Laplacians on Meshes

Misha Kazhdan

Goal

We would like to generalize the methods we had used for gradient domain image processing to the processing of triangle meshes:

- Processing functions defined over the surface of the mesh:
  - Texture maps
  - Normal maps
  - Bump maps

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We would like to generalize the methods we had used for gradient domain image processing to the processing of triangle meshes:

- Processing functions defined over the surface of the mesh:
  - Texture maps
  - Normal maps
  - Bump maps

To do this, we will have to formalize what we mean by a "mesh Laplacian".

Outline

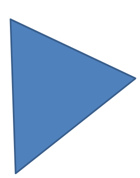
Geometry Overview

Laplacians (Combinatorial and Cotangent)

- Planar Triangulations
- Triangle Meshes

Geometry Overview

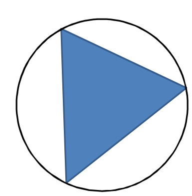
Given a Triangle:



Geometry Overview

Given a Triangle:

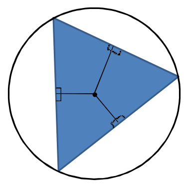
The *circumcircle* is the unique circle passing through all the vertices of the triangle.



Geometry Overview

Given a Triangle:

The *circumcenter* is at the intersection of the perpendicular bisectors.



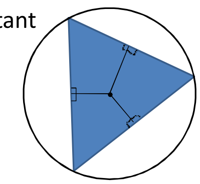
Geometry Overview

Given a Triangle:

The *circumcenter* is at the intersection of the perpendicular bisectors.

Proof:

1. The circumcenter is equidistant from the triangle vertices.



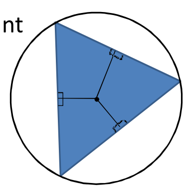
Geometry Overview

Given a Triangle:

The *circumcenter* is at the intersection of the perpendicular bisectors.

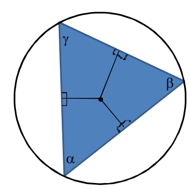
Proof:

1. The circumcenter is equidistant from the triangle vertices.
2. The perpendicular bisector of *p* and *q* is the set of points equidistant from *p* and *q*.



Geometry Overview

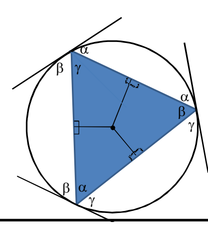
If the Angles of the Triangle are α, β, and γ:



Geometry Overview

If the Angles of the Triangle are α, β, and γ:

The angles at which the circle meet the sides of the triangle coincide with angles of the triangle.



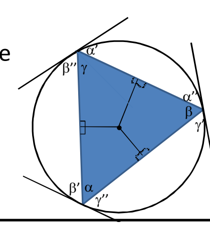
Geometry Overview

If the Angles of the Triangle are α, β, and γ:

The angles at which the circle meet the sides of the triangle coincide with angles of the triangle.

Proof:

1. Due to symmetry across the perpendicular bisector:
   - $\alpha' = \alpha''$
   - $\beta' = \beta''$
   - $\gamma' = \gamma''$



Geometry Overview

<u>If the Angles of the Triangle are α, β, and γ:</u>

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<u>Proof</u>:

2. Additionally, we have:
   - $\alpha+\beta+\gamma=\pi$
   - $\alpha+\beta'+\gamma'=\pi$
   - $\alpha'+\beta'+\gamma=\pi$
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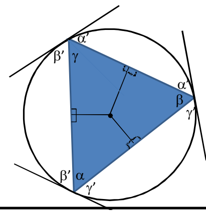

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2. Additionally, we have:
   - $\alpha+\beta+\gamma=\pi$ ⇒ $(\beta-\beta')+(\gamma-\gamma')=0$
   - $\alpha+\beta'+\gamma'=\pi$ $\quad (\beta'-\beta)+(\gamma-\gamma')=0$
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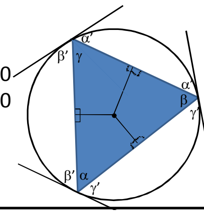

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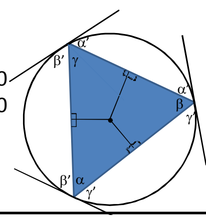
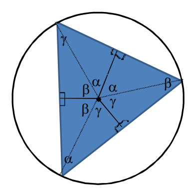


Geometry Overview

<u>If the Angles of the Triangle are α, β, and γ:</u>

The internal half-angles coincide with angles of the triangle.



