FFTs in Graphics and Vision

Differential Equations

Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

Examples

Evolving Systems

In many physical systems, the way that the system changes over time only depends on its current state.

Examples:

- Population growth
- Radioactive decay
- Vibrations of a plucked string
- Heat dissipation
- Advection of particles in a vector field

Evolving Systems

In many physical systems, the way that the system changes over time only depends on its current state.

What we would like to be able to answer is:

- Given the dependency of the change in the system to its current state, and
- Given the state of the system at some initial point in time,

How will the system evolve over time?

Evolving Systems

A Simple Case:

Consider a 1D system represented by the function $f(x,t)$, where x represents the point in space and t is a point in time.

Evolving Systems

A Simple Case:

If the change in the system can be described by the linear partial differential equation:

$$\frac{\partial f(x,t)}{\partial t} = a_0 \cdot f(x,t) + \cdots + a_n \cdot \frac{\partial^n f(x,t)}{\partial x^n}$$

And the initial state is defined by:

$$f(x,0) = g(x)$$

How do we compute the state of the system at an arbitrary point in time:

$$f(x,t) = ?$$

Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

Examples

Solving the PDE

General Approach:

1. Solve the equation $f'(x) = \lambda f(x)$.
2. Find the solutions of the differential equation.
3. Find the linear combination of solution that satisfies the initial condition.

Solve the equation *f'*(*x*)=λ*f*(*x*)

This we know how to do:

$$f'(x) = \lambda f(x)$$

Solve the equation $f'(x)=\lambda f(x)$

This we know how to do:

$$f'(x) = \lambda f(x)$$

$$\Updownarrow$$

$$f(x) = Ce^{\lambda x}$$

Find the Solutions of the Equation

Approach:

To solve for the function *f*(*x*,*t*) that satisfies:

$$\frac{\partial f(x,t)}{\partial t} = a_0 \cdot f(x,t) + \cdots + a_n \cdot \frac{\partial^n f(x,t)}{\partial x^n}$$

we will try to express *f*(*x*,*t*) as the product:

$$f(x,t) = g_\lambda(x) \cdot h_\lambda(t)$$

Find the Solutions of the Equation

Approach:

To solve for the function $f(x,t)$ that satisfies:

$$\frac{\partial f(x,t)}{\partial t} = a_0 \cdot f(x,t) + \cdots + a_n \cdot \frac{\partial^n f(x,t)}{\partial x^n}$$

we will try to express $f(x,t)$ as the product:

$$f(x,t) = g_\lambda(x) \cdot h_\lambda(t)$$

That is, we will try to solve for $g_\lambda(x)$ and $h_\lambda(t)$ s.t.:

$$\frac{\partial \left(g_\lambda(x) \cdot h_\lambda(t) \right)}{\partial t} = a_0 \cdot g_\lambda(x) \cdot h_\lambda(t) + \cdots + a_n \cdot \frac{\partial^n \left(g_\lambda(x) \cdot h_\lambda(t) \right)}{\partial x^n}$$

Find the Solutions of the Equation

Observation 1:

The map:

$$f(x) \rightarrow a_0 \cdot f(x) + \cdots + a_n \cdot \frac{\partial^n f(x)}{\partial x^n}$$

is linear.

Find the Solutions of the Equation

Observation 1:

The map:

$$f(x) \rightarrow a_0 \cdot f(x) + \cdots + a_n \cdot \frac{\partial^n f(x)}{\partial x^n}$$

is linear.

This is because taking the *k*-th derivative is a linear operator:

$$\frac{\partial^k \left(\alpha f(x) + \beta g(x)\right)}{\partial x^k} = \alpha \frac{\partial^k f(x)}{\partial x^k} + \beta \frac{\partial^k g(x)}{\partial x^k}$$

Find the Solutions of the Equation

Observation 1:

The map:

$$f(x) \to a_0 \cdot f(x) + \cdots + a_n \cdot \frac{\partial^n f(x)}{\partial x^n}$$

is linear.

We will write out this linear operator as:

$$D = a_0 + a_1 \frac{\partial}{\partial x} + \cdots + a_n \frac{\partial^n}{\partial x^n}$$

Find the Solutions of the Equation

Observation 2:

If we can find an eigenvalue/eigenvector of *D*, we can find a solution to the differential equation.

