FFTs in Graphics and Vision

Finding Sub-Representations

Outline

Review

Sub-Representations for the Circle

Sub-Representations for the Sphere

Review (Polynomials)

A polynomial of degree d in variables $\{x_1,\ldots,x_n\}$ is a linear sum of monomials in $\{x_1,\ldots,x_n\}$, each of whose degree is no greater than d.

We denote by $P^d(x_1,\ldots,x_n)$ the set of polynomials in the variables $\{x_1,\ldots,x_n\}$ of degree d.

Review (Homogenous Polynomials)

A degree d polynomial is said to be homogenous if the individual monomials all have degree d.

We denote by $HP^d(x_1,\ldots,x_n)$ the set of homogenous polynomials in the variables $\{x_1,\ldots,x_n\}$ of degree d.

Review (Polynomial Decomposition)

Any degree *d* polynomial in $\{x_1,\ldots,x_n\}$ can be uniquely expressed as the sum of homogenous polynomials in $\{x_1,\ldots,x_n\}$ of degrees 0 through *d*:

$$P^d(x_1,\ldots,x_n) = HP^0(x_1,\ldots,x_n) + \cdots + HP^d(x_1,\ldots,x_n)$$

Review (Homogenous Polynomial Decomposition)

Any homogenous polynomial in $\{x_1,\ldots,x_n\}$ of degree d can be uniquely expressed as:

- x_1 times a homogenous polynomial in $\{x_1,\ldots,x_n\}$ of degree d-1, plus
- a homogenous polynomial in $\{x_2,\ldots,x_n\}$ of degree d.

$$HP^d(x_1,\ldots,x_n) = x_1 HP^{d-1}(x_1,\ldots,x_n) + HP^d(x_2,\ldots,x_n)$$

Review (Sums of Polynomials)

Given two polynomials of degree *d*:

$$p_1(x_1,\ldots,x_n),\, p_2(x_1,\ldots,x_n) \in P^d(x_1,\ldots,x_n)$$

The linear sum of the two polynomials is a polynomial of degree *d*:

$$a \cdot p_1(x_1,\ldots,x_n) + b \cdot p_2(x_1,\ldots,x_n) \in P^d(x_1,\ldots,x_n)$$

Review (Product of Polynomials)

Given two polynomials of degrees d_1 and d_2:

$$p_1(x_1,\ldots,x_n) \in P^{d_1}(x_1,\ldots,x_n)$$

$$p_2(x_1,\ldots,x_n) \in P^{d_2}(x_1,\ldots,x_n)$$

The product of the two polynomials is a polynomial of degree d_1+d_2:

$$p_1(x_1,\ldots,x_n) \cdot p_2(x_1,\ldots,x_n) \in P^{d_1+d_2}(x_1,\ldots,x_n)$$

Review (Powers of Polynomials)

Given a polynomial of degree *d*:

$$p(x_1,\ldots,x_n) \in P^d(x_1,\ldots,x_n)$$

The *k*-th power of the polynomial is a polynomial of degree *k*·*d*:

$$p^k(x_1,\ldots,x_n) \in P^{d\cdot k}(x_1,\ldots,x_n)$$

Review (Sums of Homogenous Polynomials)

Given two homogenous polynomials of degree *d*:

$$p_1(x_1,\ldots,x_n),\, p_2(x_1,\ldots,x_n) \in HP^d(x_1,\ldots,x_n)$$

The linear sum of the two polynomials is a homogenous polynomial of degree *d*:

$$a \cdot p_1(x_1,\ldots,x_n) + b \cdot p_2(x_1,\ldots,x_n) \in HP^d(x_1,\ldots,x_n)$$

Review (Product of Homogenous Polynomials)

Given two homogenous polynomials of degrees d_1 and d_2:

$$p_1(x_1,\ldots,x_n) \in HP^{d_1}(x_1,\ldots,x_n)$$

$$p_2(x_1,\ldots,x_n) \in HP^{d_2}(x_1,\ldots,x_n)$$

The product of the two polynomials is a homogenous polynomial of degree d_1+d_2.

$$p_1(x_1,\ldots,x_n)\cdot p_2(x_1,\ldots,x_n) \in HP^{d_1+d_2}(x_1,\ldots,x_n)$$

Review (Powers of Homogenous Polynomials)