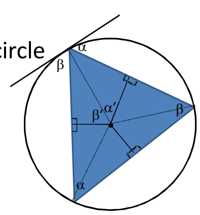
Geometry Overview

<u>If the Angles of the Triangle are α, β, and γ:</u>

The internal half-angles coincide with angles of the triangle.

<u>Proof</u>:

Using the angles at which the circle meet the sides of the triangle:



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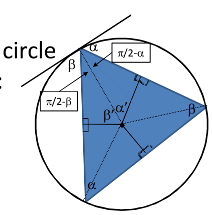

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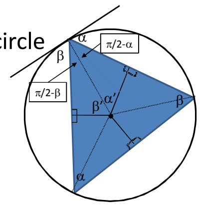
<u>Proof</u>:

Using the angles at which the circle meet the sides of the triangle:

Since the sum of the angles must be π:

$(\pi/2\text{-}\beta)+\beta'+\pi/2=\pi$
$(\pi/2\text{-}\alpha)+\alpha'+\pi/2=\pi$



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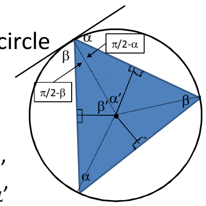
<u>Proof</u>:

Using the angles at which the circle meet the sides of the triangle:

Since the sum of the angles must be π:

$(\pi/2\text{-}\beta)+\beta'+\pi/2=\pi$ ⇒ $\beta=\beta'$
$(\pi/2\text{-}\alpha)+\alpha'+\pi/2=\pi$ ⇒ $\alpha=\alpha'$

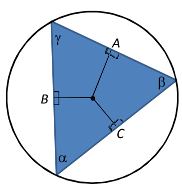


Geometry Overview

<u>If the Sides Have Lengths *A*, *B*, and *C*:</u>

By the law of sines, we have:

$$\frac{A}{\sin\alpha}=\frac{B}{\sin\beta}=\frac{C}{\sin\gamma}$$

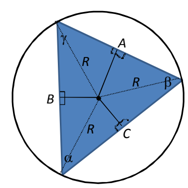


Geometry Overview

<u>If the Sides Have Lengths *A*, *B*, and *C*:</u>

If *R* is the circumradius, then:

$$\frac{A}{\sin\alpha}=\frac{B}{\sin\beta}=\frac{C}{\sin\gamma}=2R$$



Geometry Overview

<u>If the Sides Have Lengths *A*, *B*, and *C*:</u>

If *R* is the circumradius, then:

$$\frac{A}{\sin\alpha}=\frac{B}{\sin\beta}=\frac{C}{\sin\gamma}=2R$$

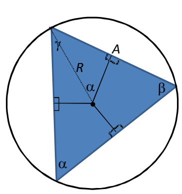
<u>Proof</u>:

Using the internal half-angles:

$$\sin\alpha=\frac{A/2}{R}$$

⇓

$$2R=\frac{A}{\sin\alpha}$$



Outline

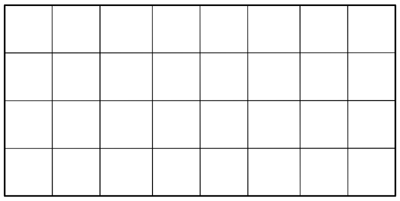
Geometry Overview

Laplacians (Combinatorial and Cotangent)

- Planar Triangulations
- Triangle Meshes

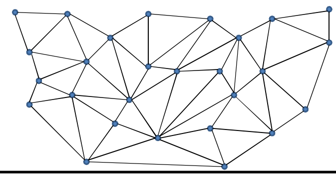
Laplacians (Planar Triangulations)

Up to now, we have been considering the "nice" case when we thought of a 2D function as a set of samples distributed on a regular grid.



Laplacians (Planar Triangulations)

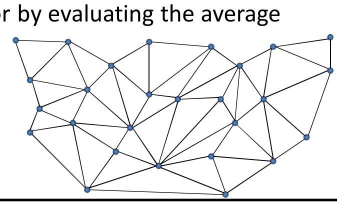
What happens when the samples lie on some arbitrary planar triangulation?



Tutte Laplacian

$$L_{ij} = \begin{cases} 1/d_i & \text{if } i \text{ and } j \text{ adjacent} \\ -1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

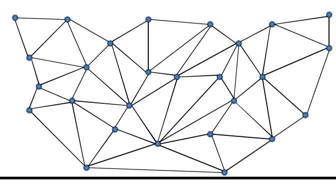
Since the Laplacian measures the difference from the local average we can define the Laplacian operator by evaluating the average over the 1-ring neighbors.



