \langle Mv, w\rangle$$

for all $v,w\in V$.

Linear Operators

Note:

If V is the space of n-dimensional, real-valued, arrays with the standard inner product:

$$\langle v[\,], w[\,] \rangle = \sum_{i=1}^{n} v[i]w[i] = v^t w$$

then the adjoint of a matrix M is just its transpose:

$$\langle v, Mw \rangle = v^t M w$$

Linear Operators

<u>Note</u>:

If *V* is the space of *n*-dimensional, real-valued, arrays with the standard inner product:

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$$= (M^t v)^t w$$

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then the adjoint of a matrix *M* is just its transpose:

$$\begin{aligned}\langle v, Mw\rangle &= v^t M w \\ &= (M^t v)^t w \\ &= \langle M^t v, w\rangle\end{aligned}$$

Linear Operators

Definition:

A real linear operator L is self-adjoint if it is its own adjoint, i.e.:

$$\langle v, Lw \rangle = \langle Lv, w \rangle$$

for all $v, w \in V$.

Linear Operators

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If *V* is the space of *n*-dimensional, real-valued, arrays with the standard inner product:

$$\langle v[\ ], w[\ ]\rangle = \sum_{i=1}^{n} v[i]w[i] = v^t w$$

then a matrix *M* is self-adjoint if it is symmetric:

$$M = M^t$$

Linear Operators

Definition:

Given a complex inner product space $(V,\langle\cdot,\cdot\rangle)$ and given a linear operator $L:V\rightarrow V$, the adjoint of the L is the linear operator M, with the property that:

$$\langle v, Lw\rangle = \langle Mv, w\rangle$$

for all $v,w\in V$.

Linear Operators

<u>Note</u>:

If *V* is the space of *n*-dimensional, complex-valued, arrays with the standard inner product:

$$\left\langle v[\ ], w[\ ] \right\rangle = \sum_{i=1}^{n} v[i]\overline{w[i]} = v^t \overline{w}$$

then the adjoint of a matrix *M* is just the complex conjugate of the transpose:

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then the adjoint of a matrix M is just the complex conjugate of the transpose:

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<u>Example</u>:

$$M = \begin{pmatrix} 2 & 3-i5 \\ 3+i5 & -7 \end{pmatrix}$$

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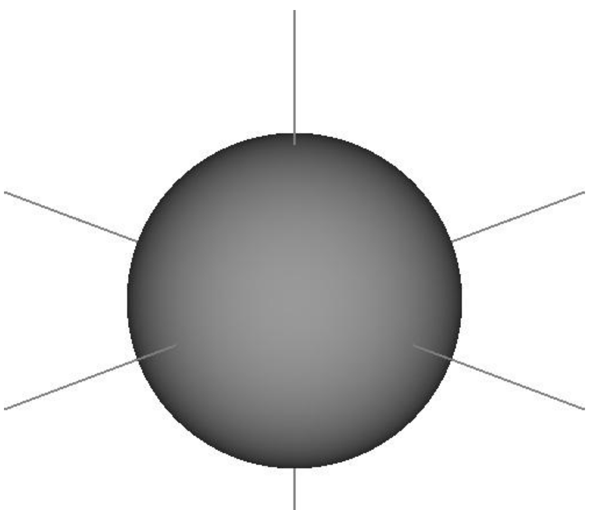
- Diagonalizing Symmetric Matrices

The Laplacian Operator

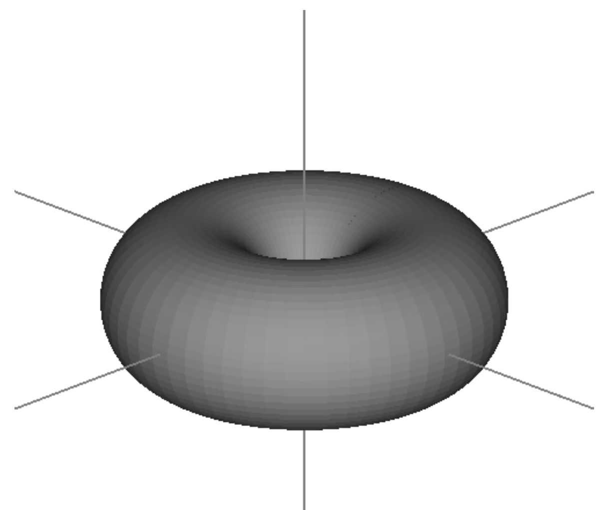
Implicit Surface

Given a function $g(x,y,z)$, the <u>implicit surface</u> or <u>iso-surface</u> defined by $g(x,y,z)$ is the set of points in 3D satisfying the condition:

$$g(x,y,z)=0$$



$g(x,y,z)=x^2+y^2+z^2-1$



$g(x,y,z)=(x^2+y^2+z^2-(R^2+r^2))^2-4R^2(r^2-z^2)$

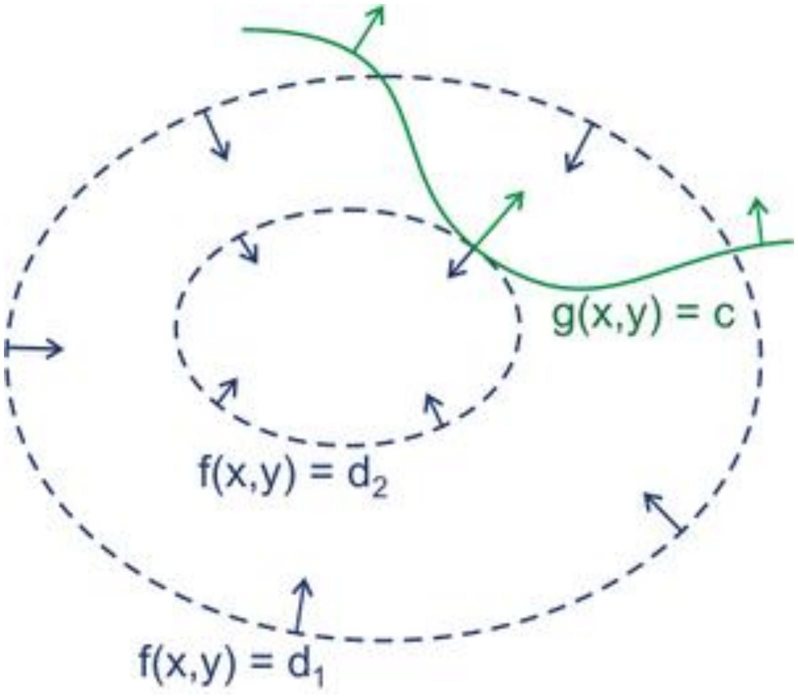
Lagrange Multipliers

Given an implicit surface defined by a function $g(x,y,z)$ and given a function $f(x,y,z)$, we would like to find the extrema of f on the surface.

