FFTs in Graphics and Vision

Polynomials and Circular Functions

Outline

The 2π Term in Assignment 1

Homogenous Polynomials

Representations of Functions on the Unit-Circle

The 2π Term in Assignment 1

Given an *n*-dimensional array of values, we would like to treat the values as the regular samples of some continuous, periodic, function:

$$f[\,]\leftarrow f(x)$$

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What is the domain/period/wavelength of $f(x)$?

The 2π Term in Assignment 1

What is the domain/period/wavelength of $f(x)$?

Two possible approaches:

- Dimension Dependent $[0,n)$:

$$f[j] = f(j)$$

- Dimension Independent $[0,\rho)$:

$$f[j] = f\left(\frac{j\rho}{n}\right)$$

The 2π Term in Assignment 1

Dimension Dependent Domain [0,*n*):

This provides a norm-preserving map from the space of *n*-dimensional arrays to the space of functions:

Vector Square Norm	Function Square Norm
$\|f[\,]\|^2 = \sum_{j=0}^{n-1} \lvert f[j] \rvert^2$	

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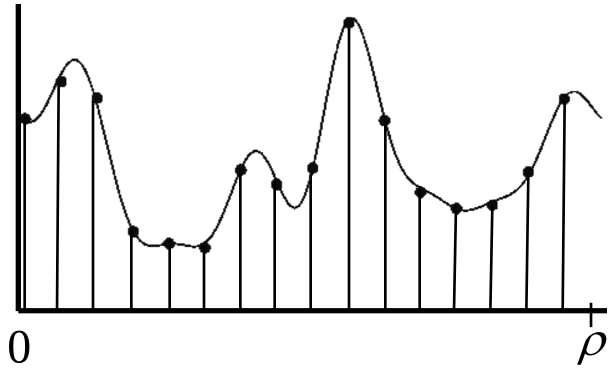
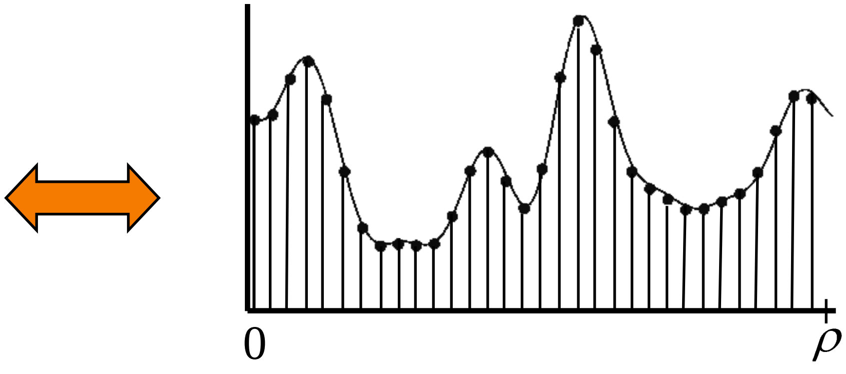
	$\approx \sum_{j=0}^{n-1} \left\vert f(j)\right\vert^2$

The 2π Term in Assignment 1

Dimension Independent Domain $[0,\rho)$:

This provides a way for treating two arrays of different dimensions as regular samplings of the same function at different resolutions:





The 2π Term in Assignment 1

Dimension Independent Domain [0,ρ):

This does not provide a norm-preserving map from the space of *n*-dimensional arrays to the space of functions:

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$\|f[\,]\|^2 = \sum_{j=0}^{n-1} \left\|f[j]\right\|^2$	

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$\left\Vert f[\;]\right\Vert^2 = \sum_{j=0}^{n-1} \left\vert f[j]\right\vert^2$	$\left\Vert f(\;)\right\Vert^2 = \int_0^\rho \left\vert f(x)\right\vert^2 dx$
$= \sum_{j=0}^{n-1} \left\vert f\left(\frac{j\rho}{n}\right)\right\vert^2$	$= \lim_{l\to\infty} \sum_{j=0}^{l-1} \left\vert f\left(\frac{j\rho}{l}\right)\right\vert^2 \frac{\rho}{l}$
	$\approx \sum_{j=0}^{n-1} \left\vert f\left(\frac{j\rho}{n}\right)\right\vert^2 \frac{\rho}{n}$

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This mapping scales the square norm by ρ/n.

