FFTs in Graphics and Vision

More Math Review

Outline

Inner Product Spaces

- Real Inner Products
- Hermitian Inner Products
- Orthogonal Transforms
- Unitary Transforms
- Function Spaces

Inner Product Spaces

Given a real vector space V, an inner product is a function $\langle \cdot, \cdot \rangle$ that takes a pair of vectors and returns a real value.

Inner Product Spaces

An inner product is a map from VxV into the real numbers that is:

1. Linear: For all $u,v,w \in V$ and any real scalar λ
$$\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$$
$$\langle \lambda v, w\rangle = \lambda\langle v,w\rangle$$
2. Symmetric: For all $u,v \in V$
$$\langle v,w\rangle = \langle w,v\rangle$$
3. Positive Definite: For all $v \in V$:
$$\langle v,v\rangle \geq 0$$
$$\langle v,v\rangle = 0 \quad \Leftrightarrow \quad v = 0$$

Inner Product Spaces

An inner product defines a notion of distance on a vector space by setting:

$$D(v,w) = \sqrt{\langle v-w, v-w \rangle} \equiv \|v-w\|^{1/2}$$

Inner Product Spaces

Examples:

1. On the space of *n*-dimensional arrays, the standard inner product is:

$$\langle (a_1,\cdots,a_n),(b_1,\cdots,b_n)\rangle = a_1b_1+\cdots+a_nb_n$$
$$= (a_1,\cdots,a_n)(b_1,\cdots,b_n)^t$$

Inner Product Spaces

Examples:

1. On the space of *n*-dimensional arrays, the standard inner product is:

$$\big\langle (a_1,\cdots,a_n),(b_1,\cdots,b_n)\big\rangle = a_1b_1+\cdots+a_nb_n$$
$$= (a_1,\cdots,a_n)(b_1,\cdots,b_n)^t$$

2. On the space of continuous, real-valued functions, defined on a circle, the standard inner product is:

$$\langle f,g\rangle = \int_0^{2\pi} f(\theta)g(\theta)\,d\theta$$

Inner Product Spaces

Examples:

3. Suppose we have the space of *n*-dimensional arrays, and suppose we have a matrix:

$$M = \begin{pmatrix} M_{11} \cdots M_{1n} \\ \vdots \quad \ddots \quad \vdots \\ M_{n1} \cdots M_{nn} \end{pmatrix}$$

Does the map:

$$\langle v, w \rangle_M = v^t M w$$

define an inner product?

Inner Product Spaces

Examples:

3. Does the map:

$$\langle v, w \rangle_M = v^t M w$$

define an inner product?

- Is it linear?
- Is it symmetric?
- Is it positive definite?

Inner Product Spaces

Examples:

$$\langle v, w \rangle_M = v^t M w$$

- Is it linear? Yes!

$$\langle u+v, w \rangle_M = (u+v)^t M w$$

$$\langle \lambda v, w \rangle_M = (\lambda v)^t M w$$

Inner Product Spaces

Examples:

$$\langle v, w \rangle_M = v^t M w$$

- Is it symmetric? Only if *M* is symmetric ($M=M^t$)

$$\langle w, v \rangle_M = w^t M v$$

Inner Product Spaces

Examples:

$$\langle v, w \rangle_M = v^t M w$$

- Is it positive definite?

If, *M* is symmetric, there exists an orthogonal basis $\{v_1,\ldots,v_n\}$ with respect to which it is diagonal:

$$M = B^t \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix} B$$

Inner Product Spaces

Examples:

$$\langle v, w \rangle_M = v^t M w$$

- Is it positive definite?

Only if the eigenvalues are all positive

If we express *v* in terms of this basis:

$$v = a_1 v_1 + \cdots + a_n v_n$$

then

$$\langle v, v \rangle_M = \lambda_1 a_1^2 + \cdots + \lambda_n a_n^2$$

Inner Product Spaces

Examples:

4. On the space of continuous, real-valued functions, defined on a circle, does the map:

$$\langle f, g \rangle = \int_0^{2\pi} f(\theta) g(\theta)\, \omega(\theta) d\theta$$

define an inner product? No!

Inner Product Spaces

<u>Examples</u>:

4. On the space of continuous, real-valued functions, defined on a circle, does the map:

$$\langle f, g \rangle = \int_0^{2\pi} f(\theta) g(\theta)\, \omega(\theta) d\theta$$

define an inner product? No!