Given a homogenous polynomial of degree *d*:

$$p(x_1,\ldots,x_n) \in HP^d(x_1,\ldots,x_n)$$

The *k*-th power of the polynomial is a homogenous polynomial of degree *k*·*d*:

$$p^k(x_1,\ldots,x_n) \in HP^{d\cdot k}(x_1,\ldots,x_n)$$

Review (Dimension of the Space of Homogenous Polynomials)

Homogenous Polynomials of Degree Zero:

The dimension of the space of homogenous polynomials of degree 0 in any number of variables is one:

$$\dim HP^0(x_1,\ldots,x_n)=1$$

Review (Dimension of the Space of Homogenous Polynomials)

<u>Homogenous Polynomials in One Variable</u>:

The dimension of the space of homogenous polynomials of degree *d* in one variable is one, for all degrees *d*:

$$\dim\left(HP^{d}(x)\right) = 1$$

Review (Dimension of the Space of Homogenous Polynomials)

Homogenous Polynomials in *n* Variables:

The dimension of the space of homogenous polynomials of degree *d* in *n* variables is:

$$\dim\left[HP^d(x_1,\ldots,x_n)\right] = \dim\left[HP^d(x_2,\ldots,x_n)\right] + \dim\left[HP^{d-1}(x_1,\ldots,x_n)\right]$$

Review (Dimension of the Space of Homogenous Polynomials)

Homogenous Polynomials in *n* Variables:

The dimension of the space of homogenous polynomials of degree *d* in *n* variables is:

$$\begin{aligned}\dim\left(HP^d(x_1,\ldots,x_n)\right) = &\dim\left(HP^d(x_2,\ldots,x_n)\right)\\ &+\dim\left(HP^{d-1}(x_2,\ldots,x_n)\right)\\ &+\dim\left(HP^{d-2}(x_1,\ldots,x_n)\right)\end{aligned}$$

Review (Dimension of the Space of Homogenous Polynomials)

<u>Homogenous Polynomials in *n* Variables</u>:

The dimension of the space of homogenous polynomials of degree *d* in *n* variables is:

$$\dim\left(HP^d(x_1,\ldots,x_n)\right) = \sum_{i=1}^{d} \dim\left(HP^i(x_2,\ldots,x_n)\right) + \dim\left(HP^0(x_1,\ldots,x_n)\right)$$

Review (Dimension of the Space of Homogenous Polynomials)

<u>Homogenous Polynomials in *n* Variables</u>:

The dimension of the space of homogenous polynomials of degree *d* in *n* variables is:

$$\dim\left(HP^d(x_1,\ldots,x_n)\right)=\sum_{i=1}^{d}\dim\left(HP^i(x_2,\ldots,x_n)\right)+1$$

Review (Dimension of the Space of Homogenous Polynomials)

<u>Homogenous Polynomials in *n* Variables</u>:

One Variable:

$$\dim\left(HP^d(x)\right) = 1$$

Review (Dimension of the Space of Homogenous Polynomials)

Homogenous Polynomials in *n* Variables:

One Variable:

$$\dim\left(HP^d(x)\right) = 1$$

Two Variables:

$$\dim\left(HP^d(x,y)\right) = 1 + \sum_{i=1}^{d} \dim\left(HP^i(x)\right)$$

Review (Dimension of the Space of Homogenous Polynomials)

Homogenous Polynomials in *n* Variables:

One Variable:

$$\dim\left(HP^d(x)\right)=1$$

Two Variables:

$$\dim\left(HP^d(x,y)\right)=1+\sum_{i=1}^{d}\dim\left(HP^i(x)\right)$$

$$=1+\sum_{i=1}^{d}1$$

Review (Dimension of the Space of Homogenous Polynomials)

<u>Homogenous Polynomials in *n* Variables</u>:

One Variable:

$$\dim\left[HP^d(x)\right] = 1$$

Two Variables:

$$\dim\left[HP^d(x,y)\right] = 1 + \sum_{i=1}^{d} \dim\left[HP^i(x)\right]$$

$$= 1 + \sum_{i=1}^{d} 1$$

$$= 1 + d$$

Review (Dimension of the Space of Homogenous Polynomials)

Homogenous Polynomials in *n* Variables:

One Variable:

$$\dim\left(HP^d(x)\right) = 1$$

Two Variables:

$$\dim\left(HP^d(x,y)\right) = 1 + d$$

Three Variables:

$$\dim\left(HP^d(x,y,z)\right) = 1 + \sum_{i=1}^{d} \dim\left(HP^i(x,y)\right)$$

Review (Dimension of the Space of Homogenous Polynomials)