Tutte Laplacian

<u>Advantages</u>:

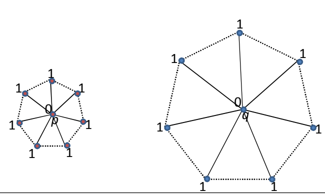
- Only depends on the topology of the mesh
- Easy to compute
- Generalizes to triangle meshes



Tutte Laplacian

<u>Disadvantages</u>:

- The Tutte Laplacian doesn't consider local scale



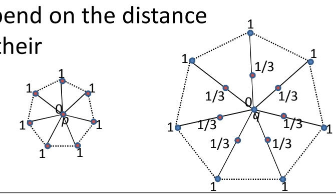
At both *p* and *q* the Tutte Laplacian has value 1.

Tutte Laplacian

<u>Disadvantages</u>:

- The Tutte Laplacian doesn't consider local scale

If the points live in the plane, the Laplacian value should depend on the distance from *p* and *q* to their neighbors.



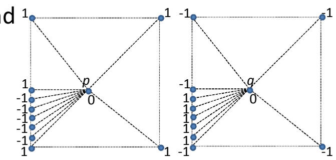
At both *p* and *q* the Tutte Laplacian has value 1.

Tutte Laplacian

Disadvantages:

- The Tutte Laplacian doesn't consider the local distribution of values

If the points live in the plane, the Laplacian value should depend on the distribution of the neighbors of *p* and *q* along the 1-ring.

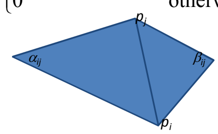


At both *p* and *q* the Tutte Laplacian has value 0.

Cotangent Laplacian

The limitations of the Tutte Laplacian can be resolved by using a Laplacian whose entries are defined in terms of the cotangent weights:

$$L_{ij} = \begin{cases} \cot(\alpha_{ij}) + \cot(\beta_{ij}) & \text{if } i \text{ and } j \text{ adjacent} \\ -\sum_k L_{ik} & i = j \\ 0 & \text{otherwise} \end{cases}$$



Cotangent Laplacian

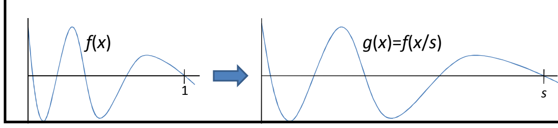
To obtain this derivation, we would like the Laplacian to consider:

1. The local scale of the samples
2. The local distribution of the samples

Cotangent Laplacian

Local Scale:

Q: How does scaling a function by a value of *s* effect the Laplacian?



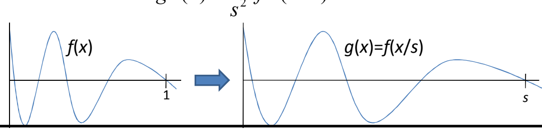
Cotangent Laplacian

Local Scale:

Q: How does scaling a function by a value of *s* effect the Laplacian?

A: Scaling by *s* results in the Laplacian being scaled by a factor of $1/s^2$:

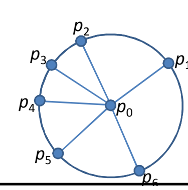
$$g''(x) = \frac{1}{s^2} f''(x/s)$$



Cotangent Laplacian

Local Distribution:

Q: Assuming all neighbors are at a fixed distance *d*, how does distribution effect the Laplacian?



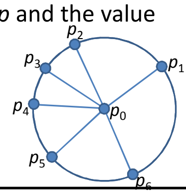
Cotangent Laplacian

Local Distribution:

Q: Assuming all neighbors are at a fixed distance *d*, how does distribution effect the Laplacian?

A: If we define the Laplacian at *p* as the average difference between the value at *p* and the value at points in disk of radius *d*:

$$\Delta f(p) \approx \int_{|q|<d} f(p-q) - f(p)dq$$



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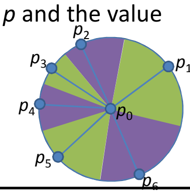
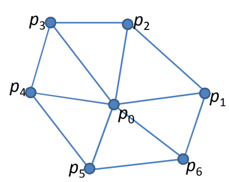
the contribution from neighbors should be weighted by area.