What if $\omega(\theta)>0$? Yes!

Hermitian Inner Product Spaces

Given a complex vector space V, a Hermitian inner product is a function $\langle\cdot,\cdot\rangle$ that takes a pair of vectors and returns a complex value.

Hermitian Inner Product Spaces

A Hermitian inner product is a map from *V*x*V* into the complex numbers that is:

1. Linear: For all $u,v,w \in V$ and any real scalar λ
$$\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$$
$$\langle \lambda v, w\rangle = \lambda\langle v, w\rangle$$
2. Conjugate Symmetric: For all $u,v \in V$
$$\langle v, w\rangle = \overline{\langle w, v\rangle}$$
3. Positive Definite: For all $v \in V$:
$$\langle v, v\rangle \geq 0$$
$$\langle v, v\rangle = 0 \quad \Leftrightarrow \quad v = 0$$

Inner Product Spaces

As in the real case, a Hermitian inner product defines a notion of distance on a complex vector space by setting:

$$D(v,w) = \sqrt{\langle v-w, v-w \rangle} \equiv \|v-w\|^{1/2}$$

Hermitian Inner Product Spaces

Examples:

1. On complex-valued, *n*-dimensional arrays, the standard Hermitian inner product is:

$$\left\langle (a_1,\cdots,a_n),(b_1,\cdots,b_n)\right\rangle = a_1\overline{b_1}+\cdots+a_n\overline{b_n}$$

$$= (a_1,\cdots,a_n)\overline{(b_1,\cdots,b_n)}^t$$

Hermitian Inner Product Spaces

Examples:

1. On complex-valued, *n*-dimensional arrays, the standard Hermitian inner product is:

$$\langle (a_1,\cdots,a_n),(b_1,\cdots,b_n)\rangle = a_1\overline{b_1}+\cdots+a_n\overline{b_n}$$

$$= (a_1,\cdots,a_n)\overline{(b_1,\cdots,b_n)}^t$$

2. On the space of continuous, complex-valued functions, defined on a circle, the standard Hermitian inner product is:

$$\langle f,g\rangle = \int_0^{2\pi} f(\theta)\overline{g(\theta)}\,d\theta$$

Structure Preservation

Recall:

If we have an n-dimensional vector space V then a linear map L is just a function from V to V that preserves the linear structure:

$$L(av_1 + bv_2) = a \cdot L(v_1) + b \cdot L(v_2)$$

for all $v,w \in V$ and all scalars a and b.

Structure Preservation

Recall:

If we have an *n*-dimensional vector space *V* then a linear map *L* is just a function from *V* to *V* that preserves the linear structure:

$$L(av_1 + bv_2) = a \cdot L(v_1) + b \cdot L(v_2)$$

for all $v, w \in V$ and all scalars *a* and *b*.

If *L* is invertible, then we can think of *L* as a function that renames all the elements in *V* while preserving the underlying vector space structure.

Structure Preservation

Orthogonal Transformations:

For a real vector space *V* that has an inner product, we would also like to consider those functions that rename the elements of *V* while preserving the underlying structure.

Structure Preservation

Orthogonal Transformations:

For a real vector space *V* that has an inner product, we would also like to consider those functions that rename the elements of *V* while preserving the underlying structure.

If *R* is such a function, then:

- *R* must be an invertible linear operator, in order to preserve the underlying vector space structure.
- *R* must also preserve the underlying inner product.

Structure Preservation

Orthogonal Transformations:

For a real vector space V, an invertible linear operator R is called orthogonal if it preserves the inner product:

$$\langle R(v), R(w)\rangle = \langle v, w\rangle$$

for all $v,w\in V$.

Structure Preservation

<u>Example</u>:

On the space of real-valued, *n*-dimensional arrays, a matrix is orthogonal if:

$$\langle Rv, Rw \rangle = \langle v, w \rangle$$

$$\Updownarrow$$

$$(Rv)^t (Rw) = v^t w$$

$$\Updownarrow$$

$$v^t R^t R w = v^t w$$

$$\Updownarrow$$

$$R^t = R^{-1}$$

Structure Preservation

Example:

On the space of real-valued, *n*-dimensional arrays, a matrix is orthogonal if:

$$R^t = R^{-1}$$

Note: The determinant of an orthogonal matrix always has absolute value 1.

Structure Preservation

Example:

On the space of real-valued, *n*-dimensional arrays, a matrix is orthogonal if:

$$R^t = R^{-1}$$

Note: The determinant of an orthogonal matrix always has absolute value 1.

