Lecture 18: Max-Flow Min-Cut

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October 28, 2021
601.433/633 Introduction to Algorithms

Introduction

Flow Network:

- Directed graph $\mathbf{G = (V, E)}$
- *Capacities* $\mathbf{c : E \to \mathbb{R}_{\geq 0}}$ (simplify notation: $\mathbf{c(x, y) = 0}$ if $\mathbf{(x, y) \notin E}$)
- Source $\mathbf{s \in V}$, sink $\mathbf{t \in V}$

Today: flows and cuts

- Flow: "sending stuff" from $\mathbf{s}$ to $\mathbf{t}$
- Cut: separating $\mathbf{t}$ from $\mathbf{s}$

Turn out to be very related!

Today: some algorithms but not efficient. Mostly structure. Better algorithms ~~Thursday~~. next week

Flows

Intuition: send "stuff" from **s** to **t**

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$$\sum_{\mathbf{u}:(\mathbf{u},\mathbf{v})\in\mathbf{E}} \mathbf{f}(\mathbf{u},\mathbf{v}) = \sum_{\mathbf{u}:(\mathbf{v},\mathbf{u})\in\mathbf{E}} \mathbf{f}(\mathbf{v},\mathbf{u})$$

for all $\mathbf{v}\in\mathbf{V}\setminus\{\mathbf{s},\mathbf{t}\}$. This constraint is known as *flow conservation*.

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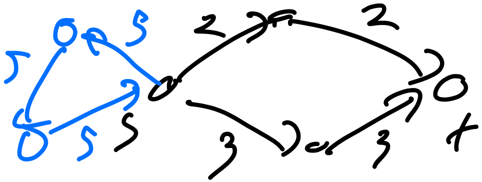
Value of flow $|\mathbf{f}|$: "total amount of stuff sent from $\mathbf{s}$ to $\mathbf{t}$"



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$$|f| = \sum_{u:(s,u)\in E} f(s,u) - \sum_{u:(u,s)\in E} f(u,s) = \sum_{u:(u,t)\in E} f(u,t) - \sum_{u:(t,u)\in E} f(t,u)$$

Feasible Flows

Capacity constraints: $0 \le f(u,v) \le c(u,v)$ for all $(u,v) \in V \times V$

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Definitions:

- An $(\mathbf{s},\mathbf{t})$-flow satisfying capacity constraints is a *feasible* flow.
- If $\mathbf{f}(\mathbf{e}) = \mathbf{c}(\mathbf{e})$ then $\mathbf{f}$ *saturates* $\mathbf{e}$.
- If $\mathbf{f}(\mathbf{e}) = \mathbf{0}$ then $\mathbf{f}$ *avoids* $\mathbf{e}$.

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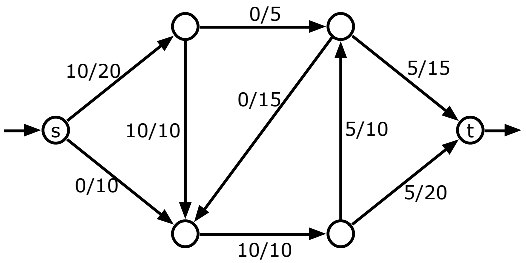

An (s,t)-flow with value 10. Each edge is labeled with its flow/capacity.

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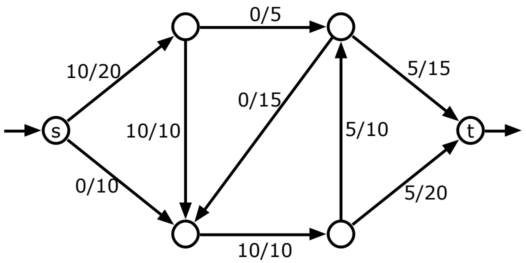


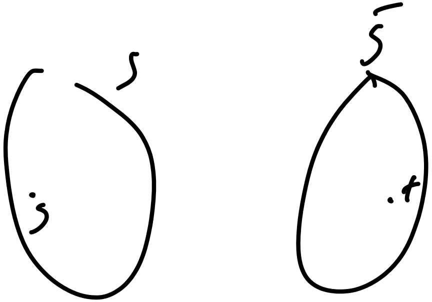
An (s,t)-flow with value 10. Each edge is labeled with its flow/capacity.