Find the Solutions of the Equation

Observation 2:

Suppose that $g_\lambda(x)$ is an eigenvector of D with eigenvalue λ:

$$D\left(g_\lambda(x)\right) = \lambda g_\lambda(x)$$

we want to find a solution to the differential equation by solving for $h_\lambda(t)$ such that:

$$\frac{\partial\left(g_\lambda(x)\cdot h_\lambda(t)\right)}{\partial t} = a_0\cdot g_\lambda(x)\cdot h_\lambda(t) + \cdots + a_n\cdot\frac{\partial^n\left(g_\lambda(x)\cdot h_\lambda(t)\right)}{\partial x^n}$$

Find the Solutions of the Equation

<u>Observation 2</u>:

To solve for $h_\lambda(t)$, we re-write the left and right hand sides of the equation:

$$\frac{\partial\left(g_\lambda(x)\cdot h_\lambda(t)\right)}{\partial t}=a_0\cdot g_\lambda(x)\cdot h_\lambda(t)+\cdots+a_n\cdot\frac{\partial^n\left(g_\lambda(x)\cdot h_\lambda(t)\right)}{\partial x^n}$$

Find the Solutions of the Equation

Observation 2:

$$\frac{\partial\left(g_\lambda(x)\cdot h_\lambda(t)\right)}{\partial t} = a_0 \cdot g_\lambda(x)\cdot h_\lambda(t) + \cdots + a_n \cdot \frac{\partial^n\left(g_\lambda(x)\cdot h_\lambda(t)\right)}{\partial x^n}$$

Using the fact that $g_\lambda(x)$ does not depend on t we can re-write the left-hand side as:

$$\text{LHS} = g_\lambda(x)\cdot\frac{\partial h_\lambda(t)}{\partial t}$$

Find the Solutions of the Equation

Observation 2:

$$\frac{\partial\left(g_\lambda(x)\cdot h_\lambda(t)\right)}{\partial t}=a_0\cdot g_\lambda(x)\cdot h_\lambda(t)+\cdots+a_n\cdot\frac{\partial^n\left(g_\lambda(x)\cdot h_\lambda(t)\right)}{\partial x^n}$$

Using the fact that $h_\lambda(t)$ does not depend on x we can re-write the right-hand side as:

$$\text{RHS}=h_\lambda(t)\cdot\left(a_0\cdot g_\lambda(x)+\cdots+a_n\cdot\frac{\partial^n g_\lambda(x)}{\partial x^n}\right)$$

Find the Solutions of the Equation

Observation 2:

$$\frac{\partial\left(g_\lambda(x)\cdot h_\lambda(t)\right)}{\partial t}=a_0\cdot g_\lambda(x)\cdot h_\lambda(t)+\cdots+a_n\cdot\frac{\partial^n\left(g_\lambda(x)\cdot h_\lambda(t)\right)}{\partial x^n}$$

Using the fact that $h_\lambda(t)$ does not depend on x we can re-write the right-hand side as:

$$\text{RHS}=h_\lambda(t)\cdot\left(a_0\cdot g_\lambda(x)+\cdots+a_n\cdot\frac{\partial^n g_\lambda(x)}{\partial x^n}\right)$$

$$=h_\lambda(t)\cdot D\left(g_\lambda(x)\right)$$

Find the Solutions of the Equation

Observation 2:

$$\frac{\partial \left(g_\lambda(x)\cdot h_\lambda(t) \right)}{\partial t} = a_0 \cdot g_\lambda(x)\cdot h_\lambda(t) + \cdots + a_n \cdot \frac{\partial^n \left(g_\lambda(x)\cdot h_\lambda(t) \right)}{\partial x^n}$$

Using the fact that $h_\lambda(t)$ does not depend on x we can re-write the right-hand side as:

$$\text{RHS} = h_\lambda(t)\cdot\left(a_0 \cdot g_\lambda(x) + \cdots + a_n \cdot \frac{\partial^n g_\lambda(x)}{\partial x^n} \right)$$

$$= h_\lambda(t)\cdot D\left(g_\lambda(x) \right)$$

$$= \lambda \cdot h_\lambda(t)\cdot g_\lambda(x)$$

Find the Solutions of the Equation

Observation 2:

So now we are left with the problem of solving for the function $h_\lambda(t)$ such that:

$$g_\lambda(x) \cdot \frac{\partial h_\lambda(t)}{\partial t} = \lambda \cdot h_\lambda(t) \cdot g_\lambda(x)$$

Find the Solutions of the Equation

Observation 2:

So now we are left with the problem of solving for the function $h_\lambda(t)$ such that:

$$g_\lambda(x) \cdot \frac{\partial h_\lambda(t)}{\partial t} = \lambda \cdot h_\lambda(t) \cdot g_\lambda(x)$$

Dividing both sides by $g_\lambda(x)$, we get:

$$\frac{\partial h_\lambda(t)}{\partial t} = \lambda \cdot h_\lambda(t)$$

Find the Solutions of the Equation

Observation 2:

So now we are left with the problem of solving for the function $h_\lambda(t)$ such that:

$$g_\lambda(x)\cdot\frac{\partial h_\lambda(t)}{\partial t}=\lambda\cdot h_\lambda(t)\cdot g_\lambda(x)$$