Cotangent Laplacian

Defining a Neighborhood:

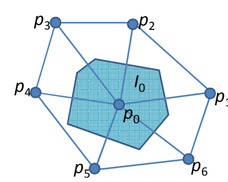
Since the neighbors of p_0 are not all at a fixed distance, we must define a neighborhood I_0 of p_0 on which to approximate the Laplacian integral.



Cotangent Laplacian

Defining a Neighborhood:

We define the neighborhood I_0 as the subset of points in the triangles adjacent to p_0 that are closer to p_0 than to any other vertex.

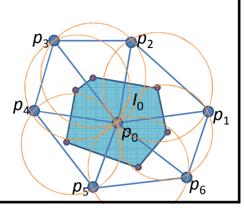


Cotangent Laplacian

Defining a Neighborhood:

We define the neighborhood I_0 as the subset of points in the triangles adjacent to p_0 that are closer to p_0 than to any other vertex.

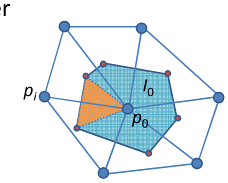
Note: The vertices of I_0 are precisely the *circumcenters* of the triangles.



Cotangent Laplacian

Defining a Neighborhood:

To compute the distribution term for a 1-ring neighbor p_i, we consider the two triangles adjacent to $\overline{p_0 p_i}$, and compute the area of I_0 closer to p_i than to any other 1-ring neighbor.



Cotangent Laplacian

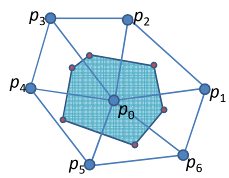
Defining the Laplacian:

Assuming that the value at vertex p_j is v_j, we define the value of the Laplacian as:

$$(\Delta v)_i = \sum_{j \in N_1(i)} \frac{v_j - v_i}{\|p_j - p_i\|^2} A_{ij}$$

where:

- $N_1(i)$ is the 1-ring of i
- $|p_j - p_i|^2$ is the scale term
- A_{ij} is the distribution term

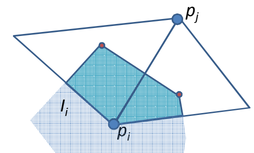


Cotangent Laplacian

$$(\Delta v)_i = \sum_{j \in N_1(i)} \frac{v_j - v_i}{\|p_j - p_i\|^2} A_{ij}$$

Computing the Laplacian:

For a fixed vertex p_i and a 1-ring neighbor p_j, we need to consider the triangles adjacent to $\overline{p_i p_j}$, to compute the scale and distribution terms.

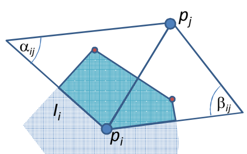


Cotangent Laplacian

$$(\Delta v)_i = \sum_{j \in N_1(i)} \frac{v_j - v_i}{\|p_j - p_i\|^2} A_{ij}$$

Notation:

- α_{ij} and β_{ij} are the two angles opposite $\overline{p_i p_j}$



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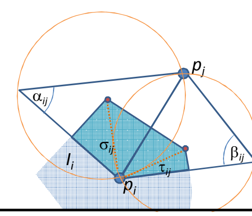
Notation:

- α_{ij} and β_{ij} are the two angles opposite $\overline{p_i p_j}$
- σ_{ij} and τ_{ij} are the two circumradii



Cotangent Laplacian

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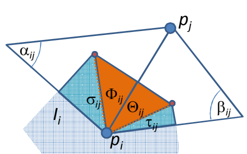
Notation:

- α_{ij} and β_{ij} are the two angles opposite $\overline{p_i p_j}$
- σ_{ij} and τ_{ij} are the two circumradii
- Φ_{ij} and Θ_{ij} are the two regions defining the distribution weights ($A_{ij} = \Phi_{ij} + \Theta_{ij}$)



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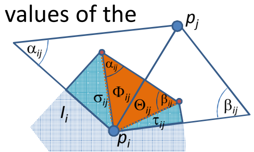
From before, we know the values of the interior half-angles.