Lagrange Multipliers

Given an implicit surface defined by a function $g(x,y,z)$ and given a function $f(x,y,z)$, we would like to find the extrema of f on the surface.

This can be done by finding the points on the surface where the gradient of f is parallel to the surface normal.



http://www.answers.com

Lagrange Multipliers

Since the implicit surface is defined as the set of points where:

$$g(x,y,z)=0$$

the normal at a point on the surface must be parallel to the gradient of *g*.

Lagrange Multipliers

Since the implicit surface is defined as the set of points where:

$$g(x, y, z) = 0$$

the normal at a point on the surface must be parallel to the gradient of *g*.

Finding the extrema amounts to finding the points (x,y,z) such that:

- $g(x,y,z)=0$ (the point is on the surface)
- $\lambda\nabla f=\nabla g$ (the point is a local extrema)

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The Laplacian Operator

Diagonalizing Symmetric Matrices

Claim:

Given the space of n-dimensional, real-valued, arrays and given a symmetric matrix M:

> M has n eigenvectors and they form an orthogonal basis

Diagonalizing Symmetric Matrices

<u>The Eigenvectors Form an Orthogonal Basis</u>:

To show this we will show two things:

1. If v is an eigenvector, then the space of vectors perpendicular to v is fixed by M.
2. At least one eigenvector must exist.

Diagonalizing Symmetric Matrices

1. If *v* is an eigenvector, then the space of vectors perpendicular to *v* is fixed by *M*.

Suppose that *v* is an eigenvector and *w* is some other vector that is perpendicular to *v*:

$$\langle v, w \rangle = 0$$

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Since *v* is an eigenvector, this implies that:

$$\langle Mv, w \rangle = \langle \lambda v, w \rangle$$

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Therefore, since *M* is symmetric, we must have:

$$\langle v, Mw \rangle = 0$$

Diagonalizing Symmetric Matrices

1. If v is an eigenvector, then the space of vectors perpendicular to v is fixed by M.

$$\boxed{\langle v, Mw \rangle = 0}$$

$\Rightarrow$ If W is the subspace of vectors perpendicular to v, then we must have:

$$Mw \in W$$

for all $w \in W$.

Diagonalizing Symmetric Matrices

1. If *v* is an eigenvector, then the space of vectors perpendicular to *v* is fixed by *M*.

<u>Implications</u>:

If we know that we can find one eigenvector *v*, we can consider the restriction of *M* to the subspace $W \subset V$ of vectors perpendicular to *v*:

$$W = \left\{ w \in V \,\middle|\, \langle w, v \rangle = 0 \right\}$$

Diagonalizing Symmetric Matrices

1. If *v* is an eigenvector, then the space of vectors perpendicular to *v* is fixed by *M*.

<u>Implications</u>:

If we know that we can find one eigenvector *v*, we can consider the restriction of *M* to the subspace $W \subset V$ of vectors perpendicular to *v*:

M maps *W* back into *W* and is still symmetric:

$$\langle Mu, w \rangle = \langle u, Mw \rangle$$

for all $u, w \in W$.

Diagonalizing Symmetric Matrices

1. If v is an eigenvector, then the space of vectors perpendicular to v is fixed by M.

Implications:

If we know that we can find one eigenvector v, we can consider the restriction of M to the subspace $W \subset V$ of vectors perpendicular to v:

M maps W back into W and is still symmetric:

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for all $u, w \in W$.

So we can repeat to find the next eigenvector.

Diagonalizing Symmetric Matrices

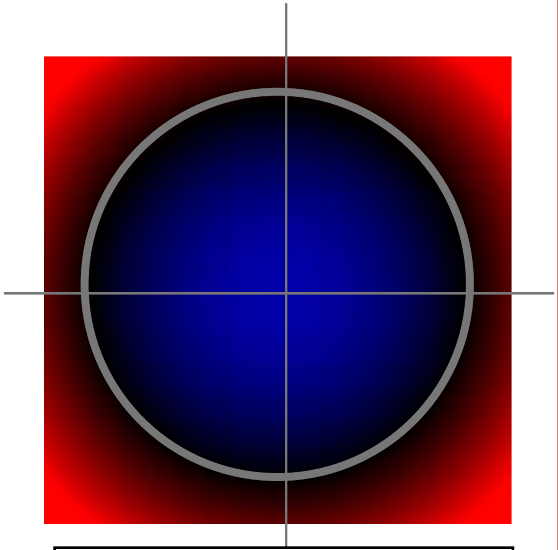
2. At least one eigenvector must exist

We will show this using Lagrange multipliers:

- The implicit surface will be the *n*-dimensional sphere:

$$g(x_1,\ldots,x_n) = \left(x_1^2 + \cdots + x_n^2\right) - 1$$

$$g(v) = \|v\|^2 - 1$$



$$g(x,y) = x^2 + y^2 - 1$$

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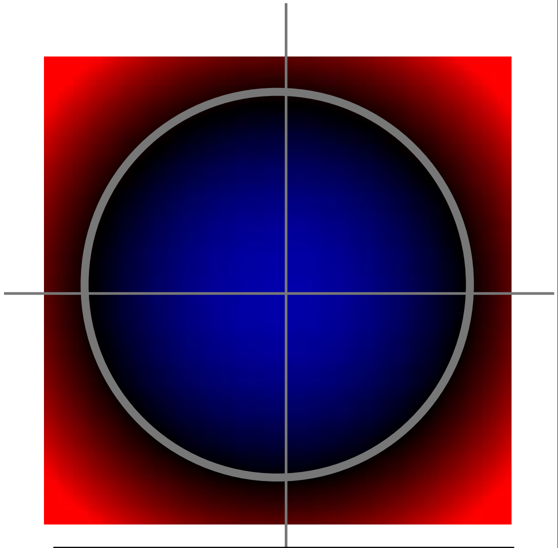
- The function we optimize will be:

$$f(v) = v^t M v$$



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- The function we optimize will be:

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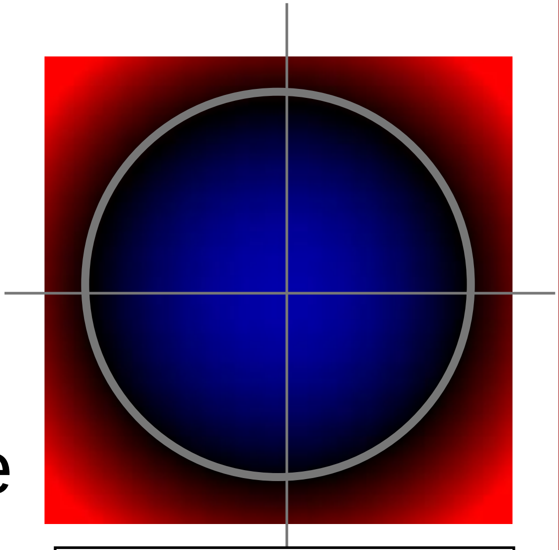
Because the sphere is compact, the extrema must exist.