Dividing both sides by $g_\lambda(x)$, we get:

$$\frac{\partial h_\lambda(t)}{\partial t}=\lambda\cdot h_\lambda(t)$$

And this must imply that:

$$h_\lambda(t)=C\cdot e^{\lambda t}$$

Find the Solutions of the Equation

Observation 2:

In sum, if $g_\lambda(x)$ is an eigenvector of the linear operator D with eigenvalue λ, then:

$$f_\lambda(x,t) = e^{\lambda t} \cdot g_\lambda(x)$$

must be a solution to the differential equation:

$$\frac{\partial f(x,t)}{\partial t} = a_0 \cdot f(x,t) + \cdots + a_n \cdot \frac{\partial^n f(x,t)}{\partial x^n}$$

Satisfying the Initial Condition

Observation 3:

If $f_1(x,t)$ and $f_2(x,t)$ are solutions to the (partial) differential equation:

$$\frac{\partial f(x,t)}{\partial t} = a_0 \cdot f(x,t) + \cdots + a_n \cdot \frac{\partial^n f(x,t)}{\partial x^n}$$

then any linear combination of the two must also be a solution.

Satisfying the Initial Condition

Since we know that for any eigenvector $g_\lambda(x)$ with eigenvalue λ, the function:

$$f_\lambda(x,t) = e^{\lambda t} \cdot g_\lambda(x)$$

is a solution and we know that any linear combination of solutions is a solution…

Satisfying the Initial Condition

Since we know that for any eigenvector $g_\lambda(x)$ with eigenvalue λ, the function:

$$f_\lambda(x,t) = e^{\lambda t} \cdot g_\lambda(x)$$

is a solution and we know that any linear combination of solutions is a solution…

Any function $f(x,t)$ expressable as the linear sum:

$$f(x,t) = \sum_\lambda c_\lambda \cdot e^{\lambda t} \cdot g_\lambda(x)$$

must be a solution to the partial differential equation.

Satisfying the Initial Condition

To satisfy the initial value conditions, we need to choose c_λ so that the function:

$$f(x,t) = \sum_{\lambda} c_\lambda \cdot e^{\lambda t} \cdot g_\lambda(x)$$

satisfies the initial value conditions:

$$f(x,0) = g(x)$$

Satisfying the Initial Condition

To satisfy the initial value conditions, we need to choose c_λ so that the function:

$$f(x,t) = \sum_\lambda c_\lambda \cdot e^{\lambda t} \cdot g_\lambda(x)$$

satisfies the initial value conditions:

$$f(x,0) = g(x)$$

But this implies that:

$$g(x) = \sum_\lambda c_\lambda \cdot g_\lambda(x)$$

Satisfying the Initial Condition

To satisfy the initial value conditions, we need to choose c_λ so that the function:

$$f(x,t) = \sum c_\lambda \cdot e^{\lambda t} \cdot g_\lambda(x)$$

sa

Satisfying the initial value conditions is equivalent to finding the coefficients of $g(x)$ with respect to the basis $\{g_\lambda(x)\}$

But this implies that:

$$g(x) = \sum_\lambda c_\lambda \cdot g_\lambda(x)$$

Outline

- A Simple PDE
- Solving the PDE
- Relationship to the Fourier Transform
- Generalizations
- Examples

Relationship to the Fourier Transform

Recall that the Fourier decomposition expresses a circular function $f(\theta)$ as a sum of complex exponentials of different frequencies:

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ik\theta}$$

Relationship to the Fourier Transform

The complex exponentials have very nice properties when it comes to derivatives – they are the eigenvectors of the derivative operator:

Relationship to the Fourier Transform

The complex exponentials have very nice properties when it comes to derivatives – they are the eigenvectors of the derivative operator:

$$\frac{\partial}{\partial\theta} e^{ik\theta} =$$

Relationship to the Fourier Transform

The complex exponentials have very nice properties when it comes to derivatives – they are the eigenvectors of the derivative operator:

$$\frac{\partial}{\partial\theta} e^{ik\theta} = ike^{ik\theta}$$

Relationship to the Fourier Transform

The complex exponentials have very nice properties when it comes to derivatives – they are the eigenvectors of the derivative operator:

$$\frac{\partial}{\partial\theta}e^{ik\theta} = ike^{ik\theta}$$

$$\frac{\partial^2}{\partial\theta^2}e^{ik\theta} =$$

Relationship to the Fourier Transform

The complex exponentials have very nice properties when it comes to derivatives – they are the eigenvectors of the derivative operator:

$$\frac{\partial}{\partial\theta} e^{ik\theta} = ike^{ik\theta}$$

$$\frac{\partial^2}{\partial\theta^2} e^{ik\theta} = -k^2 e^{ik\theta}$$

Relationship to the Fourier Transform

The complex exponentials have very nice properties when it comes to derivatives – they are the eigenvectors of the derivative operator:

$$\frac{\partial}{\partial\theta} e^{ik\theta} = ike^{ik\theta}$$

$$\frac{\partial^2}{\partial\theta^2} e^{ik\theta} = -k^2 e^{ik\theta}$$

$$\vdots$$

$$\frac{\partial^n}{\partial\theta^n} e^{ik\theta} =$$

Relationship to the Fourier Transform

The complex exponentials have very nice properties when it comes to derivatives – they are the eigenvectors of the derivative operator:

$$\frac{\partial}{\partial\theta}e^{ik\theta} = ike^{ik\theta}$$

$$\frac{\partial^2}{\partial\theta^2}e^{ik\theta} = -k^2e^{ik\theta}$$

$$\vdots$$

$$\frac{\partial^n}{\partial\theta^n}e^{ik\theta} = (ik)^n e^{ik\theta}$$

Relationship to the Fourier Transform

So, if we are given the linear map:

$$D = a_0 + a_1 \frac{\partial}{\partial\theta} + \cdots + a_n \frac{\partial^n}{\partial\theta^n}$$

it will act on $e^{ik\theta}$ as:

$$D\left(e^{ik\theta}\right) = a_0 e^{ik\theta} + a_1 (ik) e^{ik\theta} + \cdots + a_n (ik)^n e^{ik\theta}$$

Relationship to the Fourier Transform

So, if we are given the linear map:

$$D = a_0 + a_1 \frac{\partial}{\partial\theta} + \cdots + a_n \frac{\partial^n}{\partial\theta^n}$$

it will act on $e^{ik\theta}$ as:

$$\begin{aligned} D\left(e^{ik\theta}\right) &= a_0 e^{ik\theta} + a_1(ik)e^{ik\theta} + \cdots + a_n(ik)^n e^{ik\theta} \\ &= \left(a_0 + a_1(ik) + \cdots + a_n(ik)^n\right)e^{ik\theta} \end{aligned}$$

Relationship to the Fourier Transform

So, if we are given the linear map:

$$D = a_0 + a_1 \frac{\partial}{\partial\theta} + \cdots + a_n \frac{\partial^n}{\partial\theta^n}$$

it will act on $e^{ik\theta}$ as:

$$D\left(e^{ik\theta}\right) = a_0 e^{ik\theta} + a_1 (ik) e^{ik\theta} + \cdots + a_n (ik)^n e^{ik\theta}$$
$$= \left(a_0 + a_1(ik) + \cdots + a_n (ik)^n\right) e^{ik\theta}$$

So $e^{ik\theta}$ will be an eigenvector with eigenvalue:

$$\lambda_k = a_0 + a_1(ik) + \cdots + a_n (ik)^n$$

Relationship to the Fourier Transform

And in particular, this implies that the solutions to the partial differential equation:

$$\frac{\partial f(\theta,t)}{\partial t} = a_0 \cdot f(\theta,t) + \cdots + a_n \cdot \frac{\partial^n f(\theta,t)}{\partial \theta^n}$$

will be of the form:

$$f_k(\theta,t) = e^{\lambda_k t} \cdot e^{ik\theta}$$

Relationship to the Fourier Transform

And in particular, this implies that the solutions to the partial differential equation:

$$\frac{\partial f(\theta,t)}{\partial t} = a_0 \cdot f(\theta,t) + \cdots + a_n \cdot \frac{\partial^n f(\theta,t)}{\partial \theta^n}$$

will be of the form:

$$f_k(\theta,t) = e^{\lambda_k t} \cdot e^{ik\theta}$$

Thus, the solutions to the PDE will be of the form:

$$f(\theta,t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{\lambda_k t} \cdot e^{ik\theta}$$

Relationship to the Fourier Transform

To satisfy the initial condition:

$$f(\theta,0)=g(\theta)$$

we need to solve for the values of c_k such that:

$$f(\theta,0)=\sum_{k=-\infty}^{\infty}c_k e^{ik\theta}$$

Relationship to the Fourier Transform

To satisfy the initial condition:

$$f(\theta,0)=g(\theta)$$

we need to solve for the values of c_k such that:

$$f(\theta,0)=\sum_{k=-\infty}^{\infty}c_k e^{ik\theta}$$

But this just means that the c_k must be the Fourier coefficients of $g(\theta)$:

$$c_k=\hat{g}(k)$$

Relationship to the Fourier Transform

The solution to the PDE is the function $f(\theta,t)$ whose k-th Fourier coefficient at time t is just a modulation of the k-th Fourier coefficient of $g(\theta)$ by a function of t:

$$\hat{f}_t(k) = \hat{g}(k) \cdot e^{\lambda_k t}$$

Relationship to the Fourier Transform

To implement this, we start off by:

- Computing the Fourier coefficients of g(θ)

Then, at each time *t*, we:

- Compute the modulated Fourier coefficients:

$$\hat{f}_t(k) = \hat{g}(k) \cdot e^{\lambda_k t}$$

- And compute the inverse Fourier transform.

Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

- Higher dimensions
- Second order time derivatives

Examples

2D Systems

In the case that the system is two-dimensional, we want to consider functions of the form $f(\theta,\phi,t)$.

2D Systems

In the case that the system is two-dimensional, we want to consider functions of the form $f(\theta,\phi,t)$.

The linear partial differential equation becomes:

$$\frac{\partial f(\theta,\phi,t)}{\partial t} = a_{00} \cdot f(\theta,\phi,t) + \cdots + a_{nm} \cdot \frac{\partial^m \partial^n f(\theta,\phi,t)}{\partial \theta^m \partial \phi^n}$$

2D Systems

In the case that the system is two-dimensional, we want to consider functions of the form $f(\theta,\phi,t)$.

The linear partial differential equation becomes:

$$\frac{\partial f(\theta,\phi,t)}{\partial t} = a_{00} \cdot f(\theta,\phi,t) + \cdots + a_{nm} \cdot \frac{\partial^m \partial^n f(\theta,\phi,t)}{\partial \theta^m \partial \phi^n}$$

The initial state becomes:

$$f(\theta,\phi,0) = g(\theta,\phi)$$

2D Systems

In the case that the system is two-dimensional, we want to consider functions of the form *f*(*x*,*y*,*t*).

The linear partial differential equation becomes:

$$\frac{\partial f(\theta,\phi,t)}{\partial t} = a_{00} \cdot f(\theta,\phi,t) + \cdots + a_{nm} \cdot \frac{\partial^m \partial^n f(\theta,\phi,t)}{\partial \theta^m \partial \phi^n}$$

The initial state becomes:

$$f(\theta,\phi,0) = g(\theta,\phi)$$

And the challenge is to compute the state of the system at an arbitrary point in time:

$$f(\theta,\phi,t) = ?$$

2D Systems

As in the 1D case, we can use the fact that the a periodic 2D function $f(\theta,\phi)$ can be expressed in terms of its Fourier decomposition as the sum of complex exponentials:

$$f(\theta,\phi)=\sum_{k,l=-\infty}^{\infty}\hat{f}(k,l)e^{ik\theta}e^{il\phi}$$

2D Systems

And again, we use the fact that the complex exponentials are the eigenvectors of the partial derivative operator:

$$\frac{\partial}{\partial\theta} e^{ik\theta} e^{il\phi} = (ik) e^{ik\theta} e^{il\phi}$$

$$\frac{\partial}{\partial\phi} e^{ik\theta} e^{il\phi} = (il) e^{ik\theta} e^{il\phi}$$

$$\vdots$$

$$\frac{\partial^m \partial^n}{\partial\theta^m \partial\phi^n} e^{ik\theta} e^{il\phi} = (ik)^m (il)^n e^{ik\theta} e^{il\phi}$$

2D Systems

Thus, given the linear map:

$$D = a_{00} + \cdots + a_{nm} \cdot \frac{\partial^m \partial^n}{\partial \theta^m \partial \phi^n}$$

the eigenvectors of this map are:

$$f_{kl}(\theta, \phi) = e^{ik\theta} \cdot e^{il\phi}$$

with associated eigenvalues:

$$\lambda_{kl} = a_{00} + \cdots + a_{nm} \cdot (ik)^n \cdot (il)^m$$

2D Systems

Given these eigenvectors of the partial-derivative operators, we can proceed as before, obtaining a solution to the differential equation for each eigenvector:

$$f_{kl}(\theta,\phi,t) = e^{\lambda_{kl} t} \cdot \left(e^{ik\theta} \cdot e^{il\phi} \right)$$

2D Systems

And we can satisfy the initial condition:

$$f(\theta,\phi,0) = g(\theta,\phi)$$

by setting the Fourier coefficients of $f(\theta,\phi,t)$ at time t equal to:

$$\hat{f}_t(k,l) = \hat{g}(k,l) \cdot e^{\lambda_{kl} t}$$

Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

- Higher dimensions
- Second order time derivatives

Second Order Time Derivatives

What if the change in the system is characterized by the second derivative with respect to time:

$$\frac{\partial^2 f(\theta,t)}{\partial t^2} = a_0 \cdot f(\theta,t) + \cdots + a_n \cdot \frac{\partial^n f(\theta,t)}{\partial \theta^n}$$

