Lecture 6: Balanced Search Trees

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601.433/633 Introduction to Algorithms

Introduction

Today, and next few weeks: data structures.

- Since "Data Structures" a prereq, focus on advanced structures and on interesting analysis

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Today and later: data structures for *dictionaries*

Definition

A *dictionary data structure* is a data structure supporting the following operations:

- **insert(key,object)**: insert the (key, object) pair.
- **lookup(key)**: return the associated object
- **delete(key)**: remove the key and its object from the data structure. We may or may not care about this operation.

Obvious Approaches

Reminder: all running times for *worst case*

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Goal: $\mathbf{O(\log n)}$ for both.

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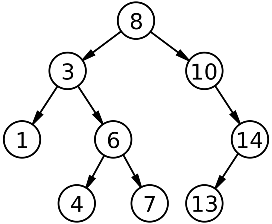
Goal: $\mathbf{O(\log n)}$ for both.
Approach today: search trees

Binary Search Tree Review

Binary search tree:

- All nodes have at most **2** children
- Each node stores (key, object) pair
- All descendants to left have smaller keys
- All descendants to the right have larger keys



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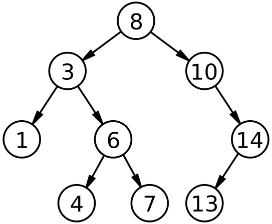

Lookup: follow path from root!

Dictionary Operations in Simple Binary Search Tree

insert(**x**):

- If tree empty, put **x** at root
- Else if **x** < **root.key** recursively insert into left child
- Else (if **x** > **root.key**) recursively insert into right child

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Example: `H O P K I N S`

Simply Binary Search Tree: Analysis

Pluses: easy to implement

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Want to make tree *balanced*.

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- If very unbalanced $\mathbf{d}$ could be $\boldsymbol{\Omega}(\mathbf{n})$!

Want to make tree *balanced*.

Rest of today:

- B-trees: perfect balance, not binary
- Red-black trees: approximate balance, binary
- Turn out to be related!

B-Trees

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Definition (B-tree with parameter $\mathbf{t}$)

1. Each node has between $\mathbf{t}-\mathbf{1}$ and $\mathbf{2t}-\mathbf{1}$ keys in it (except the root has between $\mathbf{1}$ and $\mathbf{2t}-\mathbf{1}$ keys). Keys in a node are stored in a sorted array.
2. Each non-leaf has degree (number of children) equal to the number of keys in it plus $\mathbf{1}$. If $\mathbf{v}$ is a node with keys $[\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k]$ and the children are $[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k+1}]$, then the tree rooted at $\mathbf{v}_i$ contains only keys that are at least $\mathbf{a}_{i-1}$ and at most $\mathbf{a}_i$ (except the the edge cases: the tree rooted at $\mathbf{v}_1$ has keys less than $\mathbf{a}_1$, and the tree rooted at $\mathbf{v}_{k+1}$ has keys at least $\mathbf{a}_k$).
3. All leaves are at the same depth.

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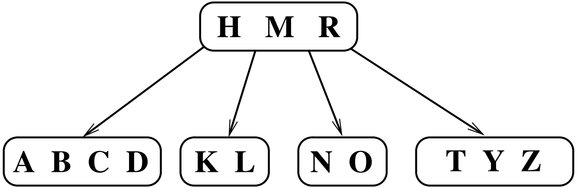
3. All leaves are at the same depth.

When $\mathbf{t} = \mathbf{2}$ known as a *2-3-4 tree*, since # children either 2, 3, or 4

B-tree: Example

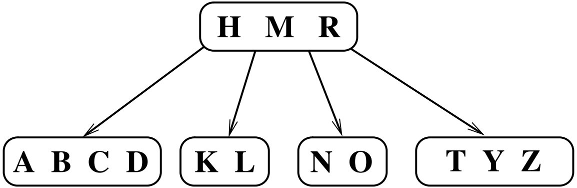
t = **3**:

- Root has between **1** and **5** keys, non-roots have between **2** and **5** keys
- Non-leaves have between **3** and **6** children (root can have fewer).

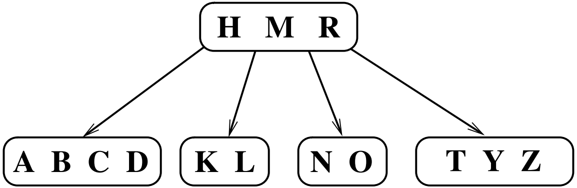


Lookups

Binary search in array at root. Finished if find item, else get pointer to appropriate child, recurse.



Insert(**x**)



Obvious approach: do a lookup, put **x** in leaf where it should be.