$g(x,y) = x^2 + y^2 - 1$

Diagonalizing Symmetric Matrices

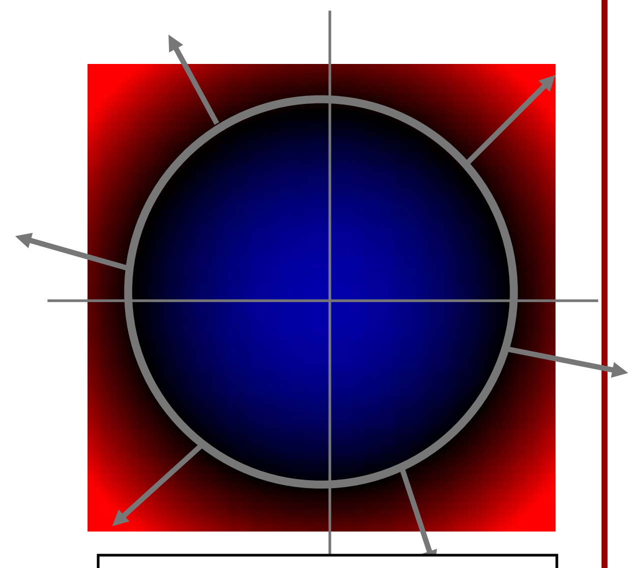
2. At least one eigenvector must exist

The normal of a point on the sphere is parallel to the gradient of *g*:

$$\nabla g(x_1,\ldots,x_n) = 2(x_1,\ldots,x_n)$$

$$\Updownarrow$$

$$\nabla g(v) = 2v$$



$$g(x,y) = x^2 + y^2 - 1$$

Diagonalizing Symmetric Matrices

2. At least one eigenvector must exist

Claim:

The gradient of *f* is:

$$\nabla f(v) = 2Mv$$

Diagonalizing Symmetric Matrices

2. At least one eigenvector must exist

<u>Proof</u>:

Let e_i be the vector with zeros everywhere but in the *i*-th entry:

$$e_i = \left(0,\ldots,0,1,0,\ldots,0\right)$$

i-th entry

Diagonalizing Symmetric Matrices

2. At least one eigenvector must exist

Proof:

Let e_i be the vector with zeros everywhere but in the *i*-th entry. Then the *i*-th coefficient of the gradient is:

$$\frac{\partial}{\partial x_i} f(v) = \left.\frac{d}{dt}\right|_{t=0} f(v + te_i)$$

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$$= \left.\frac{d}{dt}\right|_{t=0} (v+te_i)^t M (v+te_i)$$

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$$= \left.\frac{d}{dt}\right|_{t=0} v^t Mv + te_i^t Mv + tv^t Me_i + t^2 e_i^t Me_i$$

Diagonalizing Symmetric Matrices

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Proof:

Let e_i be the vector with zeros everywhere but in the i-th entry. Then the i-th coefficient of the gradient is:

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Diagonalizing Symmetric Matrices

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$$\frac{\partial}{\partial x_i} f(v) = e_i^t M v + v^t M e_i$$

$$= \langle e_i, Mv \rangle + \langle v, Me_i \rangle$$

Diagonalizing Symmetric Matrices

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Diagonalizing Symmetric Matrices

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Proof:

Let e_i be the vector with zeros everywhere but in the i-th entry. Then the i-th coefficient of the gradient is:

$$\begin{aligned}\frac{\partial}{\partial x_i} f(v) &= e_i^t M v + v^t M e_i \\ &= \langle e_i, Mv \rangle + \langle v, Me_i \rangle \\ &= \langle e_i, Mv \rangle + \langle Mv, e_i \rangle \\ &= 2\langle e_i, Mv \rangle\end{aligned}$$

Diagonalizing Symmetric Matrices

2. At least one eigenvector must exist

Proof:

Let e_i be the vector with zeros everywhere but in the *i*-th entry. Then the *i*-th coefficient of the gradient is:

$$\frac{\partial}{\partial x_i} f(v) = 2\langle e_i, Mv \rangle$$

But the dot-product of any vector *v* with e_i is equal to the *i*-th coefficient of *v*.

Diagonalizing Symmetric Matrices

2. At least one eigenvector must exist

Proof:

Thus, the *i*-th coefficient of the gradient of *f* at *v* is twice the *i*-th coefficient of *Mv*, so the gradient of *f* at *v* must be equal to twice *v*:

$$\nabla f(v) = 2Mv$$

Diagonalizing Symmetric Matrices

2. At least one eigenvector must exist

We know that the normal of the point *v* on the unit sphere is parallel to the gradient, which is:

$$\nabla g(v) = 2v$$

And we know that the gradient of the function *f* is:

$$\nabla f(v) = 2Mv$$

Diagonalizing Symmetric Matrices

2. At least one eigenvector must exist

Since the function *g* must have a maximum on the sphere, we know that there must exist a point *v* at which:

$$\lambda \nabla g(v) = \nabla f(v)$$

Diagonalizing Symmetric Matrices

2. At least one eigenvector must exist

Since the function g must have a maximum on the sphere, we know that there must exist a point v at which:

$$\lambda \nabla g(v) = \nabla f(v)$$

$$\Updownarrow$$

$$\lambda 2v = 2Mv$$

Diagonalizing Symmetric Matrices

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Diagonalizing Symmetric Matrices

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$$\Updownarrow$$

$$\lambda 2v = 2Mv$$

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$$\lambda v = Mv$$

So at the maximum, we have our eigenvalue.