Second Order Time Derivatives

Recall that the solution to the initial partial differential equation was derived from solving the first-order (time) derivative:

$$f'(t) = \lambda f(t)$$

Second Order Time Derivatives

Recall that the solution to the initial partial differential equation was derived from solving the first-order (time) derivative:

$$f'(t) = \lambda f(t)$$

In this case, we need to solve the second order time derivative:

$$f''(t) = \lambda f(t)$$

Second Order Time Derivatives

Recall that the solution to the initial partial differential equation was derived from solving the first-order (time) derivative:

$$f'(t) = \lambda f(t)$$

In this case, we need to solve the second order time derivative:

$$f''(t) = \lambda f(t)$$

$$f(t) = e^{\sqrt{\lambda} t} \quad \text{and} \quad f(t) = e^{-\sqrt{\lambda} t}$$

Second Order Time Derivatives

Following the same approach we had before, if $g_\lambda(\theta)$ is an eigenvector of the linear operator D with eigenvalue λ, then the functions:

$$f_\lambda^+(\theta,t) = e^{\sqrt{\lambda}t} \cdot g_\lambda(\theta)$$
$$f_\lambda^-(\theta,t) = e^{-\sqrt{\lambda}t} \cdot g_\lambda(\theta)$$

will both be solutions to the partial differential equation.

Second Order Time Derivatives

Note that in this case, one eigenvector of *D* gives us two solutions to the differential equation.

Second Order Time Derivatives

Note that in this case, one eigenvector of *D* gives us two solutions to the differential equation.

↓

We have more functions with which we can satisfy the initial boundary conditions.

Second Order Time Derivatives

Note that in this case, one eigenvector of *D* gives us two solutions to the differential equation.

↓

We have more functions with which we can satisfy the initial boundary conditions.

↓

We can specify more stringent boundary conditions.

Second Order Time Derivatives

In practice, this amounts to specifying two boundary conditions:

- Initial value conditions:

$$f(\theta,0) = g(\theta)$$

- Initial derivative conditions:

$$\frac{\partial}{\partial t} f(\theta,0) = h(\theta)$$

Second Order Time Derivatives

Intuitively:

In the first-order case we are given the velocity at every point. If we know the initial positions, this is enough to know where things end up.

Second Order Time Derivatives

Intuitively:

In the first-order case we are given the velocity at every point. If we know the initial positions, this is enough to know where things end up.

In the second-order case we are given the acceleration at every point. In order to know where things end up, we need to know both the initial position and the initial velocity.

Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

Examples

- The second derivative
- The 2D heat equation
- The 2D wave equation

The Second Derivative

Given a function in 1D, how do we interpret its second derivative?

The Second Derivative‡

The first derivative of $f(x)$ is approximated by looking at the difference between the value of f at x and the value of f at a neighboring point:

$$f'(x) \approx f(x+1) - f(x)$$

‡This is all very hand-wavy

The Second Derivative‡

The first derivative of *f*(*x*) is approximated by looking at the difference between the value of *f* at *x* and the value of *f* at a neighboring point:

$$f'(x) \approx f(x+1) - f(x)$$

The second derivative of *f*(*x*) is approximated by applying the process to the derivative of *f*(*x*):

$$f''(x) \approx f'(x) - f'(x-1)$$

‡This is all very hand-wavy

The Second Derivative‡

The first derivative of $f(x)$ is approximated by looking at the difference between the value of f at x and the value of f at a neighboring point:

$$f'(x) \approx f(x+1) - f(x)$$

The second derivative of $f(x)$ is approximated by applying the process to the derivative of $f(x)$:

$$\begin{aligned} f''(x) &\approx f'(x) - f'(x-1) \\ &\approx \big(f(x+1) - f(x)\big) - \big(f(x) - f(x-1)\big) \end{aligned}$$

‡This is all very hand-wavy

The Second Derivative‡

The first derivative of *f*(*x*) is approximated by looking at the difference between the value of *f* at *x* and the value of *f* at a neighboring point:

$$f'(x) \approx f(x+1) - f(x)$$

The second derivative of *f*(*x*) is approximated by applying the process to the derivative of *f*(*x*):

$$\begin{aligned} f''(x) &\approx f'(x) - f'(x-1) \\ &\approx \big(f(x+1) - f(x)\big) - \big(f(x) - f(x-1)\big) \\ &= f(x+1) + f(x-1) - 2f(x) \end{aligned}$$

‡This is all very hand-wavy

The Second Derivative‡

The first derivative of *f*(*x*) is approximated by looking at the difference between the value of *f* at *x* and the value of *f* at a neighboring point:

$$f'(x) \approx f(x+1) - f(x)$$

The second derivative of *f*(*x*) is approximated by applying the process to the derivative of *f*(*x*):

$$f''(x) \approx 2\left(\frac{f(x+1) + f(x-1)}{2} - f(x)\right)$$

i.e. it is a measure of the difference between the value of *f* at *x* and the average value of *f* at the neighbors of *x*.

