Lecture 21: NP-Completeness I

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601.433/633 Introduction to Algorithms

Introduction

Last few weeks: slower and slower algorithms for harder and harder problems

- From $\mathbf{O(m+n)}$ time algorithms for BFS/DFS/topological sort/SCCs, to $\mathbf{O(m^2 n)}$ for max flow
- Today: start of two lectures on NP-completeness.
  - The (or at least a) line between tractability and intractability

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An algorithm runs in *polynomial time* if its (worst-case) running time is $O(n^c)$ for some constant $c \geq 0$, where n is the size of the input.

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Question: When do polynomial-time algorithms exist?

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- Shortest $s - t$ path: Input is $G = (V, E), \ell : E \to \mathbb{R}, s, t \in V, k \in \mathbb{R}$. Output YES if $d(s, t) \leq k$, otherwise output NO.

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Some problems naturally decision, others naturally optimization, but can turn any optimization problem into a decision problem.

- If can solve decision, can almost always solve optimization.

Note: Can divide instances (inputs) of any decision problem into YES-instances and NO-instances

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Question: Are all decision problems in **P**?
Answer: No!

- By *time hierarchy theorem* there are problems that require super-polynomial time!
- Undecidability: there are problems which cannot be solved by *any* algorithm at all!

Verification

Different Setting: If *in addition* to the input we're given a purported solution, can we check that this solution is valid/feasible (in polynomial time)?

- Max-Flow: given $\mathbf{f : E \to \mathbb{R}_{\geq 0}}$, check that value $\geq \mathbf{k}$, flow conservation at all nodes other than $\mathbf{s, t}$, and capacity constraints obeyed

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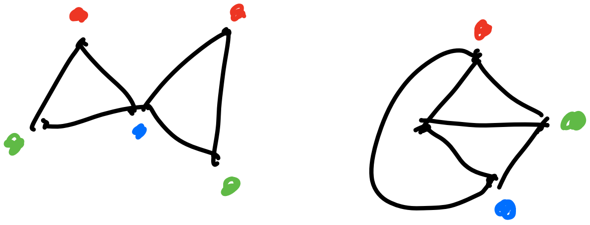
Definition (3-Coloring)

Input: Undirected graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$
Output: YES if $\exists$ *coloring* $\mathbf{f}: \mathbf{V} \to \{\mathbf{R}, \mathbf{G}, \mathbf{B}\}$ such that $\mathbf{f}(\mathbf{u}) \neq \mathbf{f}(\mathbf{v})$ for all $\{\mathbf{u}, \mathbf{v}\} \in \mathbf{E}$. NO otherwise



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Definition (3-Coloring)

Input: Undirected graph $\mathbf{G = (V, E)}$
Output: YES if $\exists$ *coloring* $\mathbf{f : V \to \{R, G, B\}}$ such that $\mathbf{f(u) \neq f(v)}$ for all $\mathbf{\{u, v\} \in E}$. NO otherwise

Verification: Given $\mathbf{f}$,

- Check that $\mathbf{f(u) \in \{R, G, B\}}$ for all $\mathbf{u \in V}$, and
- Check each edge $\mathbf{\{u, v\}}$ to make sure that $\mathbf{f(u) \neq f(v)}$

NP

NP: decision problems where solutions can be *verified* in polynomial time.

Definition

A decision problem $\mathbf{Q}$ is in **NP** (*nondeterministic polynomial time*) if there exists a polynomial time algorithm $\mathbf{V(I, X)}$ (called the *verifier*) such that

1. If $\mathbf{I}$ is a YES-instance of $\mathbf{Q}$, then there is some $\mathbf{X}$ (usually called the *witness*, *proof*, or *solution*) with size polynomial in $|\mathbf{I}|$ so that $\mathbf{V(I, X)} = \text{YES}$.
2. If $\mathbf{I}$ is a NO-instance of $\mathbf{Q}$, then $\mathbf{V(I, X)} = \text{NO}$ for all $\mathbf{X}$.