The 2π Term in Assignment 1

Dimension Independent Domain $[0,\rho)$:

When we consider periodic functions on a line – i.e. functions on a circle – we set the domain to be equal to the length of a circle: $[0,2\pi)$.

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Similarly, for periodic functions on a plane – i.e. functions on the product of two circles (a torus) – we choose the domain to be $[0,2\pi)$x $[0,2\pi)$.

The 2π Term in Assignment 1

How does this affect the Fourier coefficients?

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The Fourier coefficients of *f*[ ] are defined as the linear coefficients of *f*[ ] with respect to the Fourier basis:

$$f[\,] = \sum_{k=0}^{n-1} \hat{f}[k] v_k[\,]$$

where the v_k[ ] correspond to regular samples of the *k*-th complex exponential at *n* different positions:

$$v_k[\,] \approx \left(e^{ik2\pi 0/n}, e^{ik2\pi 1/n}, \ldots, e^{ik2\pi (n-2)/n}, e^{ik2\pi (n-1)/n} \right)$$

The 2π Term in Assignment 1

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We know that the $v_k[\ ]$ are perpendicular to each other, and we would like them to have unit-norm so that they form an orthonormal basis:

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$v_k[\ ] \rightarrow \frac{v_k[\ ]}{\sqrt{n}}$	
$(2\pi j/l)\big\|^2 \frac{n}{l}$	
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$$\left\|v_k(\,)\right\|^2 = \int_0^{2\pi} \left|e^{ikx}\right|^2 dx$$

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$v_k[\,] \to \frac{v_k[\,]}{\sqrt{n}}$ … $^{(2\pi j/l)}\Big\|^2 \frac{n}{l}$	$= \lim_{l\to\infty} \sum_{j=0}^{l-1} \left\|e^{ik(2\pi j/l)}\right\|^2 \frac{2\pi}{l}$
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$v_k[\ ] \rightarrow \frac{v_k[\ ]}{\sqrt{n}}$	$v_k[\ ] \rightarrow \frac{v_k[\ ]}{\sqrt{2\pi}}$
$\ldots {}^{(2\pi j/l)}\Big\rvert^2 \frac{n}{l}$	$\ldots {}^{j/l)}\Big\rvert^2 \frac{2\pi}{l}$
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How does this affect the Fourier coefficients?

So, if we compute the Fourier coefficients of *f*[ ] assuming that the domain is [0,*n*), to get the Fourier coefficients of *f*[ ] on the domain [0,2π), we need to scale the coefficients:

- [0,*n*)→[0,2π): $\hat{f}[k] \rightarrow \sqrt{\frac{n}{2\pi}}\hat{f}[k]$

- [0,2π)→[0,*n*): $\hat{f}[k] \rightarrow \sqrt{\frac{2\pi}{n}}\hat{f}[k]$

The 2π Term in Assignment 1

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To perform Gaussian smoothing of a signal *f*[ ][ ], we want to correlate with a signal *g*[ ][ ]whose entries "sum to one".

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$$\int_0^n\int_0^n g(x,y)\,dy\,dx = 1$$	

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$$\int_0^n\int_0^n g(x,y)\,dy\,dx=1$$ ↓ $$\lim_{l\to\infty}\sum_{j,k=0}^{l-1} g\left(\frac{jn}{l},\frac{kn}{l}\right)\left(\frac{n}{l}\right)^2=1$$	

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The Gaussian is normalized if the sum of the entries equals 1.