- Example: insert **E**

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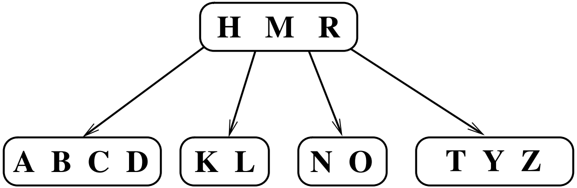

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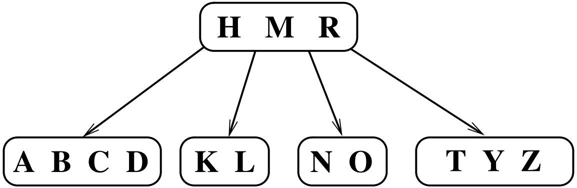
Problem: What if leaf is *full* (already has $\mathbf{2t-1}$ keys)?

Insert($\mathbf{x}$)



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- Example: insert **E**

Problem: What if leaf is *full* (already has $\mathbf{2t-1}$ keys)?

Split:

- Only used on *full* nodes (nodes with $\mathbf{2t-1}$ keys) whose parents are *not* full.
- Pull median of its keys up to its parent
- Split remaining $\mathbf{2t-2}$ keys into two nodes of $\mathbf{t-1}$ keys each. Reconnect appropriately.

Insert (continued)

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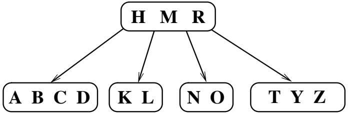

Insert **E, F** into example.

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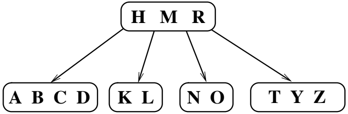

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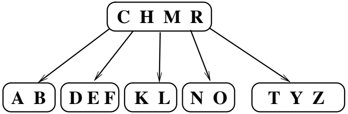

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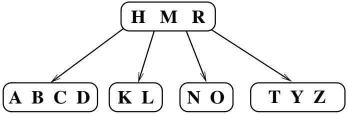

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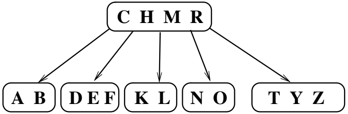
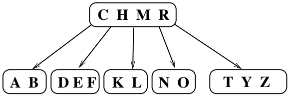


Note: since split *on the way down*, when a node is split, its parent is not full!

Example continued



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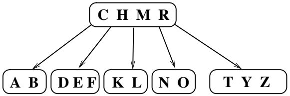

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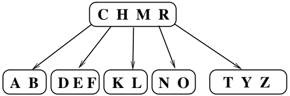

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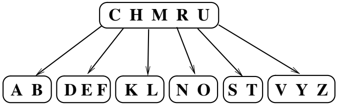

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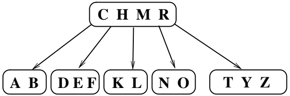

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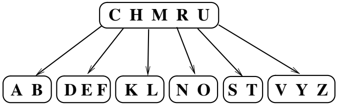

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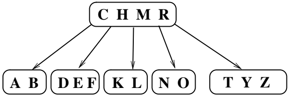

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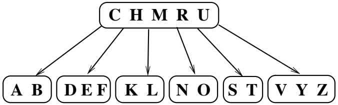

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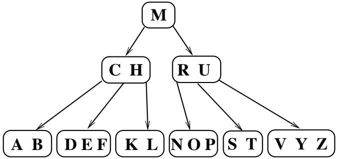

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Induction. Start with a valid B-tree, insert **x**.

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Induction. Start with a valid B-tree, insert $\mathbf{x}$.

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- No split:

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Second property (correct degrees, subtrees have keys in correct ranges): Hooked nodes up correctly after split. ✓

B-tree running time

Suppose **n** keys, depth **d**

B-tree running time

Suppose $\mathbf{n}$ keys, depth $\mathbf{d} \leq \mathbf{O}(\log_{\mathbf{t}} \mathbf{n})$

B-tree running time

Suppose $\mathbf{n}$ keys, depth $\mathbf{d} \le \mathbf{O}(\log_t \mathbf{n})$

Lookup:

- Binary search on array in each node we pass through

B-tree running time

Suppose $\mathbf{n}$ keys, depth $\mathbf{d} \leq \mathbf{O(\log_t n)}$

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Suppose $\mathbf{n}$ keys, depth $\mathbf{d} \leq \mathbf{O}(\log_t n)$

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  - Insert into leaf: $\mathbf{O}(\mathbf{t})$
  - Splitting time:

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- $\mathbf{O(t \log_t n) = O(\frac{t}{\log t} \log n)}$ total

B-tree notes

Used a lot in databases

- Large $\mathbf{t}$: shallow trees. Fits well with memory hierarchy

B-tree notes

Used a lot in databases

- Large **t**: shallow trees. Fits well with memory hierarchy

t = **2**:

- 2-3-4 tree
- Can be implemented as *binary* tree using *red-black trees*

Red-Black Trees

Red-Black Trees: Intro

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Most famous: *red-black trees*

- Default in Linux kernel, used to optimize Java HashMap, ...
- Today: Quick overview, connection to 2-3-4 trees.
- *Not* traditional or practical point of view on red-black trees. See book!