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Examples:

- 3-coloring: Witness $\mathbf{X}$ is a coloring $\mathbf{f : V \to \{R, B, G\}}$, verifier checks each edge $\{\mathbf{u, v}\}$ to make sure $\mathbf{f(u) \neq f(v)}$
    - If $\mathbf{I}$ is a YES instance, then there is a coloring so verifier will return YES
    - If $\mathbf{I}$ is a NO instance, then no valid coloring exists. Whatever $\mathbf{X}$ is, verifier returns NO.

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Examples:

- Max-Flow: Witness $\mathbf{X}$ is a flow $\mathbf{f : E \rightarrow \mathbb{R}_{\geq 0}}$, verifier checks that it's feasible of value $\geq \mathbf{k}$
  - If $\mathbf{I}$ is a YES instance, then there is a feasible flow of value at least $\mathbf{k}$ so verifier (on this flow) will return YES
  - If $\mathbf{I}$ a NO instance, then no feasible flow of value $\geq \mathbf{k}$. Whatever $\mathbf{X}$ is, verifier returns NO.

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Examples:

- Factoring: Instance is pair of integers $\mathbf{M, k}$. YES if $\mathbf{M}$ has as factor in $\{\mathbf{2, \ldots, k}\}$, NO otherwise.
  - Witness: integer $\mathbf{f}$ in $\{\mathbf{2, 3, \ldots, k}\}$. Verifier: returns YES if $\mathbf{M/f}$ is an integer and $\mathbf{f} \in \{\mathbf{2, \ldots, k}\}$, NO otherwise.
  - If YES instance, then an $\mathbf{f}$ does exist so verifier returns YES on that $\mathbf{f}$. If NO, then no such $\mathbf{f}$ exists so verifier always returns NO.

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Examples:

- Traveling Salesman: Instance is weighted graph $\mathbf{G}$ an integer $\mathbf{k}$. YES iff $\mathbf{G}$ has a tour (walk that touches very vertex at least once) of length $\leq \mathbf{k}$.
  - Witness: tour $\mathbf{P}$. Verifier checks that it is a tour, has length at most $\mathbf{k}$
  - If YES instance, then such a tour exists $\implies$ verifier returns YES on that tour.
  - If NO, no such tour exists $\implies$ verifier always returns NO.

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2. If $\mathbf{I}$ is a NO-instance of $\mathbf{Q}$, then $\mathbf{V(I, X)}$ = NO for all $\mathbf{X}$.

Important asymmetry: need a witness for YES, not a witness for NO.

P vs NP

Theorem

$\mathbf{P} \subseteq \mathbf{NP}$

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Proof.

Let $\mathbf{Q} \in \mathbf{P}$.

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$\mathbf{V(I, X)}$: Ignore $\mathbf{X}$, solve on instance $\mathbf{I}$. □

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Question: Does $\mathbf{P} = \mathbf{NP}$, i.e., is $\mathbf{NP} \subseteq \mathbf{P}$?

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Question: Does $\mathbf{P} = \mathbf{NP}$, i.e., is $\mathbf{NP} \subseteq \mathbf{P}$?

- *Almost* everyone thinks no, but we don't know for sure!
- Not even particularly close to a proof.
- Think about what $\mathbf{P} = \mathbf{NP}$ would mean. . .

Reductions

Question: How could we prove that $\mathbf{P} = \mathbf{NP}$ or $\mathbf{P} \neq \mathbf{NP}$?

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- **P** = **NP**: Need to show that *every* problem in **NP** is also in **P**!
- **P** ≠ **NP**: Need to prove that *some* problem in **NP** not in **P**.
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Definition

Problem **A** is *polytime reducible* to problem **B** (written $\mathbf{A} \leq_{\mathbf{p}} \mathbf{B}$) if, given a polynomial-time algorithm for **B**, we can use it to produce a polynomial-time algorithm for **A**.

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Means that **B** is "at least as hard" as **A**: if **B** is in **P**, then so is **A**.

- So "hardest" problems in **NP** are problems that many other problems reduce to.

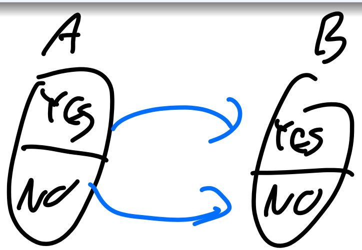
Many-One (Karp) Reductions

Almost always (and always in this course), use a special type of reduction.

