Lecture 8: Priority Queues and Heaps

Michael Dinitz

September 23, 2021
601.433/633 Introduction to Algorithms
Introduction

Priority Queues / Heaps: Like a queue/stack, but instead of FIFO/LIFO, by priority
  - Insert($H, x$): insert element $x$ into heap $H$.
  - Extract-Min($H$): remove and return an element with smallest key
  - Decrease-Key($H, x, k$): decrease the key of $x$ to $k$.
  - Meld($H_1, H_2$): replace heaps $H_1$ and $H_2$ with their union

Extra Operations:
  - Find-Min($H$): return the element with smallest key
  - Delete($H, x$): delete element $x$ from heap $H$

Min-Heap, but can also do Max-Heap.
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Min-Heap, but can also do Max-Heap.

Note: $x$ is a pointer to an element. No way to lookup, so need a pointer to an element to change it.
## Obvious Approaches

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**Goal:** get as many of these to $O(1)$ as possible

**Question:** Can we make Insert and Extract-Min both $O(1)$, even amortized? 

No! Sorting lower bound. But maybe can make one $O(1)$, other $O(\log n)$?
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Today and State of the Art

State of the art: strict Fibonacci Heaps.
  - Too complicated for this class, not practical. See CLRS 19 for Fibonacci Heaps.

Today: binary heaps (should be review), then binomial heaps
  - Binomial heaps not quite as complicated as Fibonacci heaps, many of same core ideas
Binary Heaps

- Complete binary tree, except possibly at bottom level.
- Heap order: key of any node no larger than key of its children.
Binary Heaps

- Complete binary tree, except possibly at bottom level.
- Heap order: key of any node no larger than key of its children.

Properties:
- Since (almost) complete binary tree, depth $\Theta(\log n)$
- Min must be at root

Representation:
- Pointers to root and rightmost leaf
- Every node has pointers to parent and children
**Insert**\( (H, x) \)

Preserve heap *structure*: insert \( x \) into next open spot (bottom right, or left of new level if bottom level full)

- Might violate heap *order*!

![Binary heap: insert an element into an existing heap](image)

- **Running time:** \( O(\log n) \) worst case (also amortized) via depth
**Insert($H, x$)**

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Extract-Min($H$)

Min is definitely at root. How to remove it while still have binary tree?
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- Swap root with final heap element, remove former root.
- Sink down: swap root with smaller of its children until heap order restored

Running time: $O(\log n)$ worst case (via depth). Amortized: $O(1)$ (not obvious)
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Running time: $O(\log n)$ worst case (via depth). Amortized: $O(1)$ (not obvious)
Decrease-Key($H, x, k$)

Decrease key of $x$ to $k$, “swim up” until heap order restored.

Running time: $O(\log n)$ (depth)
Meld($H_1, H_2$)

Assume both heaps have size $n$.

- Obvious approach: insert each element of $H_2$ into $H_1$. Time: $O(n \log n)$

Correctness:

Running Time:

- Inserting: $O(n)$ total

- Sinking down:
  - Nodes at height $h$ might have to sink down $h$.
  - At most $n \cdot 2^h$ nodes at height $h$.
  - $n \cdot 2^h \log n = O(n)$.
Meld($H_1, H_2$)

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Better:

- Insert all elements of $H_2$ all at once (not fixing heap order)
- Instead of fixing by swimming up: iterate from bottom up and sink down to fix heap.
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\[
\sum_{h=0}^{\log n} h \left( \frac{n}{2^h} \right) = n \sum_{h=0}^{\log n} \frac{h}{2^h} \leq O(n)
\]
Amortized Extract-Min

Weights: $w(x) = \text{depth of } x$
  - Root has weight 0, its children have weight 1, etc.

Potential: $\Phi(H) = \sum_x w(x)$
Amortized Extract-Min

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Insert: $\Delta \Phi = O(\log n) \implies \text{amortized cost} \leq O(\log n) + O(\log n) = O(\log n)$
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Extract-Min:
- True cost: height \( h = \Theta(\log n) \) of tree, plus \( O(1) \) (for initial swap).
- \( \Delta \Phi \): one less node at depth \( h \implies \Delta \Phi = -h \)
- Amortized cost: \( h + O(1) - h = O(1) \).
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Uses Inserts to “pay for” Extract-Mins.
Downsides of binary heaps:

- Do at least as many Inserts as Extract-Mins! Want $O(1)$ Insert, $O(\log n)$ Extract-Min.
- Meld in $O(n)$ is better than trivial, but still not great.
Improvements

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Binomial Heaps:

- Get Insert down to $O(1)$ (amortized)
- Meld in $O(\log n)$ (worst-case and amortized)
- Downside: $O(\log n)$ Extract-Min, $O(\log n)$ Find-Min
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Fibonacci Heaps:
- Everything $O(1)$ (amortized) except $O(\log n)$ Extract-Min (amortized)
Binomial Heaps

Not based on binary tree anymore! Based on *binomial tree*. 
Binomial Heaps

Not based on binary tree anymore! Based on binomial tree.