‡This is all very hand-wavy

The Laplacian‡

For a 2D function *f*(*x*,*y*), this is generalized by the Laplacian – the sum of unmixed second order partial derivatives:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

i.e. The Laplacian of a function is a measure of how the value of *f* at a point (*x*,*y*) differs from the average of the values of its neighbors.

‡This is all very hand-wavy

Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

Examples

- The second derivative
- The 2D heat equation
- The 2D wave equation

Newton's Law of Cooling

"*The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.*"

Newton's Law of Cooling

"*The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.*"

Translating this into the PDE setting, if $f(\theta,\phi,t)$ is the heat at position (x,y) at time t, then:

$$\frac{\partial f}{\partial t} = \zeta\left(\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial \phi^2}\right)$$

Newton's Law of Cooling

$$\frac{\partial f}{\partial t} = \zeta\left(\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial \phi^2}\right)$$

In this case, the linear operator *D* is defined by:

$$D = \zeta\left(\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \phi^2}\right)$$

And the complex exponential:

$$e^{ik\theta} e^{il\phi}$$

is an eigenvector with eigenvalue:

$$\lambda_{kl} = -\zeta(k^2 + l^2)$$

Newton's Law of Cooling

$$\frac{\partial f}{\partial t} = \zeta\left(\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial \phi^2}\right)$$

Thus, the solution to this equation, subject to the constraint that:

$$f(\theta,\phi,0) = g(\theta,\phi)$$

is the function whose (*k*,*l*)-th Fourier coefficient at time *t* is:

$$\hat{f}_t(k,l) = \hat{g}(k,l)e^{-\zeta(k^2+l^2)t}$$

Newton's Law of Cooling

$$\hat{f}_t(k,l) = \hat{g}(k,l)e^{-\lambda(k^2+l^2)t}$$

This looks remarkably like what we get when implement Gaussian smoothing…

Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

Examples

- The second derivative
- The 2D heat equation
- The 2D wave equation

The 2D Wave Equation

Consider the (simplified) example of a square rubber sheet.

The 2D Wave Equation

Consider the (simplified) example of a square rubber sheet.

If we displace the points on the sheet, then at any point (θ,ϕ), the neighbors of (θ,ϕ) will exert a force to pull the point towards them.

The 2D Wave Equation

Consider the (simplified) example of a square rubber sheet.

If we displace the points on the sheet, then at any point (θ,ϕ), the neighbors of (θ,ϕ) will exert a force to pull the point towards them.

The force that they exert on (θ,ϕ) will be proportional to the distance of (θ,ϕ) from its neighbors.

The 2D Wave Equation

If the height of the point (θ,ϕ) is given by the function $f(\theta,\phi)$ then the force at (θ,ϕ) is given by:

$$F(\theta,\phi) = \zeta \cdot \nabla^2 f(\theta,\phi) \qquad \zeta > 0$$

The 2D Wave Equation

If the height of the point (θ,ϕ) is given by the function $f(\theta,\phi)$ then the force at (θ,ϕ) is given by:

$$F(\theta,\phi) = \zeta \cdot \nabla^2 f(\theta,\phi) \qquad \zeta > 0$$

Using the fact that Force = Mass · Acceleration, we get the partial differential equation for the height at time *t*:

$$\frac{\partial^2 f}{\partial t^2} = \zeta\left(\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial \phi^2}\right)$$

The 2D Wave Equation

$$\frac{\partial^2 f}{\partial t^2} = \zeta\left(\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial \phi^2}\right)$$

We would like to solve this equation, subject to the constraints that at the initial time-step:

- The height at each point is given by:

$$f(\theta,\phi,0) = g(\theta,\phi)$$

- The sheet is not moving:

$$\frac{\partial}{\partial t} f(\theta,\phi,0) = 0$$

The 2D Wave Equation

$$\frac{\partial^2 f}{\partial t^2} = \zeta\left(\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial \phi^2}\right)$$

Again, the linear operator *D* is defined by:

$$D = \zeta\left(\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \phi^2}\right)$$

And the complex exponential:

$$e^{ik\theta} e^{il\phi}$$

is an eigenvector with eigenvalue:

$$\lambda_{kl} = -\zeta(k^2 + l^2)$$

The 2D Wave Equation

$$\frac{\partial^2 f}{\partial t^2} = \zeta\left(\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial \phi^2}\right)$$

Thus, the solutions to this equation are:

$$f_{kl}^{+}(\theta,\phi,t) = e^{i\sqrt{\zeta(k^2+l^2)}t} \cdot \left(e^{ik\theta} \cdot e^{il\phi}\right)$$

$$f_{kl}^{-}(\theta,\phi,t) = e^{-i\sqrt{\zeta(k^2+l^2)}t} \cdot \left(e^{ik\theta} \cdot e^{il\phi}\right)$$