If the determinant of an orthogonal matrix is equal to 1, the matrix is called a rotation.

Orthogonal Matrices and Eigenvalues

If R is an orthogonal transformation and R has an eigenvalue λ, then $|\lambda|=1$.

Orthogonal Matrices and Eigenvalues

If *R* is an orthogonal transformation and *R* has an eigenvalue λ, then $|\lambda|=1$.

To see this, let *v* be the eigenvector corresponding to the eigenvalue λ. Then since *R* is orthogonal, we have:

$$\langle v, v \rangle = \langle Rv, Rv \rangle$$

Structure Preservation

Unitary Transformations:

For a complex vector space V, an invertible linear operator R is called unitary if it preserves the hermitian inner product:

$$\langle R(v), R(w) \rangle = \langle v, w \rangle$$

for all $v, w \in V$.

Structure Preservation

Example:

On the space of complex-valued, *n*-dimensional arrays, a matrix is unitary if:

$$\langle Rv, Rw \rangle = \langle v, w \rangle$$

$$\Updownarrow$$

$$(Rv)^t\,\overline{(Rw)} = v^t\,\overline{w}$$

$$\Updownarrow$$

$$v^t R^t\,\overline{R}\,\overline{w} = v^t\,\overline{w}$$

$$\Updownarrow$$

$$\overline{R}^t = R^{-1}$$

Structure Preservation

Example:

On the space of complex-valued, *n*-dimensional arrays, a matrix is unitary if:

$$\overline{R}^{t} = R^{-1}$$

Note: The determinant of a unitary matrix always has norm 1.

Unitary Matrices and Eigenvalues

If R is a unitary transformation and R has an eigenvalue λ, then $|\lambda|=1$.

Unitary Matrices and Eigenvalues

If *R* is a unitary transformation and *R* has an eigenvalue λ, then $|\lambda|=1$.

To see this, let *v* be the eigenvector corresponding to the eigenvalue λ. Then since *R* is unitary, we have:

$$\langle v, v \rangle = \langle Rv, Rv \rangle$$

Function Spaces

In this course, the vector spaces we will be looking at most often are the vector spaces of functions defined on some domain:

- Continuous functions on the unit circle (S^1)
- Continuous functions on the unit disk (D^2)
- Continuous, periodic functions on the plane ($\mathbb{R}^2$)
- Continuous functions on the unit sphere (S^2)
- Continuous functions on the unit ball (B^3)

Function Spaces

Continuous functions on the unit circle (S^1):

This is the set of points (x,y) such that $x^2+y^2=1$.

If we have functions $f(x,y)$, and $g(x,y)$ the inner product is:

$$\langle f,g\rangle = \int_{p\in S^1} f(p)\overline{g}(p)\,dp$$

Function Spaces

Continuous functions on the unit circle (S^1):

This is the set of points (x,y) such that $x^2+y^2=1$.

If we have functions $f(x,y)$, and $g(x,y)$ the inner product is:

$$\langle f, g \rangle = \int_{p \in S^1} f(p)\bar{g}(p)\,dp$$

Or, we can represent points on the circle in terms of angle $\theta \in [0,2\pi]$:

$$\theta \to (\cos\theta, \sin\theta)$$

For functions $f(\theta)$ and $g(\theta)$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} f(\theta)\bar{g}(\theta)\,d\theta$$

Function Spaces

Continuous functions on the unit disk (D^2):

This is the set of points (x,y) such that $x^2+y^2\leq 1$.

If we have functions $f(x,y)$, and $g(x,y)$ the inner product is:

$$\langle f, g\rangle = \int_{p\in D^2} f(p)\overline{g}(p)\,dp$$

Function Spaces

Continuous functions on the unit disk (D^2):

This is the set of points (x,y) such that $x^2+y^2\leq 1$.