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$\sum_{j,k=0}^{n-1} g[j][k] = 1$	$\lim_{l\to\infty}\sum_{j,k=0}^{l-1} g\left(\frac{j2\pi}{l},\frac{k2\pi}{l}\right)\left(\frac{2\pi}{l}\right)^2 = 1$

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$$\sum_{j,k=0}^{n-1} g[j][k]=1$$	$$\sum_{j,k=0}^{n-1} g\left(\frac{j2\pi}{n},\frac{k2\pi}{n}\right)\left(\frac{2\pi}{n}\right)^2=1$$

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$\int_0^n\int_0^n g(x,y)\,dy\,dx = 1$	$\sum_{j,k=0}^{n-1} g\left(\frac{j2\pi}{n},\frac{k2\pi}{n}\right)\left(\frac{2\pi}{n}\right)^2 = 1$
$\sum_{j,k=0}^{n-1} g[j][k] = 1$	$\sum_{j,k=0}^{n-1} g[j][k]\left(\frac{2\pi}{n}\right)^2 = 1$

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To perform Gaussian smoothing of a signal *f*[ ][ ], we want to correlate with a signal *g*[ ][ ]whose entries "sum to one".

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$$\int_0^n\int_0^n g(x,y)\,dy\,dx=1$$	$$\int_0^{2\pi}\int_0^{2\pi} g(x,y)\,dy\,dx=1$$
The Gaussian is normalized if the sum of the entries equals 1.	The Gaussian is normalized if the sum of the entries equals (*n*/2π)[2].
$$\sum_{j,k=0}^{n-1} g[j][k]=1$$	$$\sum_{j,k=0}^{n-1} g[j][k]\left(\frac{2\pi}{n}\right)^2=1$$

Outline

The 2π Term in Assignment 1

Homogenous Polynomials

Representations of Functions on the Unit-Circle

Monomials

Definition:

A monomial in variables $\{x_1,\ldots,x_n\}$ is a product of non-negative integer powers of the variables.

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Definition:

A monomial in variables $\{x_1, \ldots, x_n\}$ is a product of non-negative integer powers of the variables.

The degree of a monomial is the sum of the powers.

Monomials

Examples:

- Degree 0: 1
- Degree 1: x, y, z
- Degree 2: x^2, y^2, z^2, xy, xz, yz
- Degree 3: x^3, x^2y, x^2z, xy^2, xz^2, xyz, y^3, y^2z, yz^2, z^3

Polynomials

Definition:

A polynomial of degree d in variables $\{x_1,\ldots,x_n\}$ is a linear sum of monomials in $\{x_1,\ldots,x_n\}$, each of whose degree is no greater than d.

Polynomials

Notation:

Denote by $P^d(x_1,\ldots,x_n)$ the set of polynomials in $\{x_1,\ldots,x_n\}$ of degree d.

Polynomials

Examples:

- $d=0$:
  - » $P^0(x)= P^0(x,y)= P^0(x,y,z)=a$
- $d=1$:
  - » $P^1(x)=ax+c$
  - » $P^1(x,y)=ax+by+c$
  - » $P^1(x,y,z)=ax+by+cz+d$
- $d=2$:
  - » $P^2(x)=ax^2+bx+c$
  - » $P^2(x,y)=ax^2+by^2+cxy+dx+ey+f$
  - » $P^2(x,y,z)=ax^2+by^2+cz^2+dxy+exz+gyz+hx+iy+jz+k$
- …

Homogenous Polynomials

Definition:

A degree *d* polynomial is said to be homogenous if the individual monomials all have degree *d*.

Homogenous Polynomials

Notation:

Denote by $HP^d(x_1,\ldots,x_n)$ the set of homogenous polynomials in $\{x_1,\ldots,x_n\}$ of degree d.