<u>Homogenous Polynomials in *n* Variables</u>:

One Variable:

$$\dim\left(HP^d(x)\right)=1$$

Two Variables:

$$\dim\left(HP^d(x,y)\right)=1+d$$

Three Variables:

$$\dim\left(HP^d(x,y,z)\right)=1+\sum_{i=1}^{d}\dim\left(HP^i(x,y)\right)$$

$$=1+\sum_{i=1}^{d}\left(1+i\right)$$

Review (Dimension of the Space of Homogenous Polynomials)

<u>Homogenous Polynomials in *n* Variables</u>:

One Variable:

$$\dim\left(HP^d(x)\right) = 1$$

Two Variables:

$$\dim\left(HP^d(x,y)\right) = 1 + d$$

Three Variables:

$$\dim\left(HP^d(x,y,z)\right) = 1 + \sum_{i=1}^{d} \dim\left(HP^i(x,y)\right)$$

$$= 1 + \sum_{i=1}^{d} \left(1 + i\right) = \frac{(d+2)(d+1)}{2}$$

Review (Dimension of the Space of Homogenous Polynomials)

Homogenous Polynomials in *n* Variables:

One Variable:

$$\dim\left(HP^d(x)\right) = 1$$

Two Variables:

$$\dim\left(HP^d(x,y)\right) = 1 + d$$

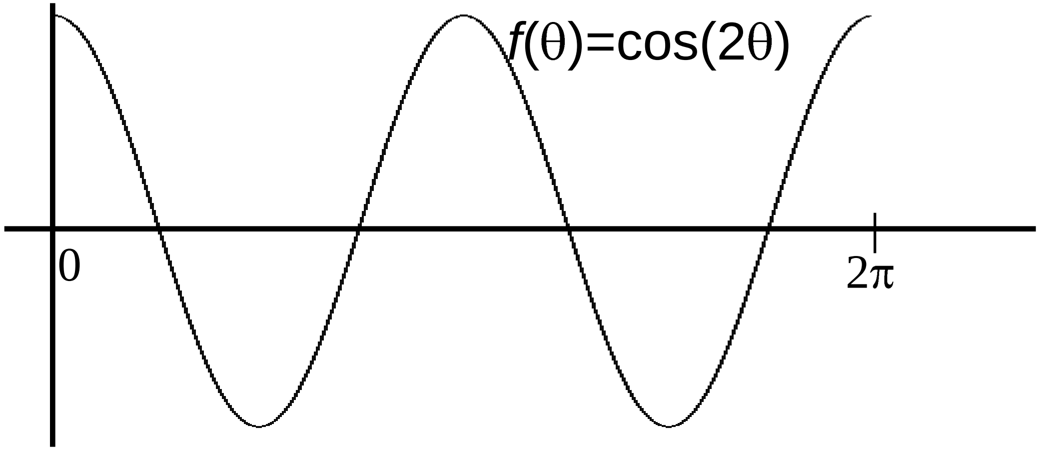
Three Variables:

$$\dim\left(HP^d(x,y,z)\right) = \frac{(d+2)(d+1)}{2}$$

Review (Representing Circular Functions)

There are two ways we can represent a function on the unit-circle:

1. By Parameter: Every point on the circle can be represented by an angle in the range $[0,2\pi)$.
   $\Rightarrow$ We can represent circular functions as 1D functions on the domain $[0,2\pi)$.



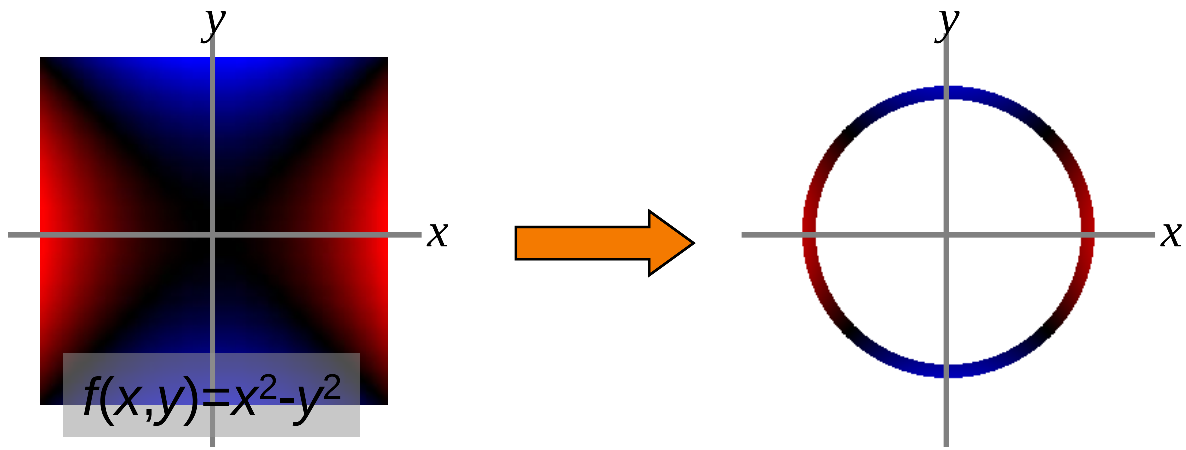
Review (Representing Circular Functions)