If we have functions $f(x,y)$, and $g(x,y)$ the inner product is:

$$\langle f,g\rangle = \int_{p\in S^1} f(p)\bar{g}(p)\,dp$$

Or, we can represent points on the circle in terms of radius $r\in[0,1]$ and angle $\theta\in[0,2\pi]$:

$$(r,\theta)\rightarrow (r\cos\theta, r\sin\theta)$$

For functions $f(r,\theta)$ and $g(r,\theta)$ the inner product is:

$$\langle f,g\rangle = \int_0^{2\pi}\int_0^1 f(r,\theta)\bar{g}(r,\theta)\,r\,dr\,d\theta$$

Function Spaces

Continuous, periodic functions on the plane ($\mathbb{R}^2$):

This is the set of functions $f(x,y)$ satisfying the property that:

$$f(x,y) = f(x+2\pi, y) = f(x, y+2\pi)$$

If we have functions $f(x,y)$, and $g(x,y)$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi}\int_0^{2\pi} f(x,y)\,\overline{g}(x,y)\,dy\,dx$$

Function Spaces

Continuous functions on the unit sphere (S^2):

This is the set of points (x,y,z) such that $x^2+y^2+z^2=1$.

If we have functions $f(x,y,z)$, and $g(x,y,z)$ the inner product is:

$$\langle f,g\rangle = \int_{p\in S^2} f(p)\bar{g}(p)\,dp$$

Function Spaces

Continuous functions on the unit sphere (S^2):

This is the set of points (x,y,z) such that $x^2+y^2+z^2=1$.

If we have functions $f(x,y,z)$, and $g(x,y,z)$ the inner product is:

$$\langle f, g \rangle = \int_{p \in S^2} f(p)\bar{g}(p)\,dp$$

Or, we can represent points on the sphere in terms of spherical angle $\theta \in [0,\pi]$ and $\varphi \in [0,2\pi]$:

$$(\theta,\varphi) \to (\sin\theta\cos\varphi, \cos\theta, \sin\theta\sin\varphi)$$

For functions $f(\theta,\varphi)$ and $g(\theta,\varphi)$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi)\bar{g}(\theta,\varphi)\sin\theta\,d\theta\,d\varphi$$

Function Spaces

Continuous functions on the unit ball (B^3):

This is the set of points (x,y,z) such that $x^2+y^2+z^2\leq 1$.

If we have functions $f(x,y,z)$, and $g(x,y,z)$ the inner product is:

$$\langle f,g\rangle = \int_{p\in B^3} f(p)\overline{g}(p)\,dp$$

Function Spaces

Continuous functions on the unit ball (B^3):

This is the set of points (x,y,z) such that $x^2+y^2+z^2\leq 1$.

If we have functions $f(x,y,z)$, and $g(x,y,z)$ the inner product is:

$$\langle f,g\rangle = \int_{p\in B^3} f(p)\bar{g}(p)\,dp$$

Or, we can represent points in the ball in terms of radius $r\in[0,1]$ and spherical angle $\theta\in[0,\pi]$, $\varphi\in[0,2\pi]$:

$$(r,\theta,\varphi)\rightarrow\left(r\sin\theta\cos\varphi, r\cos\theta, r\sin\theta\sin\varphi\right)$$

For functions $f(\theta,\varphi)$ and $g(\theta,\varphi)$ the inner product is:

$$\langle f,g\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} f(r,\theta,\varphi)\bar{g}(r,\theta,\varphi)r^2\sin\theta\,dr\,d\theta\,d\varphi$$

Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- Is the map:

$$f(p) \rightarrow f(p) + 1$$

a linear transformation? No!