Homogenous Polynomials

<u>Examples</u>:

- $d=0$:
  - » $HP^0(x)= HP^0(x,y)= HP^0(x,y,z)=a$
- $d=1$:
  - » $HP^1(x)=ax$
  - » $HP^1(x,y)=ax+by$
  - » $HP^1(x,y,z)=ax+by+cz$
- $d=2$:
  - » $HP^2(x)=ax^2$
  - » $HP^2(x,y)=ax^2+by^2+cxy$
  - » $HP^2(x,y,z)=ax^2+by^2+cz^2+dxy+exz+gyz$
- …

Homogenous Polynomials

Note 1:

The set of polynomials of degree *d* and the set of homogenous polynomials of degree *d* are both vector spaces.

Homogenous Polynomials

Note 2:

Any degree d polynomial in $\{x_1,\ldots,x_n\}$ can be uniquely expressed as the sum of homogenous polynomials in $\{x_1,\ldots,x_n\}$ of degrees 0 through d:

$$P^d(x_1,\ldots,x_n) = HP^0(x_1,\ldots,x_n) + \cdots + HP^d(x_1,\ldots,x_n)$$

Homogenous Polynomials

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Example:

- $p(x,y)=2x^2 + 3y^2 - xy + 5x - 7y + 2$

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Homogenous Polynomials

Note 3:

Any homogenous polynomial in $\{x_1,\ldots,x_n\}$ of degree d can be uniquely expressed as:

- x_1 times a homogenous polynomial in $\{x_1,\ldots,x_n\}$ of degree d-1, plus
- a homogenous polynomial in $\{x_2,\ldots,x_n\}$ of degree d.

$$HP^d(x_1,\ldots,x_n) = x_1 HP^{d-1}(x_1,\ldots,x_n) + HP^d(x_2,\ldots,x_n)$$

Homogenous Polynomials

Note 3:

$$HP^d(x_1,\ldots,x_n) = x_1 HP^{d-1}(x_1,\ldots,x_n) + HP^d(x_2,\ldots,x_n)$$

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Homogenous Polynomials

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Example:

- $$\underbrace{p(x,y)}_{HP^2(x,y)} = 2x^2 + 3y^2 - xy$$

  $$= x\underbrace{(2x - y)}_{HP^1(x,y)} + \underbrace{3y^2}_{HP^2(y)}$$

Dimensions

What is the dimension of $P^d(x_1, \ldots, x_n)$?

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Since every polynomial of degree *d* can be uniquely expressed as the sum of homogenous polynomials of degrees 0 through *d*:

$$\dim\left(P^d(x_1,\ldots,x_n)\right) = \dim\left(HP^0(x_1,\ldots,x_n)\right) + \cdots + \dim\left(HP^d(x_1,\ldots,x_n)\right)$$

Dimensions

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Dimensions

Three properties give us a recursive definition:

1. A homogenous polynomial of degree *d* factors as:

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2. The space of homogenous polynomials in $\{x_1,\ldots,x_n\}$ of degree 0 is one-dimensional:
$$HP^0(x_1,\ldots,x_n) = a$$

3. The space of homogenous polynomials in $\{x\}$ of degree *d* is one-dimensional:
$$HP^d(x) = ax^d$$

Dimensions

Combining these:

- We get the recursive definition for the dimension:

$$HP^d(x_1,\ldots,x_n) = x_1 HP^{d-1}(x_1,\ldots,x_n) + HP^d(x_2,\ldots,x_n)$$

$$\dim\left[HP^d(x_1,\ldots,x_n)\right] = \dim\left[HP^{d-1}(x_1,\ldots,x_n)\right] + \dim\left[HP^d(x_2,\ldots,x_n)\right]$$

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$$HP^{d}(x_1,\ldots,x_n) = x_1 HP^{d-1}(x_1,\ldots,x_n) + HP^{d}(x_2,\ldots,x_n)$$

$$\dim\left(HP^{d}(x_1,\ldots,x_n)\right) = \dim\left(HP^{d-1}(x_1,\ldots,x_n)\right) + \dim\left(HP^{d}(x_2,\ldots,x_n)\right)$$