Definition

A *Many-one* or *Karp* reduction from **A** to **B** is a function **f** which takes arbitrary instances of **A** and transforms them into instances of **B** so that

1. If **x** is a YES-instance of **A** then **f(x)** is a YES-instance of **B**.
2. If **x** is a NO-instance of **A** then **f(x)** is a NO-instance **B**.
3. **f** can be computed in polynomial time.



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3. $\mathbf{f}$ can be computed in polynomial time.

So given instance $\mathbf{x}$ of $\mathbf{A}$, compute $\mathbf{f(x)}$ and use polytime algorithm for $\mathbf{B}$ on $\mathbf{f(x)}$

- Polytime, since $\mathbf{f}$ in polytime and algorithm for $\mathbf{B}$ in polytime
- Correct by first two properties of many-one reduction.

NP-Completeness

So what is "hardest problem" in **NP**?

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Problem $\mathbf{Q}$ is **NP**-*hard* if $\mathbf{Q}' \leq_{\mathbf{p}} \mathbf{Q}$ for all problems $\mathbf{Q}'$ in **NP**.

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Problem $\mathbf{Q}$ is **NP**-*complete* if it is **NP**-hard and in **NP**.

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So suppose **Q** is **NP**-complete.

- To prove $\mathbf{P} \neq \mathbf{NP}$: Hardest problem in **NP**! If anything in **NP** is not in **P**, then **Q** is not in **P**
- To prove $\mathbf{P} = \mathbf{NP}$: Just need to prove that $\mathbf{Q} \in \mathbf{P}$.

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Is anything **NP**-complete?

Circuit-SAT

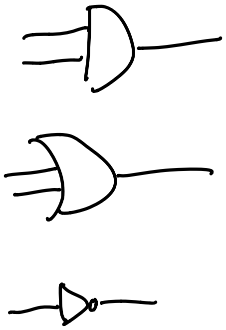
Definition

Circuit-SAT: Given a boolean circuit with a single output and no loops (some inputs might be hardwired), is there a way of setting the inputs so that the output of the circuit is **1**?

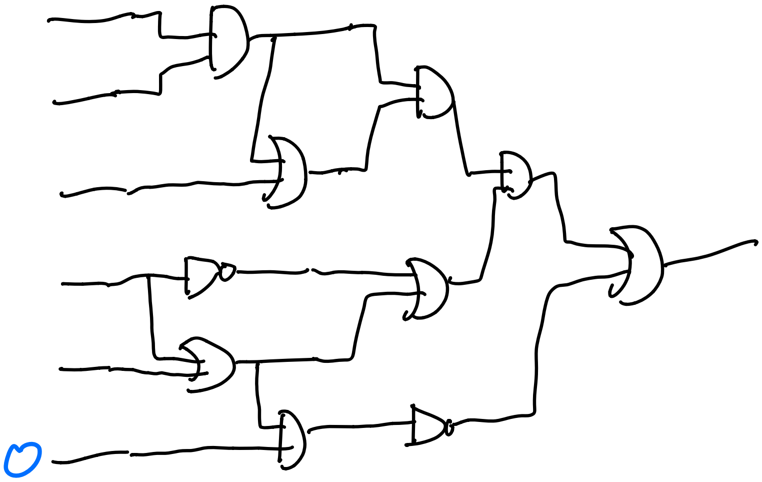
Gates: AND

OR

NOT



Arbitrary fan-out



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- If input is a YES instance then there is some assignment so circuit outputs **1**. When verifier run on that assignment, returns YES.
- In input is a NO instance then in every assignment circuit outputs **0**. So verifier returns NO on every witness.

□

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Let **A** $\in$ **NP**. Want to show **A** $\leq_p$ Circuit-SAT (construct a many-one reduction).