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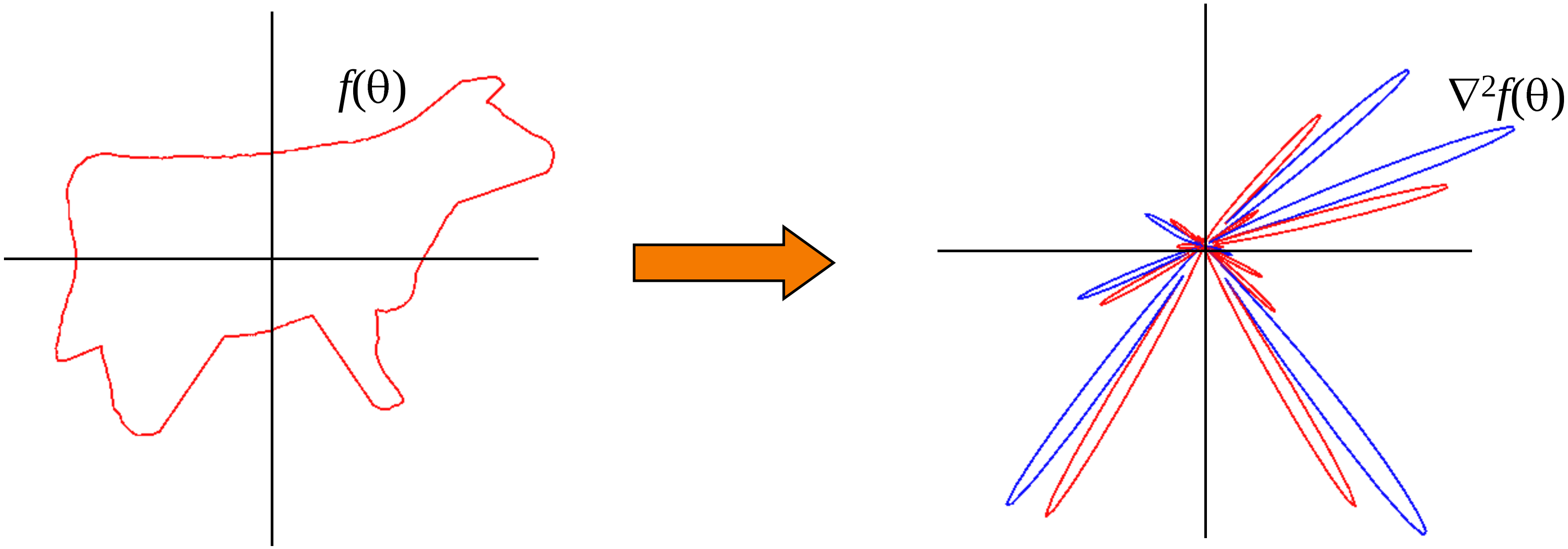
Recall:

The Laplacian of a function *f* at the point (*x*,*y*) is a measure of how similar the value of *f* at (*x*,*y*) is to the average values of its neighbors.



The Laplacian Operator

Recall:

Formally, for a function in 2D, the Laplacian is the sum of unmixed partial second derivatives:

$$\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

The Laplacian Operator

Observation 1:

The Laplacian is a self-adjoint operator.

The Laplacian Operator

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The Laplacian is a self-adjoint operator.

To show this, we need to show that for any two functions *f* and *g*, we have:

$$\left\langle f, \nabla^2 g \right\rangle = \left\langle \nabla^2 f, g \right\rangle$$

The Laplacian Operator

Observation 1:

First, we show this in the 1D case, for functions $f(\theta)$ and $g(\theta)$:

$$\langle f, g'' \rangle = \langle f'', g \rangle$$

The Laplacian Operator

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First, we show this in the 1D case, for functions $f(\theta)$ and $g(\theta)$:

$$\langle f, g'' \rangle = \langle f'', g \rangle$$

Writing the dot-product as an integral gives:

$$\langle f, g'' \rangle = \int_0^{2\pi} f(\theta) g''(\theta)\, d\theta$$

The Laplacian Operator

Observation 1:

Using the product rule for derivatives:

$$(fg)' = f'g + fg'$$

we know that:

$$\int_0^{2\pi} \left(fg\right)'(\theta)\,d\theta = \int_0^{2\pi} f'(\theta)g(\theta)\,d\theta + \int_0^{2\pi} f(\theta)g'(\theta)\,d\theta$$

The Laplacian Operator

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Using the product rule for derivatives:

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we know that:

$$\int_0^{2\pi} (fg)'(\theta)d\theta = \int_0^{2\pi} f'(\theta)g(\theta)d\theta + \int_0^{2\pi} f(\theta)g'(\theta)d\theta'$$

Furthermore, since we are assuming that *f* and *g* are functions on a circle, so that $(fg)(0)=(fg)(2\pi)$:

$$\int_0^{2\pi} (fg)'(\theta)d\theta = (fg)(2\pi) - (fg)(0) = 0$$

The Laplacian Operator

Observation 1:

Thus, we have:

$$\int_0^{2\pi} f'(\theta)g(\theta)d\theta = -\int_0^{2\pi} f(\theta)g'(\theta)d\theta$$

The Laplacian Operator

Observation 1:

Thus, we have:

$$\int_0^{2\pi} f'(\theta)g(\theta)d\theta = -\int_0^{2\pi} f(\theta)g'(\theta)d\theta$$

Taking the second derivative, this gives:

$$\int_0^{2\pi} f''(\theta)g(\theta)d\theta = \int_0^{2\pi} f(\theta)g''(\theta)d\theta$$

$$\Updownarrow$$

$$\langle f'', g \rangle = \langle f, g'' \rangle$$

The Laplacian Operator

Observation 1:

To generalize this to higher dimensions, we write out the dot-product as:

$$\left\langle \nabla^2 f, g \right\rangle = \int_0^{2\pi}\int_0^{2\pi} \left(\frac{\partial^2 f}{\partial \theta^2} \right) g \, d\theta \, d\phi + \int_0^{2\pi}\int_0^{2\pi} \left(\frac{\partial^2 f}{\partial \phi^2} \right) g \, d\phi \, d\theta$$

The Laplacian Operator

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$$= \int_0^{2\pi}\int_0^{2\pi} f \left(\frac{\partial^2 g}{\partial \theta^2} \right) d\theta \, d\phi + \int_0^{2\pi}\int_0^{2\pi} f \left(\frac{\partial^2 g}{\partial \phi^2} \right) d\phi \, d\theta$$

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$$= \int_0^{2\pi}\int_0^{2\pi} f \left(\frac{\partial^2 g}{\partial \theta^2} \right) d\theta \, d\phi + \int_0^{2\pi}\int_0^{2\pi} f \left(\frac{\partial^2 g}{\partial \phi^2} \right) d\phi \, d\theta$$