There are two ways we can represent a function on the unit-circle:

2. By Restriction: We know that the unit-circle "lives" in 2D, i.e. it is the set of points (x,y) satisfying:

$$x^2 + y^2 = 1$$

$\Rightarrow$ We can represent circular functions by looking at the restriction of 2D functions to the unit-circle.



Outline

Review

Sub-Representations for the Circle

Sub-Representations for the Sphere

Irreducible Representations

Recall:

In considering many essential shape/image analysis tasks:

- Rotation invariant representation
- Image filtering
- Symmetry detection
- (2D) Rotational alignment

we needed to consider the representation of the group of 2D rotations on the space of circular functions.

Irreducible Representations

Recall:

In order to perform these tasks efficiently and/or effectively, we depended on Schur's Lemma:

> Since the group was commutative, the irreducible representations were all one (complex) dimensional

Irreducible Representations

<u>Challenge</u>:

We know that the irreducible representations exist. How do we find them?

Sub-Representations

How do we find a sub-space of functions that is also a sub-representation?

Sub-Representations

How do we find a sub-space of functions that is also a sub-representation?

That is, how do we find a space of functions with the property that a rotation of a function from this space, will give some other function in the space.

Fourier Basis

For the case of the circle, we already know that these spaces are the one-dimensional sub-spaces spanned by the complex exponentials:

$$f_k(\theta) = e^{ik\theta}$$

Fourier Basis

For the case of the circle, we already know that these spaces are the one-dimensional sub-spaces spanned by the complex exponentials:

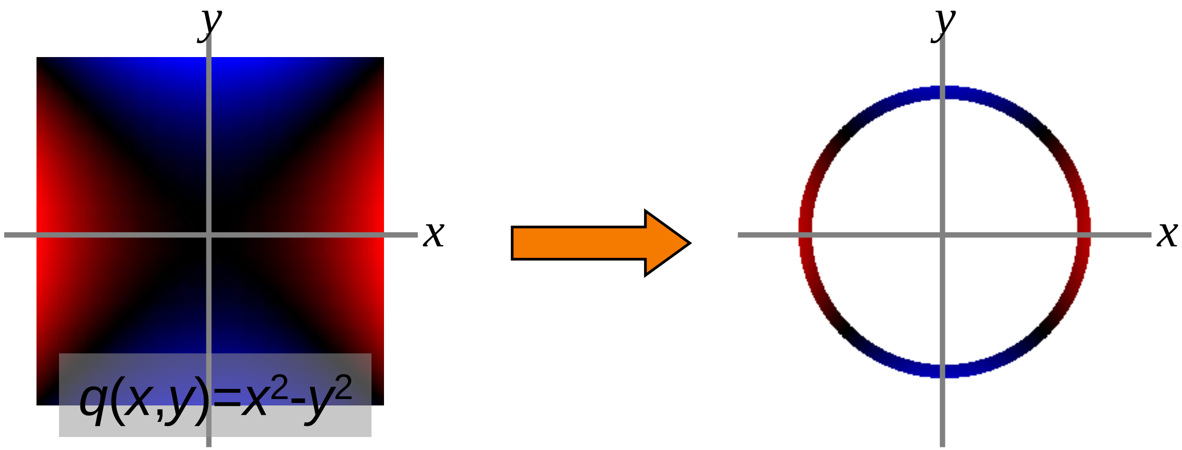
$$f_k(\theta) = e^{ik\theta}$$

But how would we go about finding them if we didn't know?

Polynomials

Consider the circular functions that are obtained by restricting degree *d* polynomials to the circle:

$$q(x,y)=\sum_{j+k\leq d} a_{jk}x^j y^k$$



Polynomials

Consider the circular functions that are obtained by restricting degree *d* polynomials to the circle:

$$q(x,y) = \sum_{j+k \le d} a_{jk} x^j y^k$$

How does a rotation act on this function?