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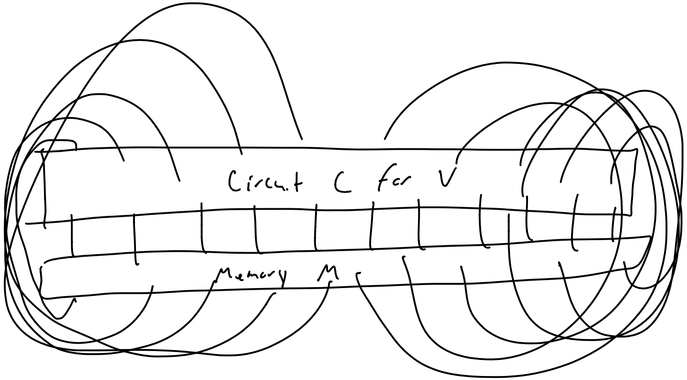

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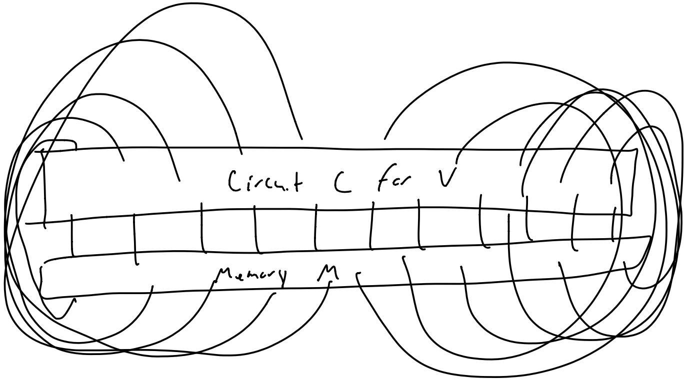

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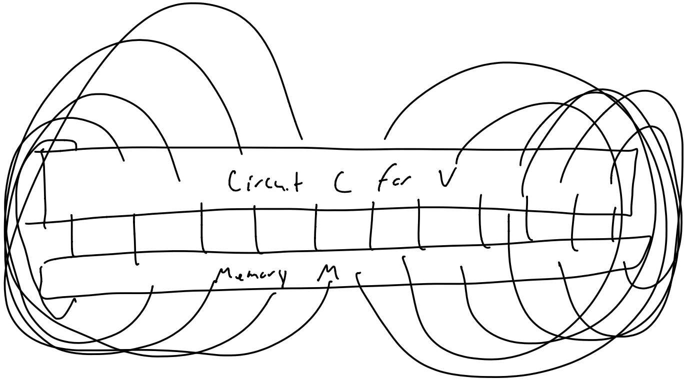