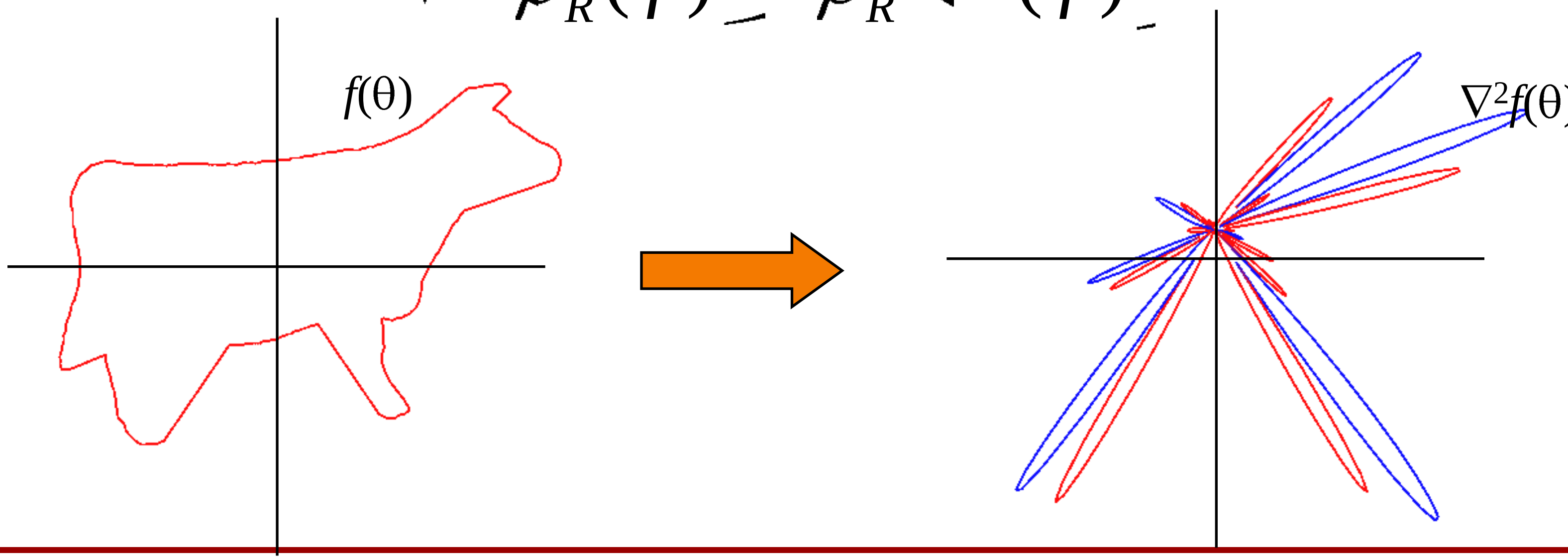
$$= \left\langle f, \nabla^2 g \right\rangle$$

The Laplacian Operator

Observation 2:

The Laplacian operator commutes with rotation – i.e. computing the Laplacian and rotating gives the same function as first rotating and then computing the Laplacian:

$$\nabla^2\left(\rho_R(f)\right) = \rho_R\left(\nabla^2(f)\right)$$



The Laplacian Operator

Implications:

- **Observation 1**: Since the Laplacian operator is self-adjoint, it must be diagonalizable.

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  $\Rightarrow$ There is an orthogonal basis of eigenvectors.

The Laplacian Operator

Implications:

- **Observation 1**: Since the Laplacian operator is self-adjoint, it must be diagonalizable.
  $\Rightarrow$ There is an orthogonal basis of eigenvectors.
  $\Rightarrow$ If we group the eigenvectors with the same eigenvalues together, we get a set of vector spaces F_λ such that for any function $f \in F_\lambda$:

$$\nabla^2 f = \lambda f$$

The Laplacian Operator

Implications:

- **Observation 2**: Since the Laplacian operator commutes with rotation, if *f* is an eigenvector of the Laplacian with eigenvalue λ, any rotation of *f* must also be an eigenvector of the Laplacian with eigenvalue λ.

The Laplacian Operator

Implications:

- **Observation 2**: Since the Laplacian operator commutes with rotation, if *f* is an eigenvector of the Laplacian with eigenvalue λ, any rotation of *f* must also be an eigenvector of the Laplacian with eigenvalue λ.

  $\Rightarrow$ The spaces F_λ must be fixed under the action of rotation.

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  $\Rightarrow$ The spaces F_λ must be fixed under the action of rotation.

  $\Rightarrow$ The spaces F_λ are sub-representations for the group of rotation.

The Laplacian Operator

Going back to the problem of finding the irreducible representations, this indicates that we can begin by looking for the eigenspaces of the Laplacian operator.

Computing the Laplacian

We know how to compute the Laplacian of a circular function represented by parameter:

$$\nabla^2 f(\theta) = f''(\theta)$$

How do we compute the Laplacian for a function represented by restriction?

Computing the Laplacian

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Consider the function $f(x,y)=x$:

- In the plane, the Laplacian is:

$$\nabla^2 f(x,y) = 0$$

Computing the Laplacian

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Example:

Consider the function $f(x,y)=x$:

- In the plane, the Laplacian is:
$$\nabla^2 f(x,y) = 0$$
- On the circle this is the function $f(\theta)=\cos(\theta)$:
$$\nabla^2 f(\theta) = -\cos(\theta)$$

Computing the Laplacian

The problem is that if we define a function on a circle as the restriction of a 2D function, the 2D Laplacian is not the same as the circular Laplacian.

Intuitively:

The Laplacian measures the difference between the value of a point and the average value of the "neighbors".

Who the "neighbors" are changes depending on whether we are considering the plane or the circle.

Computing the Laplacian

Recall:

Using the fact that for a vector field:

$$F(x,y) = \left(F_1(x,y), F_2(x,y)\right)$$

the divergence is defined:

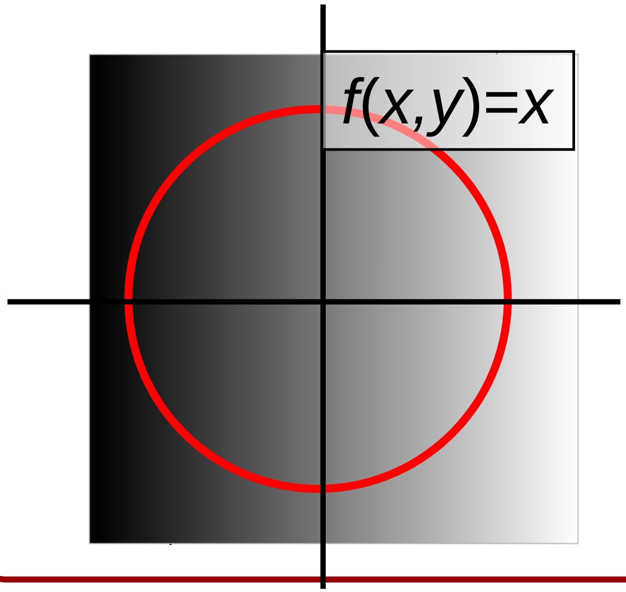
$$\mathrm{div}(F) = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$