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Polynomials

Rotations act on the space of functions by rotating the domain of evaluation:

$$\left(\rho_R(q)\right)(x,y) = q\left(R^{-1}(x,y)\right)$$

Polynomials

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$$\left(\rho_R(q)\right)(x, y) = q\left(R^{-1}(x, y)\right)$$

Since the inverse of a rotation is its transpose, the rotation R^{-1}, acts on the 2D space by sending:

$$R^{-1}(x, y) = (ax + cy, bx + dy)$$

Polynomials

This means that the rotation acts on the polynomial by sending:

$$q(x,y)=\sum_{j+k\le d} a_{jk}\,x^j y^k$$

$$\left(\rho_R(q)\right)(x,y)=\sum_{j+k\le d} a_{jk}\,(ax+cy)^j(bx+dy)^k$$

Polynomials

This means that the rotation acts on the polynomial by sending:

$$q(x,y) = \sum_{j+k\le d} a_{jk} x^j y^k$$

$$\big(\rho_R(q)\big)(x,y) = \sum_{j+k\le d} a_{jk} (\underbrace{ax+cy}_{\text{Degree 1}})^j (\underbrace{bx+dy}_{\text{Degree 1}})^k$$

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Polynomials

This means that the rotation acts on the polynomial by sending:

$$q(x,y) = \sum_{j+k\le d} a_{jk} x^j y^k$$

$$\left[\rho_R(q)\right](x,y) = \sum_{j+k\le d} a_{jk} \underbrace{(ax+cy)^j (bx+dy)^k}_{\text{Degree } j+k}$$

Since $j+k\le d$, the rotation of q must also be a polynomial of degree d.

Polynomials

In sum, if we start with a polynomial of degree *d*:

$$q(x,y) \in P^d(x,y)$$

and we apply any rotation *R* to the polynomial, the rotated polynomial will also be a polynomial of degree *d*:

$$\rho_R(q) \in P^d(x,y)$$

Polynomials

Thus, the space of functions obtained by restricting polynomials of degree *d* to the unit circle is a sub-representation.

Homogenous Polynomials

It turns out that we can repeat the same argument for the restrictions of homogenous polynomials of degree *d* to the unit circle:

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Homogenous Polynomials

Thus, the space of functions obtained by restricting homogenous polynomials of degree *d* to the unit circle is a sub-representation.

Homogenous Polynomials

How small are these sub-representations?

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The space of homogenous polynomials of degree d in two variables is $(d+1)$-dimensional.

Homogenous Polynomials

How small are these sub-representations?