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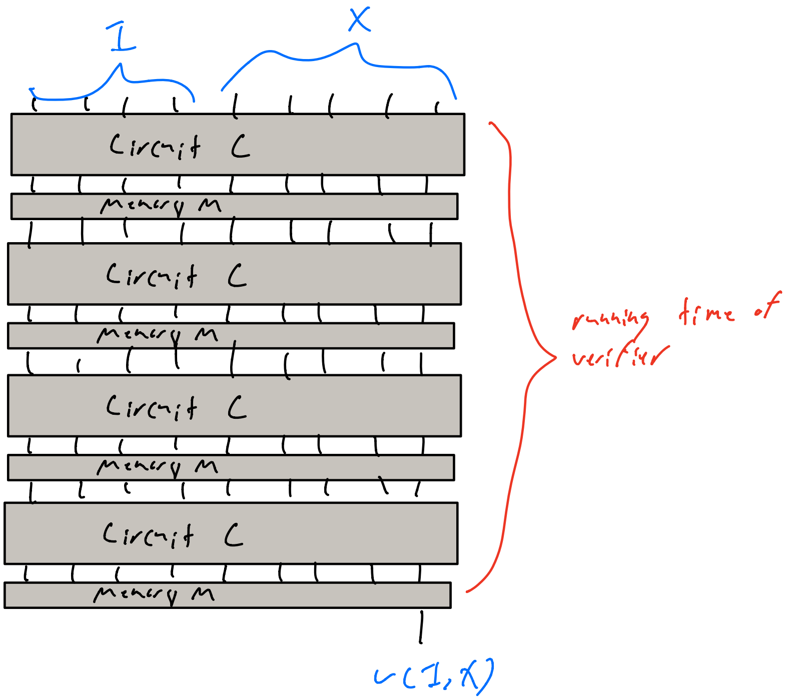
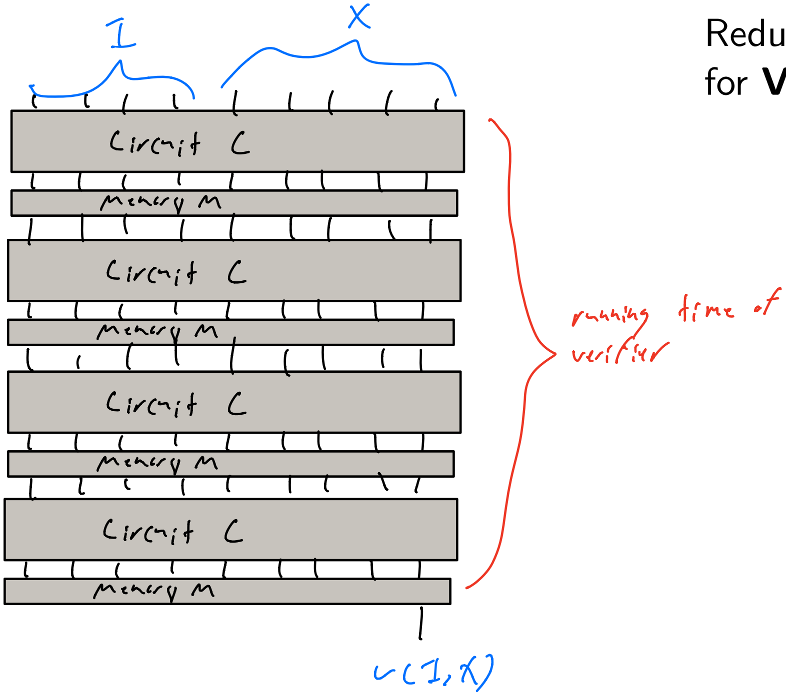
Fix: "Unroll" circuit using fact that **V** runs in polynomial time

Reduction



I
X
Circuit C
Memory M
Circuit C
Memory M
Circuit C
Memory M
Circuit C
Memory M
running time of verifier
v(I, X)

Reduction



Reduction: given instance **I** of **A**, construct this circuit for **V**, hardwire **I**. Combined circuit **f(I)**

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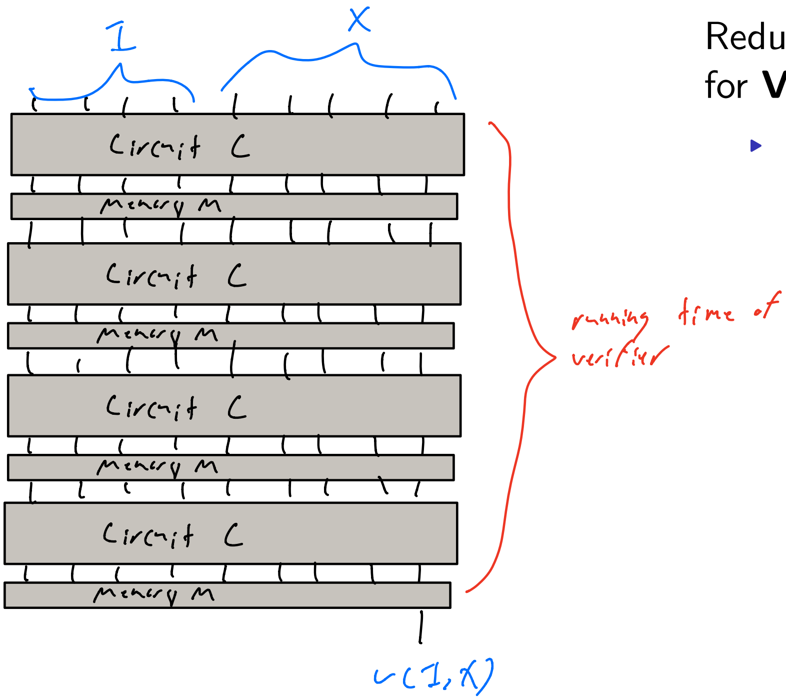