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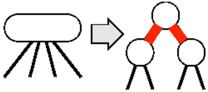

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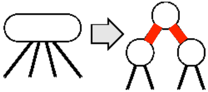

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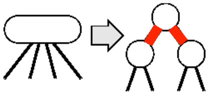

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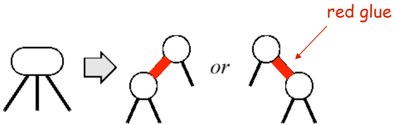
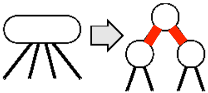
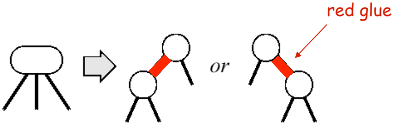


Important Properties





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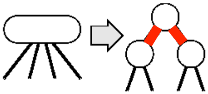

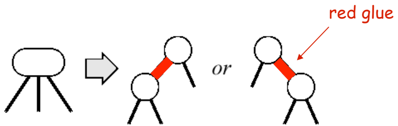

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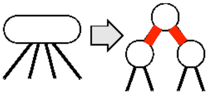

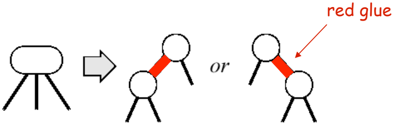

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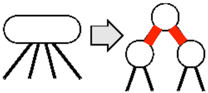

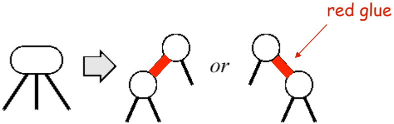

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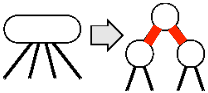

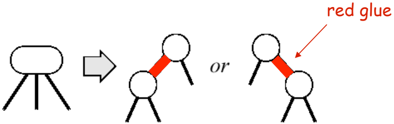

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$\implies$ depth $\leq \mathbf{O(\log n)}$

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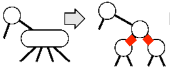

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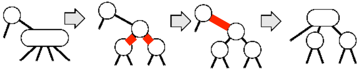

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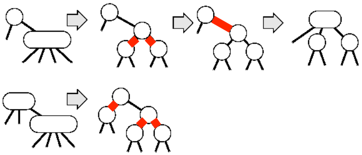

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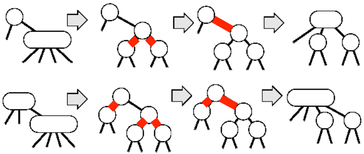

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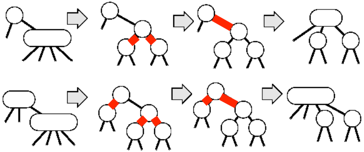
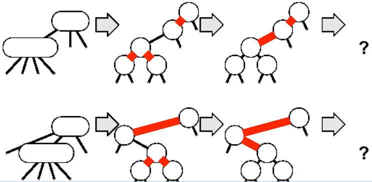


Harder cases:



Tree Rotations

Used in many different tree constructions.

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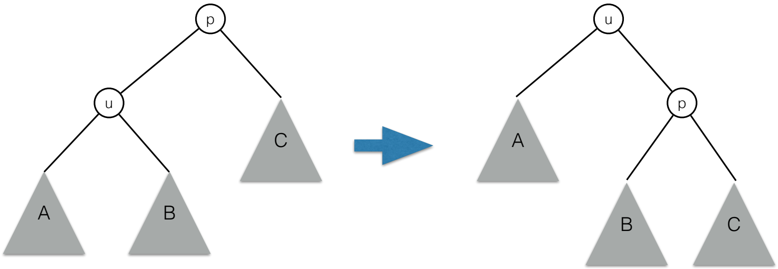
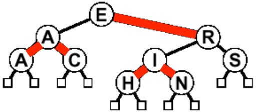


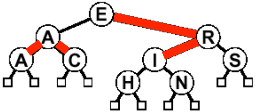
Using Rotations

Can use rotations to “fix” hard cases. Example:

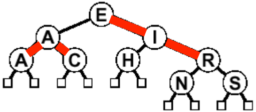
inserting G



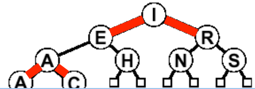
change colors



right rotate R →



left rotate E →



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Main points:

- Red-Black trees can be thought of as a binary implementation of 2-3-4 trees
- Approximately balanced, so $\mathbf{O(\log n)}$ lookup time
- Insert time (basically) same as 2-3-4 tree, so also $\mathbf{O(\log n)}$.
- See book for direct approach (not through 2-3-4 trees).