- Plus the initial conditions:

$$\dim\left(HP^{0}(x_1,\ldots,x_n)\right) = \dim\left(HP^{d}(x)\right) = 1$$

Dimensions

$$\dim\left[HP^d(x_1,\ldots,x_n)\right] = \dim\left[HP^{d-1}(x_1,\ldots,x_n)\right] + \dim\left[HP^d(x_2,\ldots,x_n)\right]$$

$$\dim\left[HP^0(x_1,\ldots,x_n)\right] = \dim\left[HP^d(x)\right] = 1$$

- One Variable:
  - » $\dim[HP^0(x)]=1$
  - » $\dim[HP^1(x)]=1$
  - » $\dim[HP^2(x)]=1$
  - » …
  - » $\dim[HP^d(x)]=1$

Dimensions

$$\dim\left[HP^{d}(x_1,\ldots,x_n)\right]=\dim\left[HP^{d-1}(x_1,\ldots,x_n)\right]+\dim\left[HP^{d}(x_2,\ldots,x_n)\right]$$

$$\dim\left[HP^{0}(x_1,\ldots,x_n)\right]=\dim\left[HP^{d}(x)\right]=1$$

- Two Variables:
  - » $\dim[HP^0(x,y)]=1$

Dimensions

$$\dim\left[HP^{d}(x_1,\ldots,x_n)\right]=\dim\left[HP^{d-1}(x_1,\ldots,x_n)\right]+\dim\left[HP^{d}(x_2,\ldots,x_n)\right]$$

$$\dim\left[HP^{0}(x_1,\ldots,x_n)\right]=\dim\left[HP^{d}(x)\right]=1$$

- Two Variables:
  - » $\dim[HP^0(x,y)]=1$
  - » $\dim[HP^1(x,y)]=\dim[HP^0(x,y)]+\dim[HP^1(y)]$

Dimensions

$$\dim\left[HP^d(x_1,\ldots,x_n)\right] = \dim\left[HP^{d-1}(x_1,\ldots,x_n)\right] + \dim\left[HP^d(x_2,\ldots,x_n)\right]$$

$$\dim\left[HP^0(x_1,\ldots,x_n)\right] = \dim\left[HP^d(x)\right] = 1$$

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  - » $\dim[HP^0(x,y)]=1$
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Dimensions

$$\dim\left[HP^{d}(x_1,\ldots,x_n)\right]=\dim\left[HP^{d-1}(x_1,\ldots,x_n)\right]+\dim\left[HP^{d}(x_2,\ldots,x_n)\right]$$

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Dimensions

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  - » $\dim[HP^0(x,y)]=1$
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Dimensions

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  - » dim[$HP^1(x,y)$]=2
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  - » …

$$\boxed{\dim\left[HP^d(x,y)\right] = \dim\left[HP^{d-1}(x,y)\right] + 1}$$

Dimensions

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  - » …

$$\dim\left[HP^d(x,y)\right]=\dim\left[HP^{d-1}(x,y)\right]+1$$

$$\dim\left[HP^d(x,y)\right]=1+\sum_{i=1}^{d}1=d+1$$

Dimensions

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$$\dim\left[HP^{0}(x_1,\ldots,x_n)\right]=\dim\left[HP^{d}(x)\right]=1$$

- Three Variables:
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Dimensions

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Dimensions

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- Three Variables:
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Dimensions

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  - » dim[$HP^2(x,y,z)$]=3+3=6

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- Three Variables:
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  - » dim[$HP^1(x,y,z)$]=3
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$$\boxed{\dim\left[HP^d(x,y,z)\right] = \dim\left[HP^{d-1}(x,y,z)\right] + (d+1)}$$

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  - » $\dim[HP^2(x,y,z)]=6$
  - » …

$$\dim\left[HP^d(x,y,z)\right] = \dim\left[HP^{d-1}(x,y,z)\right] + (d+1)$$

$$\dim\left[HP^d(x,y,z)\right] = 1 + \sum_{i=1}^{d}(i+1) = \frac{(d+2)(d+1)}{2}$$

