Multigrid Solvers

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Anouncements

The reading seminar starts this week:

- Usually it will be held in NEB 317
- This week it will be in Maryland 310

Multigrid Solvers

Recall:

To compute the solution to the Poisson equation $Ax=b$ using a Jacobi solver we:

1. We start with an initial guess x^0,
2. We generate a sequence of improved guesses

$$\{x^0, x^1, \ldots, x^i, \ldots\}$$

converging to the solution:

$$\lim_{i\to\infty} \|Ax^i - b\| \to 0$$

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Recall:

To do this, we decompose the matrix A as the sum $A=D+P$:

- D is the diagonal part of A.
- P is everything else.

And define the update rule as:

$$x^{i+1} = D^{-1}\left(b - \left(Px^i\right)\right)$$

which has the fixed-point property that if x is the solution $Ax=b$, then:

$$x = D^{-1}\left(b - \left(Px\right)\right)$$

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Goal:

To evaluate the convergence properties of the update rule:

Given a matrix A, a vector b, and an initial guess x^0, how quickly will the series $\{x^0,\ldots,x^i,\ldots\}$ converge to the correct answer?

$$x^{i+1} = D^{-1}\left(b - \left(Px^i\right)\right)$$

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Goal:

Equivalently, we can think of the question in terms of the error. If we have the vector x such that $Ax=b$ we can ask:

Given a matrix A, and an initial guess $y^0=x-x^0$, how quickly will the series $\{y^0,\ldots,y^i,\ldots\}$ converge to zero?

$$y^{i+1} = -D^{-1}Py^i$$

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Proof of Equivalence:

To show the equivalence, we need to show that if x is the solution to the equation $Ax=b$, then if:

- $\{x^0,\ldots,x^i,\ldots\}$ is the Jacobi sequence generated solving $Ax=b$ with initial guess x^0, and
- $\{y^0,\ldots,y^i,\ldots\}$ is the Jacobi sequence generated solving $Ay=0$ with initial guess $y^0=x^0-x$

Then:

$$y^i = x^i - x$$

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Proof (by induction):

(i=0):

Clearly true, by definition of y^0.

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Proof (by induction):

Assume true for i:

$$y^{i+1} = -D^{-1}Py^i$$

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$$y^i = x^i - x$$

Proof (by induction):

Assume true for i:

$$\begin{aligned} y^{i+1} &= -D^{-1}Py^i \\ &= -D^{-1}\left(P\left(x^i - x\right)\right) \end{aligned}$$

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Assume true for i:

$$\begin{aligned} y^{i+1} &= -D^{-1}Py^i \\ &= -D^{-1}\left(P\left(x^i - x\right)\right) \\ &= -D^{-1}\left(Px^i\right) + D^{-1}\left(P(x)\right) \end{aligned}$$

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Convergence:

The question now can be formulated as follows:

Given a matrix A, and an initial guess y^0, how quickly will the series $\{y^0,\dots,y^i,\dots\}$ converge to zero?

$$y^{i+1} = -D^{-1}Py^i$$

Multigrid Solvers

In the 1D case, the Laplacian matrix can be expressed as:

$$\underbrace{\begin{pmatrix} -2 & 1 & \cdots & 0 & 0 \\ 1 & -2 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & -2 & 1 \\ 0 & 0 & \cdots & 1 & -2 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} -2 & 0 & \cdots & 0 & 0 \\ 0 & -2 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & -2 & 0 \\ 0 & 0 & \cdots & 0 & -2 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}}_{P}$$

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How does the operator:

$$y^{i+1} = -D^{-1}Py^i$$

act on a vector y^i?

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How does the operator:

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It defines a vector y^{i+1} whose k-th coefficient is the average of the (k-1)-th and (k+1)-th coefficients of y^i.

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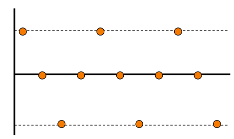
Example 1:

When the initial error y^0 is high-frequency:



Multigrid Solvers

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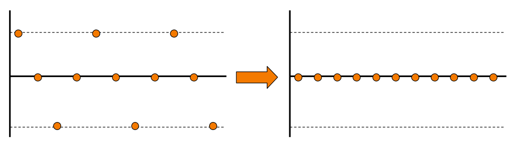

the convergence is very fast!

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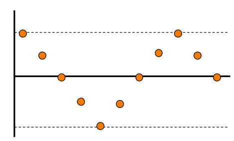
Example 2:

When the initial error y^0 is lower-frequency:



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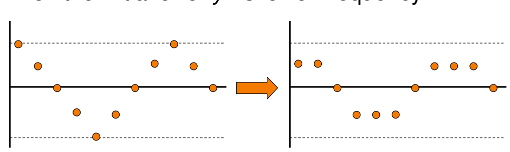

the convergence slows down.

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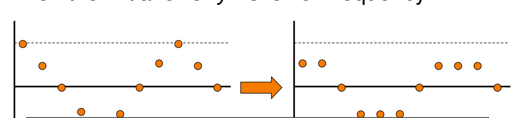

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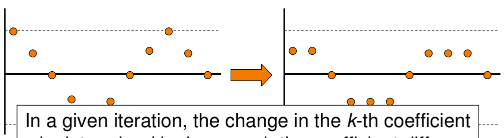

In a given iteration, the change in the k-th coefficient is determined by how much the coefficient differs from the average of its neighbors.

So the smoother the error, the slower the convergence

the

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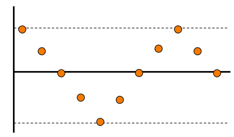
Key Idea:

Transform error so low-frequencies become high-frequencies:



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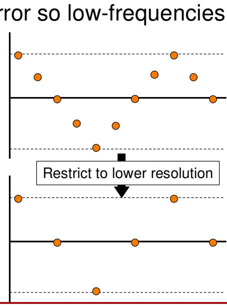

Multigrid Solvers

General Approach:

Given the equation $Ax=b$:

1. Restriction:
   Compute the low-resolution equation:
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(Step 4: Solves for the high-res part of x.)