Function Spaces

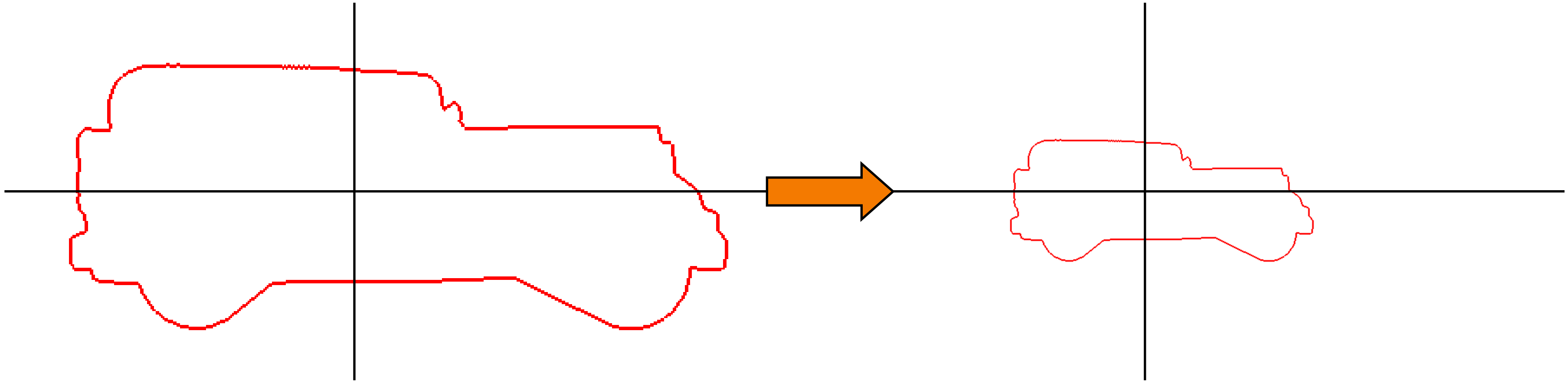
<u>Examples</u>

If we consider the space of continuous, complex-valued functions on the unit circle:

- For any scalar value λ, is:

$$f(p) \rightarrow \lambda f(p)$$

a linear transformation? Yes!



Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- For any scalar value λ, is:

$$f(p) \rightarrow \lambda f(p)$$

  a linear transformation? Yes!

- Is it unitary? No!

Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- For any scalar value λ, is:

  $$f(p) \rightarrow \lambda f(p)$$

  a linear transformation? Yes!

- Is it unitary? No!

- How about if $|\lambda|=1$? Yes!

Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- Is the differentiation operator:

$$f(p) \rightarrow f'(p)$$

a linear transformation? No!

Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- Is the differentiation operator:

$$f(p) \to f'(p)$$

  a linear transformation? No!

- What if we only consider the functions that are infinitely differentiable? Yes!

Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- Is the differentiation operator:

$$f(p) \to f'(p)$$

  a linear transformation? No!

- What if we only consider the functions that are infinitely differentiable? Yes!

- Is it unitary? No!

Function Spaces

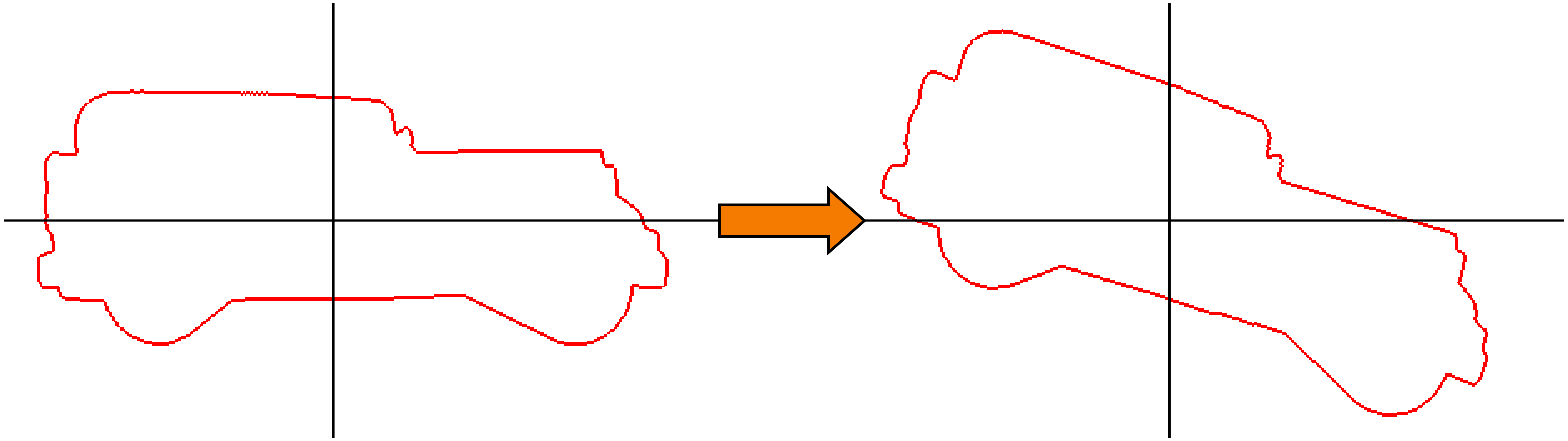
Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- For any 2D rotation *R* is the transformation:

$$f(p) \rightarrow f(R^{-1}p)$$

  a linear transformation? Yes!



Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- For any 2D rotation *R* is the transformation:

  $$f(p) \rightarrow f(R^{-1}p)$$

  a linear transformation? Yes!

- Is it unitary? Yes!