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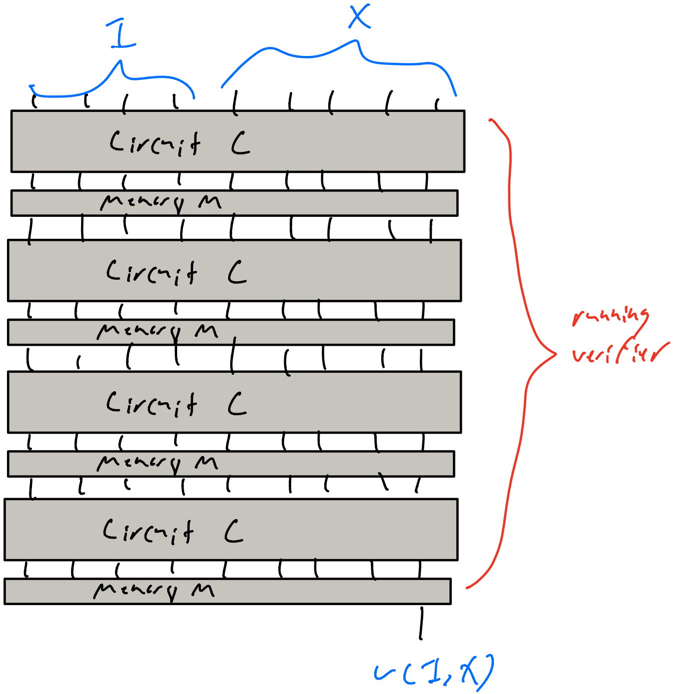

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- If **I** YES of **A**: there is some **X** so that **V(I, X)** = YES
  $\implies$ some **X** so that when **X** input to **f(I)**, outputs **1**
  $\implies$ **f(I)** YES instance of Circuit-SAT.

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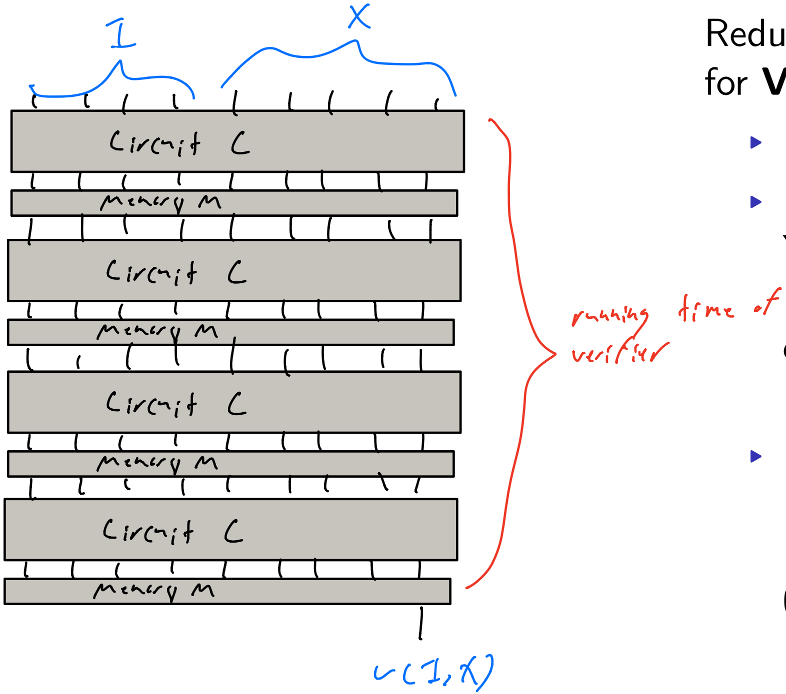

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  $\implies$ **f(I)** YES instance of Circuit-SAT.
- If **I** NO of **A**: For every **X**, know that **V(I, X)** = NO
  $\implies$ for every **X**, when **X** input to **f(I)**, outputs **0**
  $\implies$ **f(I)** NO instance of Circuit-SAT