The 2D Wave Equation

$$\frac{\partial^2 f}{\partial t^2} = \zeta\left(\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial \phi^2}\right)$$

Thus, the solutions to this equation are:

$$f_{kl}^{+}(\theta,\phi,t) = e^{i\sqrt{\zeta(k^2+l^2)}t} \cdot \left(e^{ik\theta} \cdot e^{il\phi}\right)$$

$$f_{kl}^{-}(\theta,\phi,t) = e^{-i\sqrt{\zeta(k^2+l^2)}t} \cdot \left(e^{ik\theta} \cdot e^{il\phi}\right)$$

and a general solution takes the form:

$$f(\theta,\phi,t) = \sum_{k,l}\left(A_{kl} \cdot e^{i\sqrt{\zeta(k^2+l^2)}t} + B_{kl} \cdot e^{-i\sqrt{\zeta(k^2+l^2)}t}\right)e^{ik\theta} \cdot e^{il\phi}$$

The 2D Wave Equation

$$f(\theta,\phi,t)=\sum_{k,l}\left(A_{kl}\cdot e^{i\sqrt{\zeta(k^2+l^2)}t}+B_{kl}\cdot e^{-i\sqrt{\zeta(k^2+l^2)}t}\right)e^{ik\theta}\cdot e^{il\phi}$$

To satisfy the initial value condition:

$$f(\theta,\phi,0)=g(\theta,\phi)$$

we need to have

$$g(\theta,\phi)=\sum_{k,l}(A_{kl}+B_{kl})\cdot\left(e^{ik\theta}\cdot e^{il\phi}\right)$$

The 2D Wave Equation

$$f(\theta,\phi,t)=\sum_{k,l}\left(A_{kl}\cdot e^{i\sqrt{\zeta(k^2+l^2)}t}+B_{kl}\cdot e^{-i\sqrt{\zeta(k^2+l^2)}t}\right)e^{ik\theta}\cdot e^{il\phi}$$

Thus, we must have:

$$\hat{g}(k,l)=A_{kl}+B_{kl}$$

The 2D Wave Equation

$$f(\theta,\phi,t)=\sum_{k,l}\left(A_{kl}\cdot e^{i\sqrt{\zeta(k^2+l^2)}t}+B_{kl}\cdot e^{-i\sqrt{\zeta(k^2+l^2)}t}\right)e^{ik\theta}\cdot e^{il\phi}$$

To satisfy the initial derivative condition:

$$\frac{\partial}{\partial t}f(\theta,\phi,0)=0$$

we need to have

$$0=\sum_{k,l}(A_{kl}-B_{kl})\cdot\left(\sqrt{\zeta(k^2+l^2)}\right)\cdot e^{ik\theta}\cdot e^{il\phi}$$

The 2D Wave Equation

$$f(\theta,\phi,t)=\sum_{k,l}\left(A_{kl}\cdot e^{i\sqrt{\zeta(k^2+l^2)}t}+B_{kl}\cdot e^{-i\sqrt{\zeta(k^2+l^2)}t}\right)e^{ik\theta}\cdot e^{il\phi}$$

Thus, we must have:

$$A_{kl}=B_{kl}$$

The 2D Wave Equation

Putting all of this together, we find that the solution to the 2D wave equation, with initial position $g(\theta,\phi)$ and zero initial derivative is the function $f(\theta,\phi,t)$ whose Fourier coefficients at time t are equal to:

$$\hat{f}_t(k,l) = \hat{g}(k,l)\frac{\left(e^{i\sqrt{\zeta(k^2+l^2)}t} + e^{-i\sqrt{\zeta(k^2+l^2)}t}\right)}{2}$$

The 2D Wave Equation

Putting all of this together, we find that the solution to the 2D wave equation, with initial position $g(\theta,\phi)$ and zero initial derivative is the function $f(\theta,\phi,t)$ whose Fourier coefficients at time t are equal to:

$$\hat{f}_t(k,l) = \hat{g}(k,l)\frac{\left(e^{i\sqrt{\zeta(k^2+l^2)}t} + e^{-i\sqrt{\zeta(k^2+l^2)}t}\right)}{2}$$

$$= \hat{g}(k,l)\cos\left(\sqrt{\zeta(k^2+l^2)}t\right)$$

The 2D Wave Equation

$$\hat{f}_t(k,l) = \hat{g}(k,l)\frac{\left(e^{i\sqrt{\zeta(k^2+l^2)}t} + e^{-i\sqrt{\zeta(k^2+l^2)}t}\right)}{2}$$

This looks nothing like what we get when implement Gaussian smoothing…