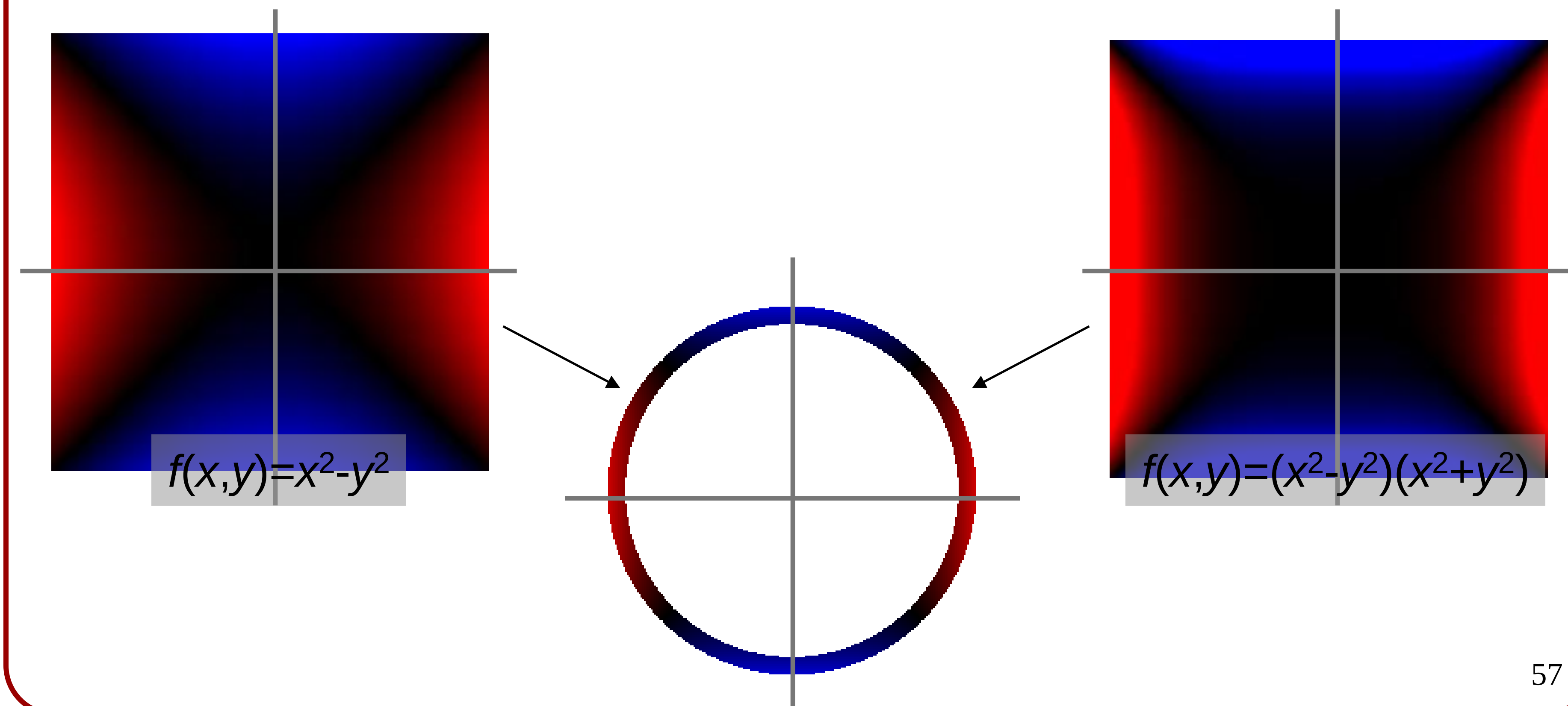
The space of homogenous polynomials of degree d in two variables is $(d+1)$-dimensional.

We know that the irreducible representations all have to be one-dimensional – what's going on?

Homogenous Polynomials

Recall:

Two different functions in 2D, can have the same restriction to the unit-circle.



Homogenous Polynomials

In particular, since any point (x,y) on the circle satisfies the condition:

$$x^2 + y^2 = 1$$

Homogenous Polynomials

In particular, since any point (x,y) on the circle satisfies the condition:

$$x^2 + y^2 = 1$$

for any degree d homogenous polynomial $q(x,y)$:

$$q(x,y) \in HP^d(x,y)$$

the homogenous polynomial of degree d+2:

$$q(x,y)(x^2+y^2) \in HP^{d+2}(x,y)$$

will have the same restriction to the unit circle.

Homogenous Polynomials

Thus, when we consider the restriction of the homogenous polynomials to the unit circle, the degree *d* polynomials are "contained" in the space of degree (*d*+2) polynomials.

Homogenous Polynomials

Thus, when we consider the restriction of the homogenous polynomials to the unit circle, the degree d polynomials are "contained" in the space of degree $(d+2)$ polynomials.

Since we already know that the restrictions of degree d polynomials to the unit circle are a sub-representation, we only want to consider the polynomials of degree $(d+2)$ whose restrictions are perpendicular to those of polynomials of degree d.

Homogenous Polynomials

Example:

- d=0:

  The space of homogenous polynomials is spanned by {1} so the restriction is just the space of constant functions.

Homogenous Polynomials

Example:

- $d=1$:

  The space of homogenous polynomials is spanned by $\{x,y\}$ so the restriction is just the space of functions of the form $ax+by$.

Homogenous Polynomials

<u>Example</u>:

- *d*=1:

  The space of homogenous polynomials is spanned by {*x*,*y*} so the restriction is just the space of functions of the form *ax*+*by*.

  Since we can write out the *x* and *y* coordinates in terms of the circular angle θ:

$$x = \cos(\theta) \qquad y = \sin(\theta)$$

  this gives the space of circular functions of the form:

$$f(\theta) = a\cos(\theta) + b\sin(\theta)$$

Homogenous Polynomials

<u>Example</u>:

- $d=2$:

  The space of homogenous polynomials is spanned by $\{x^2, xy, y^2\}$ so the restriction is just the space of functions of the form $ax^2+bxy+cy^2$.

Homogenous Polynomials

Example:

- d=2:

  The space of homogenous polynomials is spanned by $\{x^2, xy, y^2\}$ so the restriction is just the space of functions of the form $ax^2+bxy+cy^2$.