- $B_0 =$ single node.
- $B_k =$ one $B_{k-1}$ linked to another $B_{k-1}$.
The order $k$ binomial tree $B_k$ has the following properties:

1. *Its height is* $k$.
2. *It has* $2^k$ *nodes*.
3. *The degree of the root is* $k$.
4. *If we delete the root, we get* $k$ *binomial trees* $B_{k-1}, \ldots, B_0$. 

![Diagram showing the structure lemma with binomial trees $B_k$, $B_{k+1}$, $B_1$, $B_0$, and $B_4$.]
Binomial Heap

Definition

A *binomial heap* is a collection of binomial trees so that each tree is heap ordered, and there is exactly 0 or 1 tree of order $k$ for each integer $k$.

Keep roots of trees in linked list, from smallest order (not key!) to largest

Write $n$ in binary: $b_a b_{a-1} ... b_1 b_0$.

Tree $B_k$ exists if and only if $b_k = 1$.

⇒ at most $\log n$ trees, and by lemma each has height $\leq \log n$. 

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Analysis: Beginning

Analyze all operations both worst-case and amortized.
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Potential function: $\Phi(H) = \# \text{ trees in } H$
  
  - Initially 0
  
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$\text{Find-Min}(H)$: Scan through roots of trees in $H$, return min
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Find-Min($H$): Scan through roots of trees in $H$, return min

- Correct: each tree heap-ordered, so global min one of the roots
- Worst-case: $O(\log n)$
- Amortized: doesn’t change potential, also $O(\log n)$. 
Meld($H_1, H_2$): Link

Key operation: we’ll use Meld to do Insert and Extract-Min
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Warmup: $H_1, H_2$ both single trees of same order $k$.
- Union has size $2^k + 2^k = 2^{k+1}$: just a single $B_{k+1}$
- Easy to make a $B_{k+1}$ out of two $B_k$’s!
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Link of two trees.
  - Worst-case time: $O(1)$ (create a single link). Normalize: call $1$
  - $\Delta \Phi$: two trees to one: $-1$
  - Amortized cost:
    $1 - 1 = 0 = O(1)$. 

![Diagram showing the meld operation with two binomial heaps $H_1$ and $H_2$. The roots of the trees are connected to form a single tree.]
Meld($H_1, H_2$): General Case

(Almost) just like binary addition!
Meld($H_1, H_2$): Analysis

Easy to prove correct (exercise for home).

Running time:
- Worst case: $O(1)$ per “order” $k \implies \leq O(\log n)$
- Amortized: Potential does not go up, but could stay the same
  $\implies O(\log n)$ amortized
Insert($H, x$)

Use Meld:
- Create new heap $H'$ with one $B_0$ consisting of just $x$
- Meld($H, H'$)

Correctness: Obvious
Insert($H, x$)

Use Meld:
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Correctness: Obvious

Running Time:
  - Worst case: $O(\log n)$ (via Meld)
**Insert**($H, x$)

Use Meld:
- Create new heap $H'$ with one $B_0$ consisting of just $x$
- Meld($H, H'$)

Correctness: Obvious

Running Time:
- Worst case: $O(\log n)$ (via Meld)
- Amortized:
  - Like incrementing a binary counter!
Insert($H, x$)

Use Meld:
- Create new heap $H'$ with one $B_0$ consisting of just $x$
- $\text{Meld}(H, H')$

Correctness: Obvious

Running Time:
- Worst case: $O(\log n)$ (via Meld)
- Amortized:
  - Like incrementing a binary counter!
  - If we link $k$ trees, potential goes down by $k - 1$
  - Cost = # links plus 1 (for making new heap)
  - Amortized cost = $k + 1 + \Delta \Phi = k + 1 - (k - 1) = 2 = O(1)$
Extract-Min($H$)

Use Meld again!

- $O(\log n)$ to Find-Min: one of the roots.
- Delete and return this root: tree turns into a new heap!
- Meld with original heap (minus the tree)

Correctness: Obvious
Use Meld again!

- \( O(\log n) \) to Find-Min: one of the roots.
- Delete and return this root: tree turns into a new heap!
- Meld with original heap (minus the tree)

Correctness: Obvious

Running Time:

- Worst-Case: \( O(\log n) \) from creating new heap, Meld
- Amortized:
  - Potential can go up! But by at most \( \log n \)
  - Amortized time at most \( O(\log n) + \log n = O(\log n) \)