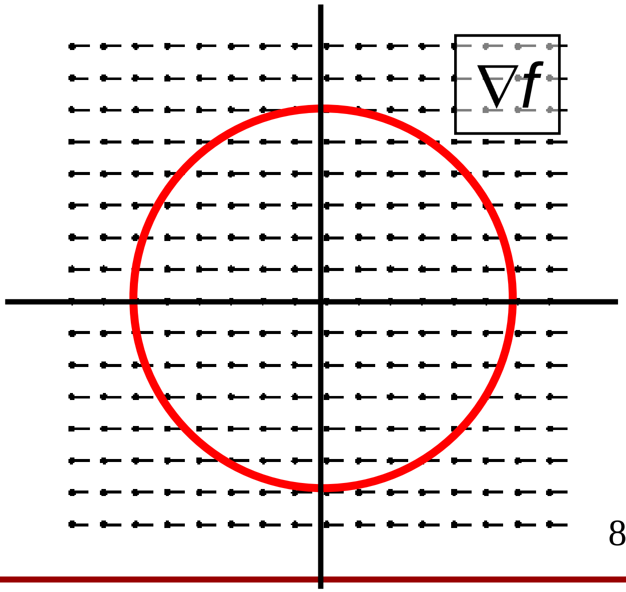
We can also express the Laplacian as the divergence of the gradient:

$$\nabla^2 f = \nabla \cdot (\nabla f)$$

Computing the Gradient

In general, the gradient of the function $f(x,y)$ need not lie along the unit-circle:



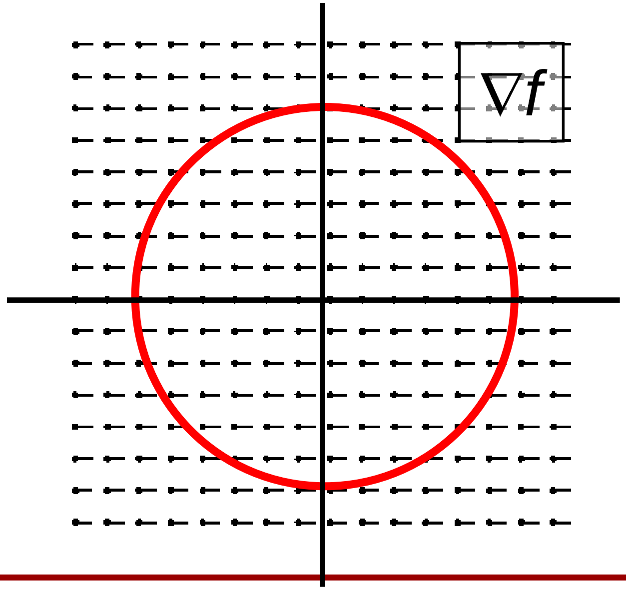


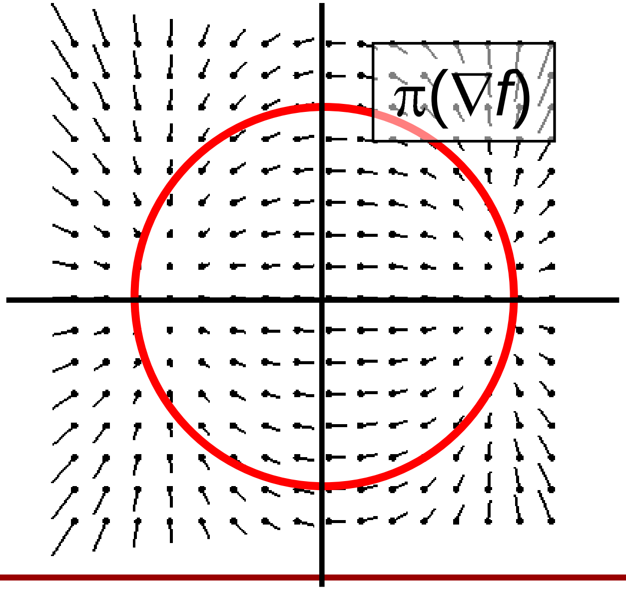
Computing the Gradient

In general, the gradient of the function *f*(*x*,*y*) need not lie along the unit-circle.

We can fix this by projecting the gradient on to the unit circle:

$$\nabla f \rightarrow \nabla f - \langle \nabla f, (x,y) \rangle (x,y)$$





Computing the Laplacian

The divergence of a vector field *F*:

$$F(x,y) = \left(F_1(x,y), F_2(x,y)\right)$$

can be expressed as the sum of partials:

$$\operatorname{div}(F) = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$

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That is, the divergence is the derivative of the *x*-component of the vector field in the *x*-direction, plus the derivative of the *y*-component of the vector field in the *y*-direction.

Computing the Laplacian

It turns out that we don't have to restrict ourselves to the coordinate axis:

For any orthogonal basis $\{v,w\}$, the divergence can be expressed as the derivative of the v-component of the vector field in the v-direction, plus the derivative of the w-component of the vector field in the w-direction:

$$\mathrm{div}(F) = \nabla \cdot F = \frac{\partial \langle F, v \rangle}{\partial v} + \frac{\partial \langle F, w \rangle}{\partial w}$$

Computing the Laplacian

Thus, to compute the divergence of the vector field along the circle, we can compute the 2D divergence, and subtract off the contribution from the normal direction:

$$\text{div}_{\text{circle}}(F) = \text{div}_{\text{2D}}(F) - \frac{\partial \langle F, n \rangle}{\partial n}$$

Computing the Laplacian

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$$\mathrm{div}_{\mathrm{circle}}(F) = \mathrm{div}_{\mathrm{2D}}(F) - \frac{\partial\langle F, n\rangle}{\partial n}$$

Since the component of the vector field in the normal direction is a scalar function, its derivative in the normal direction can be expressed as a gradient:

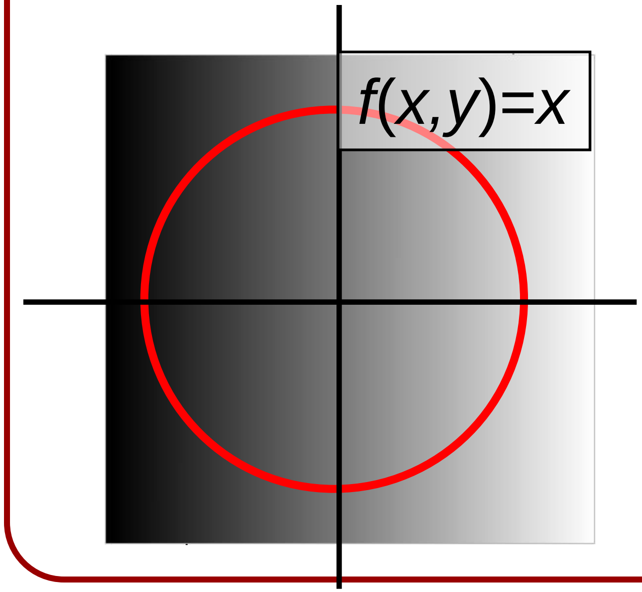
$$\frac{\partial\langle F, n\rangle}{\partial n} = \langle \nabla\langle F, n\rangle, n\rangle$$

Computing the Laplacian

<u>Example</u>:

Consider the function:

$$f(x, y) = x$$

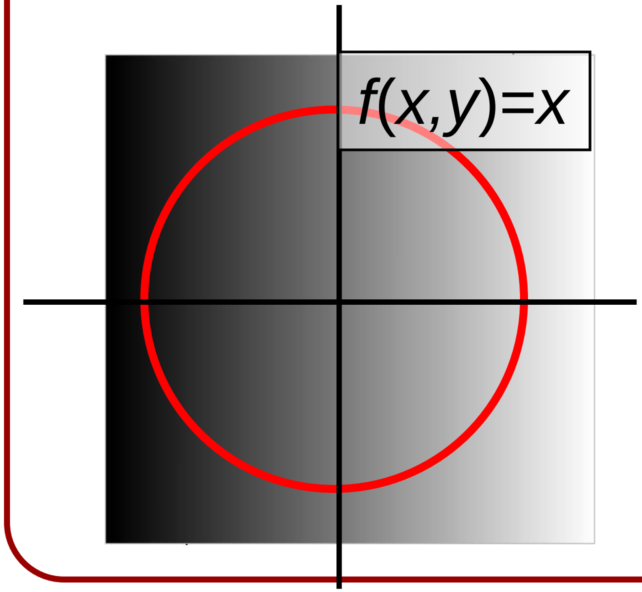


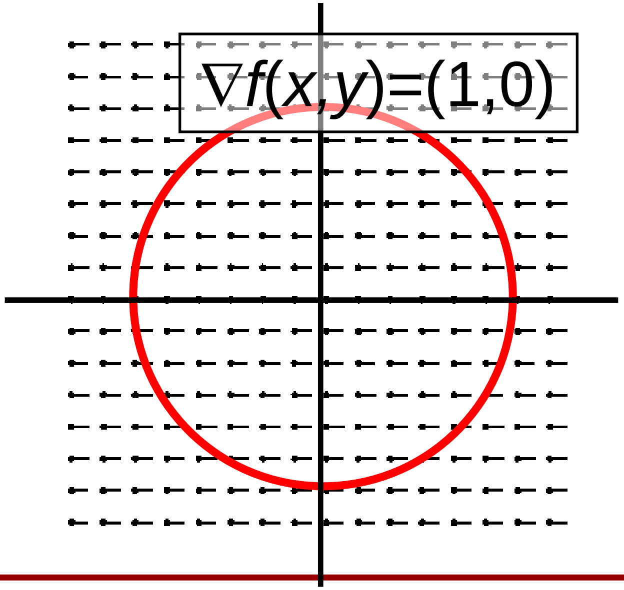
Computing the Laplacian

Example: $f(x,y)=x$

Its gradient is:

$$\nabla f = (1,0)$$



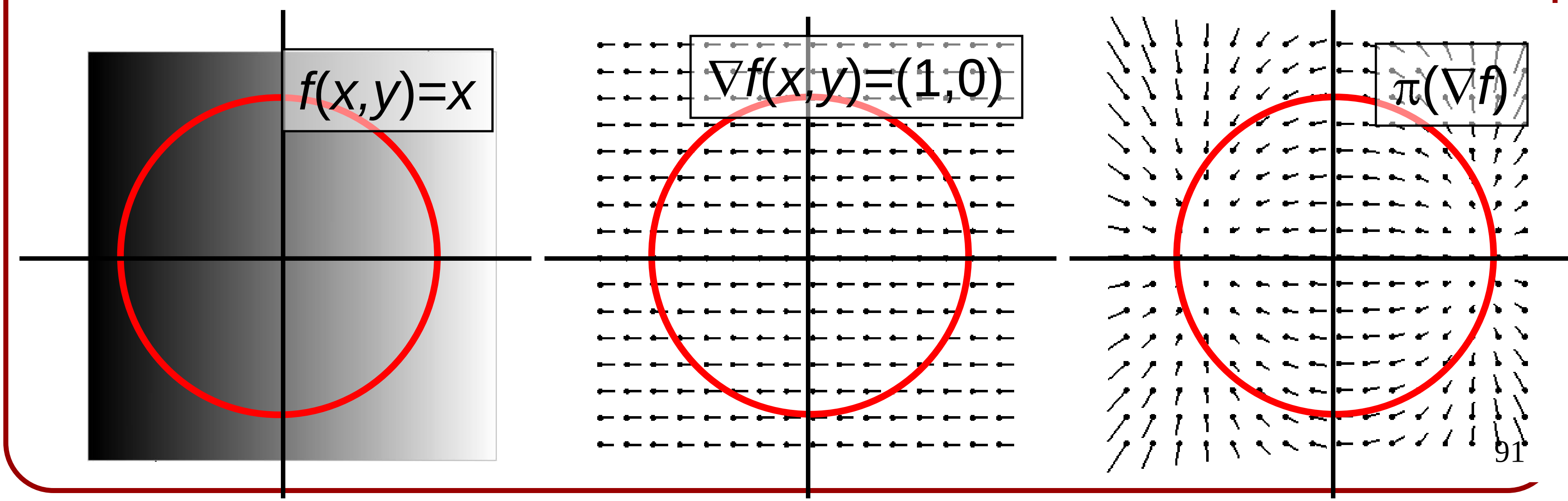


Computing the Laplacian

<u>Example</u>: $f(x,y)=x$ $\nabla f(x,y)=(1,0)$

Projecting the gradient onto the unit-circle we get:

$$\pi_n^{\perp}(\nabla f) = \nabla f - \langle \nabla f, n \rangle n$$



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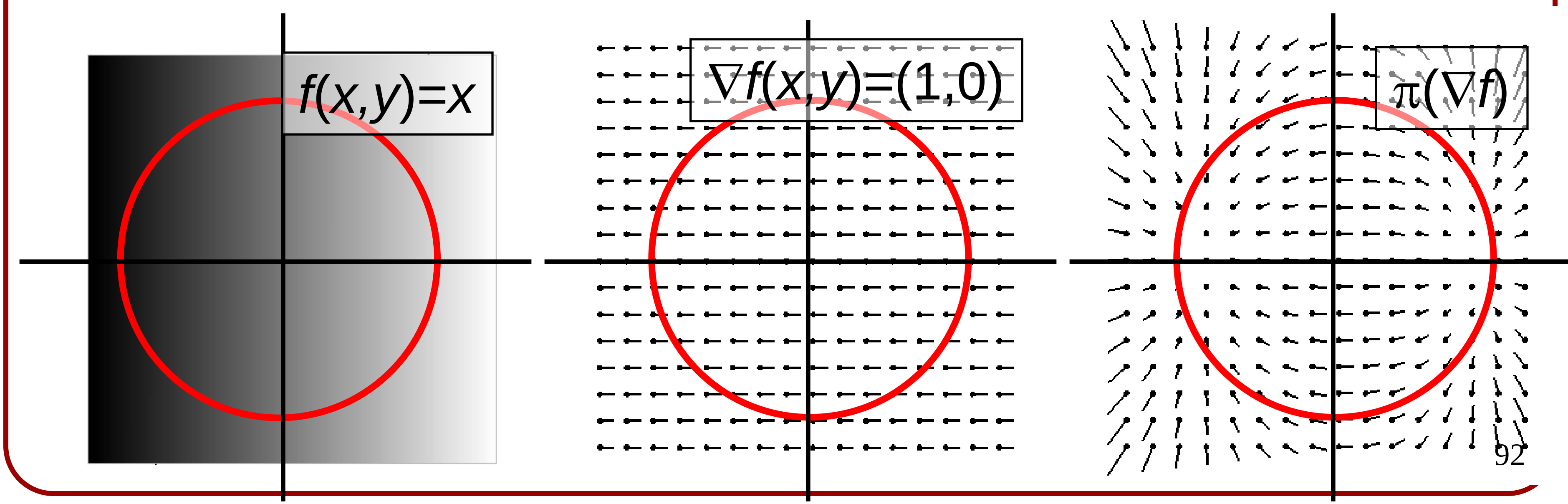
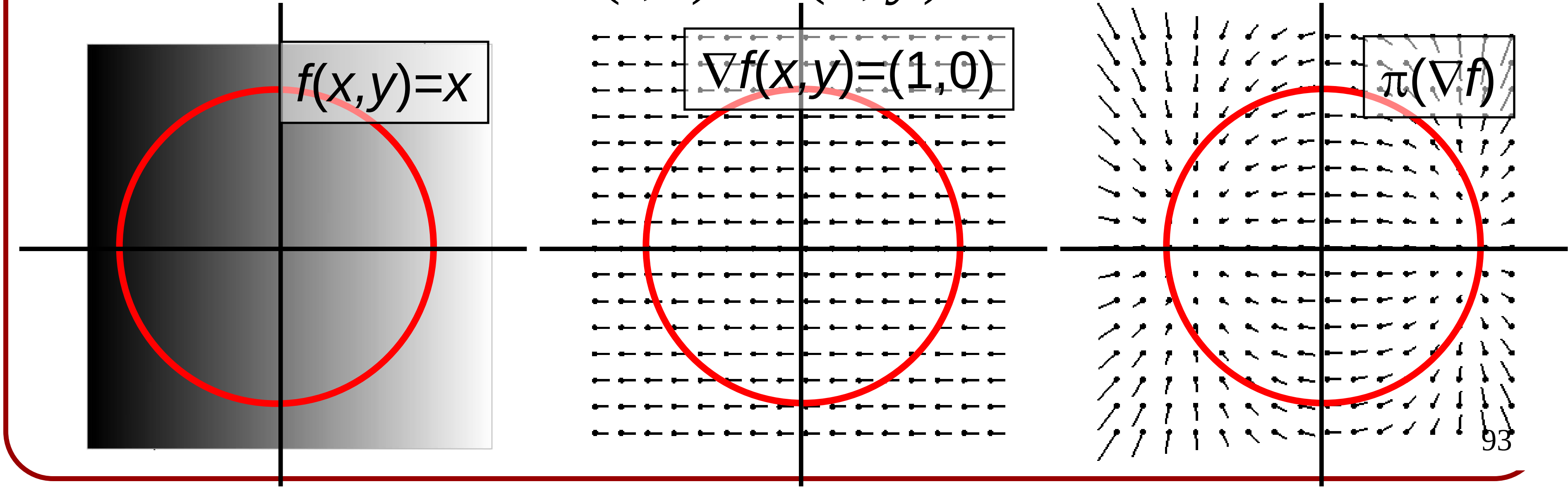


Computing the Laplacian

Example: $f(x,y)=x$ $\nabla f(x,y)=(1,0)$

Projecting the gradient onto the unit-circle we get:

$$\pi_n^{\perp}(\nabla f) = \nabla f - \langle \nabla f, n \rangle n$$
$$= \nabla f - \langle \nabla f, (x,y) \rangle (x,y)$$
$$= (1,0) - x(x,y)$$



Computing the Laplacian

<u>Example</u>: $f(x,y)=x$ $\nabla f(x,y)=(1,0)$ $\pi_n^{\perp}(\nabla f)(x,y)=(1,0)-x(x,y)$

The divergence of the vector field $\pi_n^{\perp}(\nabla f)$ is:

$$\mathrm{div}_{2D}\left(\pi_n^{\perp}(\nabla f)\right) = -2x - x$$

Computing the Laplacian

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The projection of the vector field onto the normal direction is:

$$\left\langle \pi_n^{\perp}(\nabla f),n\right\rangle=\left\langle (1,0)-x(x,y),(x,y)\right\rangle$$

Computing the Laplacian

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$$= x - x\left(x^2 + y^2\right)$$

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$$\left\langle \pi_n^{\perp}(\nabla f)(n),n\right\rangle = x-x\left(x^2+y^2\right)$$

Its gradient is:

$$\nabla\left\langle \pi_n^{\perp}(\nabla f)(n),n\right\rangle=(1,0)-(3x^2+y^2,2xy)$$

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So the divergence in the normal direction is:

$$\mathrm{div}_{n}\left(\pi_n^{\perp}(\nabla f)\right) = \left\langle \nabla\left\langle \pi_n^{\perp}(\nabla f), n\right\rangle, n\right\rangle$$

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$$= \left\langle (1,0) - (3x^2 + y^2, 2xy), (x, y)\right\rangle$$

Computing the Laplacian

Example: $f(x,y)=x$ $\nabla f(x,y)=(1,0)$ $\pi_n^\perp(\nabla f)(x,y)=(1,0)-x(x,y)$

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Computing the Laplacian

Example: $f(x,y)=x$ $\nabla f(x,y)=(1,0)$ $\pi_n^{\perp}(\nabla f)(x,y)=(1,0)-x(x,y)$

Thus the circular Laplacian can be expressed as the difference between the 2D divergence and the divergence in the normal direction:

$$\nabla^2_{\text{circle}} f(x,y) = \text{div}_{2D}\left(\pi_n^{\perp}(\nabla f)\right) - \text{div}_n\left(\pi_n^{\perp}(\nabla f)\right)$$

Computing the Laplacian

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Since points on the circle satisfy $x^2+y^2=1$, this implies that for (x,y) on the circle:

$$\nabla^2_{\text{circle}} f(x,y) = -x$$

Computing the Laplacian

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Since points on the circle satisfy $x^2+y^2=1$, this implies that for (x,y) on the circle:

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Computing the Laplacian

Example: $f(x,y)=x$ $\nabla f(x,y)=(1,0)$ $\pi_n^{\perp}(\nabla f)(x,y)=(1,0)-x(x,y)$

Thus just as in the parameter case we get that the function $f(x,y)=x$, is an eigenvector of the circular Laplacian operator, with eigenvalue -1.