Problem we'll talk about: find feasible flow of maximum value (max flow)

Cuts

Definition

- An $(\mathbf{s},\mathbf{t})$-*cut* is a partition of $\mathbf{V}$ into $(\mathbf{S},\bar{\mathbf{S}})$ such that $\mathbf{s} \in \mathbf{S}$, $\mathbf{t} \notin \mathbf{S}$



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$$\mathbf{cap}(\mathbf{S}, \bar{\mathbf{S}}) = \sum_{(\mathbf{u},\mathbf{v}) \in \mathbf{E} : \mathbf{u} \in \mathbf{S}, \mathbf{v} \in \bar{\mathbf{S}}} \mathbf{c}(\mathbf{u}, \mathbf{v}) = \sum_{\mathbf{u} \in \mathbf{S}} \sum_{\mathbf{v} \in \bar{\mathbf{S}}} \mathbf{c}(\mathbf{u}, \mathbf{v})$$

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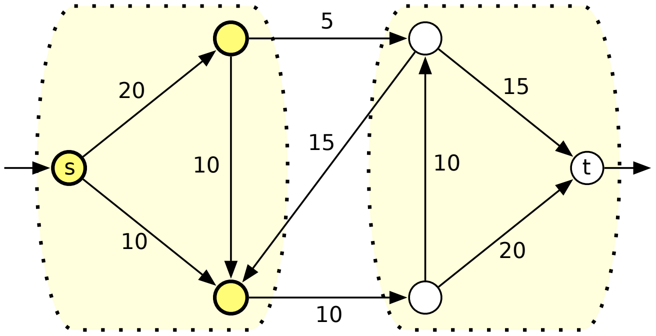

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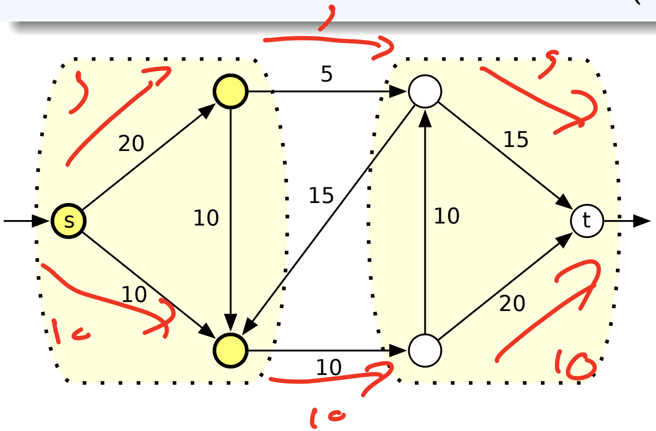
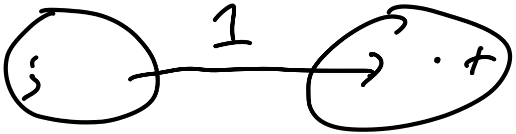


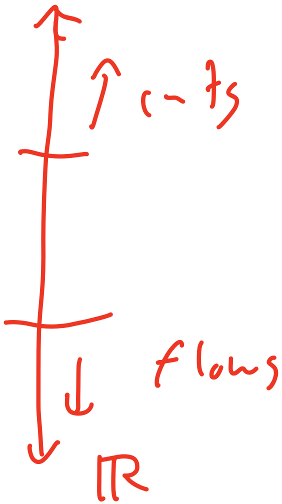
Problem we'll talk about: find (s,t)-cut of minimum capacity (min cut)



Warmup Theorem

Theorem

Let $\mathbf{f}$ *be a feasible* $(\mathbf{s}, \mathbf{t})$*-flow, and let* $(\mathbf{S}, \bar{\mathbf{S}})$ *be an* $(\mathbf{s}, \mathbf{t})$*-cut. Then* $|\mathbf{f}| \leq \mathbf{cap}(\mathbf{S}, \bar{\mathbf{S}})$.