Function Spaces

Examples

If we consider the space of continuous, periodic, complex-valued functions on the plane:

- For any 2D point (x_0, y_0), is the transformation:

$$f(x, y) \rightarrow f(x - x_0, y - y_0)$$

a linear transformation? Yes!

Function Spaces

Examples

If we consider the space of continuous, periodic, complex-valued functions on the plane:

- For any 2D point (x_0, y_0), is the transformation:

  $$f(x,y) \rightarrow f(x-x_0, y-y_0)$$

  a linear transformation? Yes!

- Is it unitary? Yes!

Function Spaces

Examples

If we consider the space of continuous, infinitely-differentiable, periodic, complex-valued functions on the plane:

- Is differentiation with respect to *x*:

$$f(x,y) \rightarrow \frac{\partial}{\partial x} f(x,y)$$

  a linear transformation? Yes!

Function Spaces

Examples

If we consider the space of continuous, infinitely-differentiable, periodic, complex-valued functions on the plane:

- Is differentiation with respect to *x*:

$$f(x,y) \rightarrow \frac{\partial}{\partial x} f(x,y)$$

  a linear transformation? Yes!

- Is it unitary? No!

Function Spaces

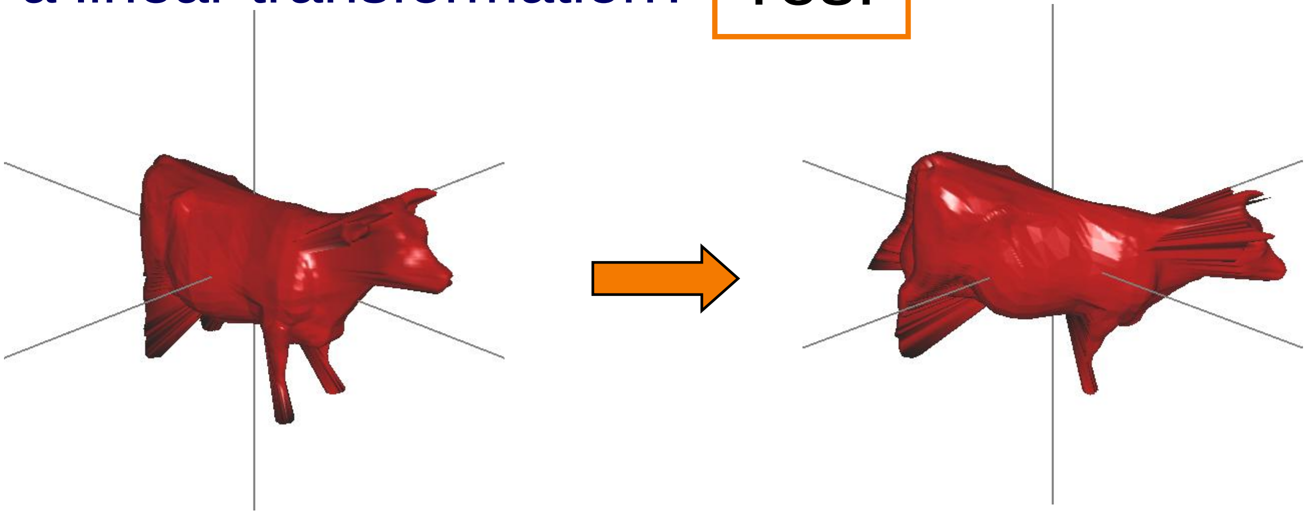
<u>Examples</u>

If we consider the space of continuous, complex-valued functions on the sphere:

- For any rotation *R*, is the transformation:

$$f(p) \rightarrow f(R^{-1}p)$$

a linear transformation? Yes!



Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the sphere:

- For any rotation *R*, is the transformation:

$$f(p) \rightarrow f(R^{-1}p)$$

  a linear transformation? Yes!

- Is it unitary?

Function Spaces

Change of Variables:

Given a real/complex-valued function *f* defined on some domain *D*, and given some differentiable, invertible, map:

$$\Phi : D \rightarrow \Phi(D)$$

We have:

$$\int_{x\in D} f\left(\Phi(x)\right)\left|\partial\Phi\right| dx = \int_{y\in\Phi(x)} f(y)\, dy$$

where $|\partial\Phi|$ denotes the Jacobian of Φ (i.e. the determinant of the derivative matrix)

Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the sphere:

- For any rotation *R*, is the transformation:

$$f(p) \to f(R^{-1}p)$$

  a linear transformation? Yes!

- Is it unitary? Yes!