Cotangent Laplacian

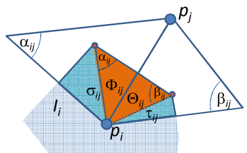
$$(\Delta v)_i = \sum_{j \in N_1(i)} \frac{v_j - v_i}{\|p_j - p_i\|^2} A_{ij}$$

Computing the Laplacian:

Using the law of sines, we know that:

$$\frac{\|p_j - p_i\|}{\sin \alpha_{ij}} = 2\sigma_{ij} \quad \text{and} \quad \frac{\|p_j - p_i\|}{\sin \beta_{ij}} = 2\tau_{ij}$$

$$\frac{A}{\sin \alpha} = \frac{B}{\sin \beta} = \frac{C}{\sin \gamma} = 2R$$



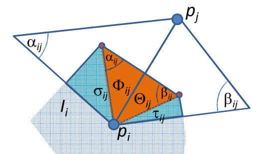
Cotangent Laplacian

$$(\Delta v)_i = \sum_{j \in N_1(i)} \frac{v_j - v_i}{\|p_j - p_i\|^2} A_{ij}$$

Computing the Laplacian: $\frac{\|p_j - p_i\|}{\sin \alpha_{ij}} = 2\sigma_{ij} \quad \text{and} \quad \frac{\|p_j - p_i\|}{\sin \beta_{ij}} = 2\tau_{ij}$

Using the formula for the area of a triangle (base x height / 2) gives:

$$\Phi_{ij} = \frac{\sin \alpha_{ij} \cos \alpha_{ij} \sigma_{ij}^2}{2} \quad \text{and} \quad \Theta_{ij} = \frac{\sin \beta_{ij} \cos \beta_{ij} \tau_{ij}^2}{2}$$



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Putting it all together, we get the expression: $\Phi_{ij} = \frac{\sin \alpha_{ij} \cos \alpha_{ij} \sigma_{ij}^2}{2} \quad \text{and} \quad \Theta_{ij} = \frac{\sin \beta_{ij} \cos \beta_{ij} \tau_{ij}^2}{2}$

$$\frac{A_{ij}}{\|p_j - p_i\|^2} = \frac{\Phi_{ij}}{\|p_j - p_i\|^2} + \frac{\Theta_{ij}}{\|p_j - p_i\|^2}$$

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$$= \frac{1}{2}\left(\frac{\sin \alpha_{ij} \cos \alpha_{ij} \sigma_{ij}^2}{4\sigma_{ij}^2 \sin^2 \alpha_{ij}} + \frac{\sin \beta_{ij} \cos \beta_{ij} \tau_{ij}^2}{4\tau_{ij}^2 \sin^2 \beta_{ij}} \right)$$

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$$= \frac{1}{8}\left(\frac{\cos \alpha_{ij}}{\sin \alpha_{ij}} + \frac{\cos \beta_{ij}}{\sin \beta_{ij}} \right)$$

Cotangent Laplacian

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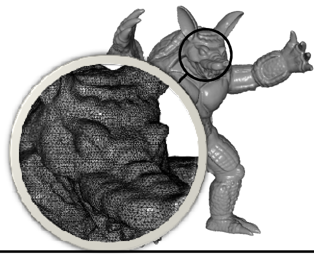
$$= \frac{1}{8}\left(\frac{\cos \alpha_{ij}}{\sin \alpha_{ij}} + \frac{\cos \beta_{ij}}{\sin \beta_{ij}} \right) = \frac{1}{8}\left(\cot \alpha_{ij} + \cot \beta_{ij} \right)$$

Cotangent Laplacian

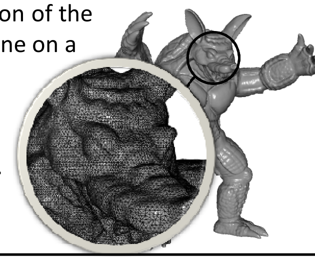
Q: How do things change when we move from a triangulation of the plane to a triangle mesh?



Cotangent Laplacian

Q: How do things change when we move from a triangulation of the plane to a triangle mesh?

A: Since the definition of the Laplacian can be done on a triangle-by-triangle basis, the same definition will work for triangle meshes.



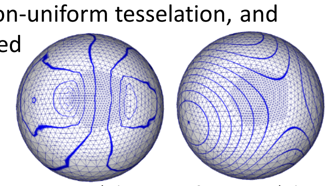
Cotangent Laplacian

Q: What would happen if we just used the simpler Tutte Laplacian?

Cotangent Laplacian

Q: What would happen if we just used the simpler Tutte Laplacian?

A: We would end up with a Laplacian that does not adapt well to non-uniform tesselation, and processing performed using this Laplacian would be biased by the triangulation.



Tutte Laplacian Cotangent Laplacian

Bruno Levy, SMI '06