  In terms of the circular angle, this gives the space of circular functions of the form:

$$f(\theta) = a\cos^2(\theta) + b\cos(\theta)\sin(\theta) + c\sin^2(\theta)$$

Homogenous Polynomials

<u>Example</u>:

- *d*=2:

$$f(\theta) = a\cos^2(\theta) + b\cos(\theta)\sin(\theta) + c\sin^2(\theta)$$

Since we know that:

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

is a constant function already accounted for by the *d*=0 case, what we want is the space of homogenous polynomial restrictions that are perpendicular to those accounted for by the *d*=0 case.

Homogenous Polynomials

Example:

- d=2:

A function of the form:

$$f(\theta) = a\cos^2(\theta) + b\cos(\theta)\sin(\theta) + c\sin^2(\theta)$$

is perpendicular to the function:

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

if and only if:

$$0 = \left\langle \cos^2(\theta) + \sin^2(\theta), a\cos^2(\theta) + b\cos(\theta)\sin(\theta) + c\sin^2(\theta) \right\rangle$$

Homogenous Polynomials

<u>Example</u>:

- *d*=2:

$$0 = \left\langle \cos^2(\theta) + \sin^2(\theta), a\cos^2(\theta) + b\cos(\theta)\sin(\theta) + c\sin^2(\theta) \right\rangle$$

$$0 = \int_0^{2\pi} \left(\cos^2(\theta) + \sin^2(\theta)\right)\left(a\cos^2(\theta) + b\cos(\theta)\sin(\theta) + c\sin^2(\theta)\right) d\theta$$

Homogenous Polynomials

Example:

- d=2:

$$0=\int_0^{2\pi}\left(\cos^2(\theta)+\sin^2(\theta)\right)\left(a\cos^2(\theta)+b\cos(\theta)\sin(\theta)+c\sin^2(\theta)\right)d\theta$$

$$0=\int_0^{2\pi}a\cos^4(\theta)+b\cos^3(\theta)\sin(\theta)+c\cos^2(\theta)\sin^2(\theta)\,d\theta+$$

$$+\int_0^{2\pi}a\cos^2(\theta)\sin^2(\theta)+b\cos(\theta)\sin^3(\theta)+c(\theta)\sin^4(\theta)\,d\theta$$

Homogenous Polynomials

Example:

- *d*=2:

$$0 = \int_0^{2\pi} a\cos^4(\theta) + b\cos^3(\theta)\sin(\theta) + c\cos^2(\theta)\sin^2(\theta)\, d\theta +$$

$$+ \int_0^{2\pi} a\cos^2(\theta)\sin^2(\theta) + b\cos(\theta)\sin^3(\theta) + c(\theta)\sin^4(\theta)\, d\theta$$

$$\Updownarrow$$

$$0 = a\frac{2\pi}{3} + (a+c)\frac{\pi}{4} + c\frac{2\pi}{3}$$

Homogenous Polynomials

Example:

- *d*=2:

$$0 = a\frac{2\pi}{3} + (a+c)\frac{\pi}{4} + c\frac{2\pi}{3}$$

$$\Updownarrow$$

$$c = -a$$

Homogenous Polynomials

Example:

- d=2:

  Since a homogenous polynomials of degree 2 can be expressed as:

$$f(\theta) = a\cos^2(\theta) + b\cos(\theta)\sin(\theta) + c\sin^2(\theta)$$

  and since the orthogonality condition implies that:

$$c = -a$$

  A basis for the sub-representation is:

$$\cos^2(\theta) - \sin^2(\theta), \cos(\theta)\sin(\theta)$$

Homogenous Polynomials

Example:

- *d*=2:

Since a homogenous polynomials of degree 2 can be expressed as:

$$f(\theta) = a\cos^2(\theta) + b\cos(\theta)\sin(\theta) + c\sin^2(\theta)$$

and since the orthogonality conditions imply that:

$$c = -a$$

A basis for the sub-representation is:

$$\cos^2(\theta) - \sin^2(\theta),\ \cos(\theta)\sin(\theta)$$

$$\Updownarrow$$

$$\cos(2\theta),\ \sin(2\theta)$$

Homogenous Polynomials

Example:

- $d \geq 2$:

  As in the $d=2$ case, we start with the space of homogenous polynomials of degree d.

Homogenous Polynomials

Example:

- $d \geq 2$:

  As in the d=2 case, we start with the space of homogenous polynomials of degree d.