Outline

The 2π Term in Assignment 1

Homogenous Polynomials

Representations of Functions on the Unit-Circle

Representing Functions on the Unit-Circle

There are two ways we can represent a function on the unit-circle:

1. By Parameter: Every point on the circle can be represented by an angle in the range $[0,2\pi)$.

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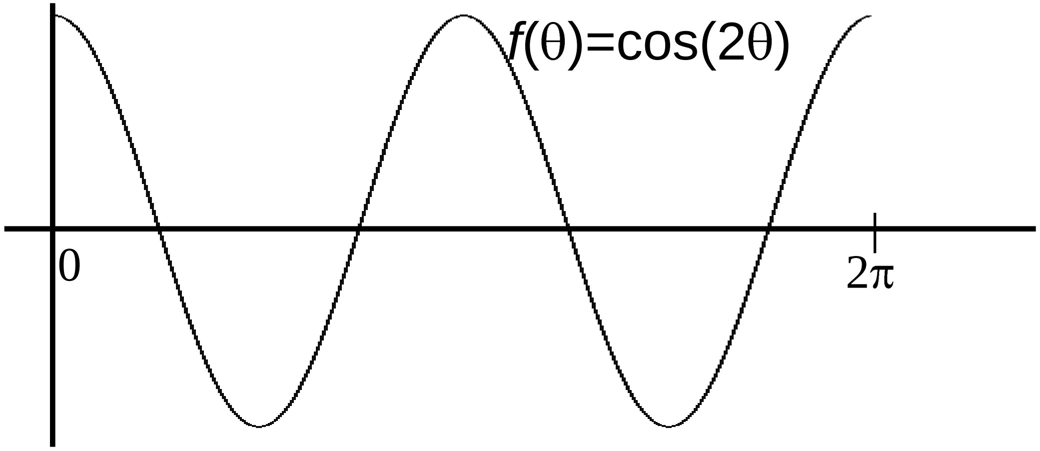
1. By Parameter: Every point on the circle can be represented by an angle in the range $[0,2\pi)$.
   $\Rightarrow$ We can represent circular functions as 1D functions on the domain $[0,2\pi)$.



Representing Functions on the Unit-Circle

There are two ways we can represent a function on the unit-circle:

2. By Restriction: We know that the unit-circle “lives” in 2D, i.e. it is the set of points (x,y) satisfying:

$$x^2 + y^2 = 1$$

Representing Functions on the Unit-Circle

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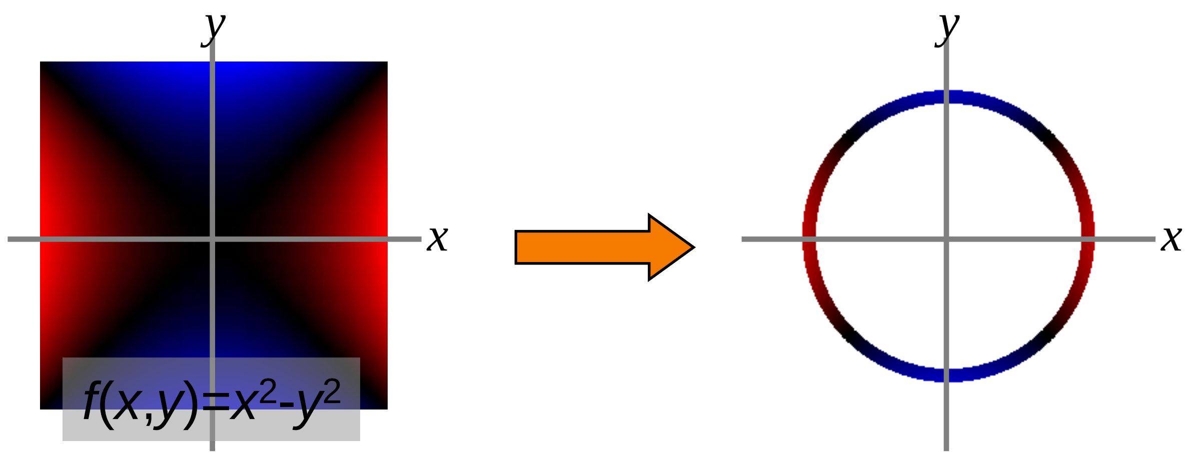
2. By Restriction: We know that the unit-circle "lives" in 2D, i.e. it is the set of points (x,y) satisfying:

$$x^2 + y^2 = 1$$

$\Rightarrow$ We can represent circular functions by looking at the restriction of 2D functions to the unit-circle.



Representing By Restriction

Observation 1:

On a circle, a point with angle θ has *x* and *y* coordinates given by:

$$x = \cos(\theta) \qquad y = \sin(\theta)$$

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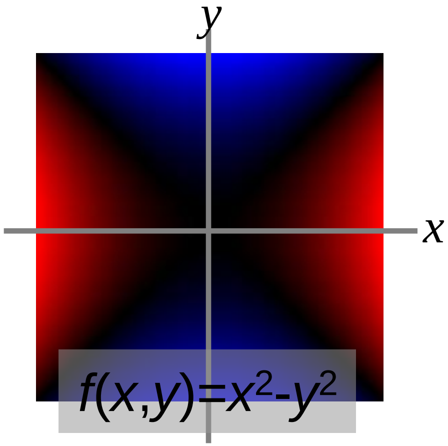
This lets us transform a circular function represented by the restriction of a 2D function *f*(*x*,*y*) to a circular function represented by parameter:

$$f(x, y) \rightarrow g(\theta) \equiv f(\cos\theta, \sin\theta)$$

Representing By Restriction

Example: If the circular function is defined as the restriction of the 2D function:

$$f(x,y)=x^2-y^2$$



Representing By Restriction

Example: If the circular function is defined as the restriction of the 2D function:

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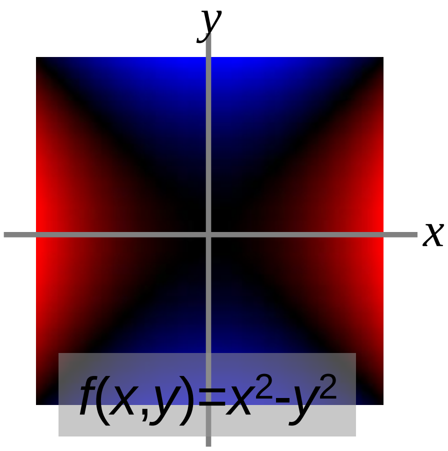
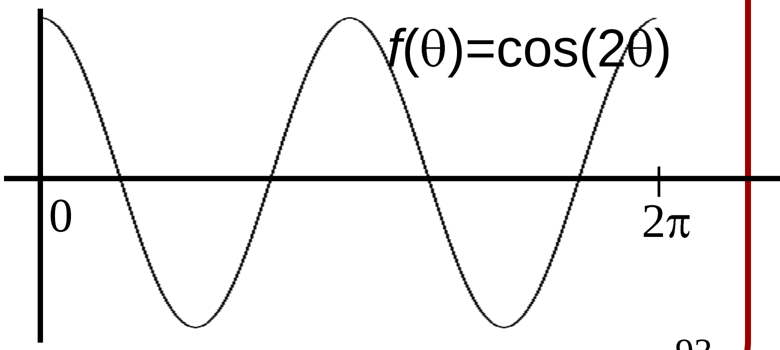
Then the representation in terms of angle is:

$$g(\theta)=\cos^2(\theta)-\sin^2(\theta)$$
$$=\cos(2\theta)$$





Representing By Restriction

Observation 2:

Two different functions in 2D, can have the same restriction to the unit-circle.