  Since the space of homogenous polynomials of degree d-2 is contained in this space, we need to "throw out" the degree d-2 polynomials.

Homogenous Polynomials

<u>Example</u>:

- *d*≥2:

  As in the *d*=2 case, we start with the space of homogenous polynomials of degree *d.*

  Since the space of homogenous polynomials of degree *d*-2 is contained in this space, we need to "throw out" the degree *d*-2 polynomials.

  Thus, the final dimension of the sub-representation is:

$$\dim\left(HP^{d}(x,y)\right)-\dim\left(HP^{d-2}(x,y)\right)=(d+1)-(d-1)$$

Homogenous Polynomials

<u>Example</u>:

- *d*≥2:

  As in the *d*=2 case, we start with the space of homogenous polynomials of degree *d.*

  Since the space of homogenous polynomials of degree *d*-2 is contained in this space, we need to “throw out” the degree *d*-2 polynomials.

  Thus, the final dimension of the sub-representation is:

$$\dim\left(HP^{d}(x,y)\right)-\dim\left(HP^{d-2}(x,y)\right)=(d+1)-(d-1)$$
$$=2$$

Homogenous Polynomials

<u>Example</u>:

- *d*≥2:

  As in the *d*=2 case, one can show that the two functions:

$$\cos(d\theta),\ \sin(d\theta)$$

  are a basis for the sub-representation.

Homogenous Polynomials

Note:

These sub-representations are not irreducible.

Homogenous Polynomials

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These sub-representations are not irreducible.

By Schur's lemma, we know that the irreducible representations are all one-dimensional and for $d>0$, we are getting two-dimensional sub-representations.

Homogenous Polynomials

<u>Note</u>:

These sub-representations are not irreducible.

By Schur's lemma, we know that the irreducible representations are all one-dimensional and for *d*>0, we are getting two-dimensional sub-representations.

To get the irreducible representations, we need to further break apart these sub-representations.

$$\cos(d\theta),\ \sin(d\theta) \rightarrow \cos(d\theta)+i\sin(d\theta) = e^{id\theta}$$

$$\cos(d\theta),\ \sin(d\theta) \rightarrow \cos(d\theta)-i\sin(d\theta) = e^{-id\theta}$$

Outline

Review

Sub-Representations for the Circle

Sub-Representations for the Sphere

Spherical Functions

As in the case of circular functions, we would like to find the sub-representations of the spherical functions – sub-spaces of spherical functions which get rotated back into themselves.

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As in the case of circular functions, we would like to find the sub-representations of the spherical functions – sub-spaces of spherical functions which get rotated back into themselves.

In this case, the group of rotations is not commutative, so we do not expect the sub-representations to be one-dimensional.

Homogenous Polynomials

As in the case of circular functions, we will consider spherical functions that are obtained by restricting homogenous polynomials of degree *d* to the unit sphere:

$$q(x,y,z) = \sum_{j+k+l=d} a_{jkl} x^j y^k z^l$$

Homogenous Polynomials

If *R* is a rotation:

$$R = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

then *R* will rotate the polynomial *q* by:

$$\big(\rho_R(q)\big)(x,y,z) = \sum_{j+k+l=d} a_{jkl}(ax+dy+gz)^j(bx+ey+hz)^k(cx+fy+iz)^l$$

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Again, rotations fix homogenous polynomials – mapping homogenous polynomials of degree d back into homogenous polynomials of degree d.

Homogenous Polynomials

As in the 2D case, we know that the restrictions of homogenous polynomials of degree *d* to the unit sphere contain the restrictions of homogenous polynomials of degree *d*-2 to the unit sphere.

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If *q* is a homogenous polynomial of degree *d*:

$$q(x,y,z) \in HP^d(x,y,z)$$

the homogenous polynomial of degree *d*+2:

$$q(x,y,z)(x^2+y^2+z^2) \in HP^{d+2}(x,y,z)$$

will have the same restriction to the unit sphere.

Homogenous Polynomials

Thus, the sub-representations can be obtained by considering the restrictions of homogenous polynomials of degree d to the unit sphere, and then removing those functions that were already introduced at degree d-2.

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Thus, the dimension of the space obtained from the degree *d* homogenous polynomials will be:

$$\dim\left[HP^{d}(x,y,z)\right] - \dim\left[HP^{d-2}(x,y,z)\right] = 2d+1$$

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It turns out that for spherical functions, these are the irreducible representations for the group of rotations.