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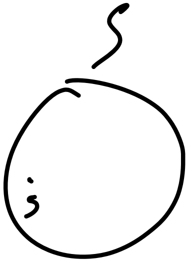

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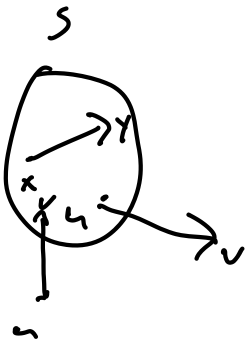

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&\leq \sum_{u \in S} \sum_{v \in \bar{S}} f(u,v) && \text{(flow is nonnegative)} \\
&\leq \sum_{u \in S} \sum_{v \in \bar{S}} c(u,v) = \mathrm{cap}(S, \bar{S}) && \text{(flow is feasible)}
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Max-Flow Min-Cut

Corollary

If $\mathbf{f}$ *avoids every* $\bar{\mathbf{S}} \rightarrow \mathbf{S}$ *edge and saturates every* $\mathbf{S} \rightarrow \bar{\mathbf{S}}$ *edge, then* $\mathbf{f}$ *is a maximum flow and* $(\mathbf{S}, \bar{\mathbf{S}})$ *is a minimum cut.*

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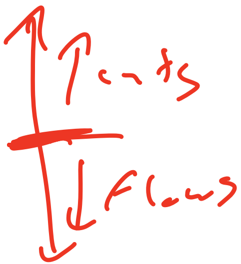
Theorem (Max-Flow Min-Cut Theorem)

In any flow network, value of max $(\mathbf{s}, \mathbf{t})$*-flow* $=$ *capacity of min* $(\mathbf{s}, \mathbf{t})$*-cut.*



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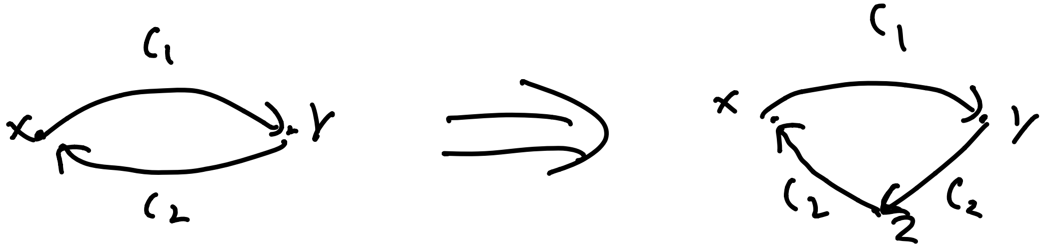
Spend rest of today proving this.

- Many different valid proofs.
- We'll see a classical proof which will naturally lead to algorithms for these problems.

One Direction

Cycles of length **2** will turn out to be annoying. Get rid of them.



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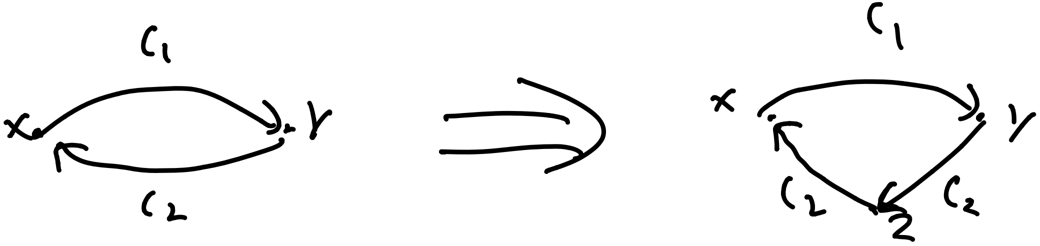
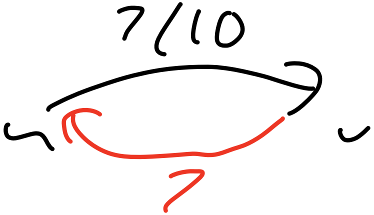


- Doesn't change max-flow or min-cut
- Increases #edges by constant factor, # nodes to original # edges.

Residual

Let $\mathbf{f}$ be feasible $(\mathbf{s}, \mathbf{t})$-flow. Define *residual capacities*:

$$\mathbf{c_f}(\mathbf{u}, \mathbf{v}) = \begin{cases} \mathbf{c}(\mathbf{u}, \mathbf{v}) - \mathbf{f}(\mathbf{u}, \mathbf{v}) & \text{if } (\mathbf{u}, \mathbf{v}) \in \mathbf{E} \\ \mathbf{0} & \text{otherwise} \end{cases}$$



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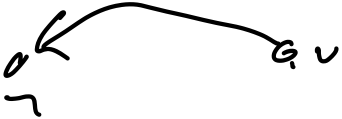

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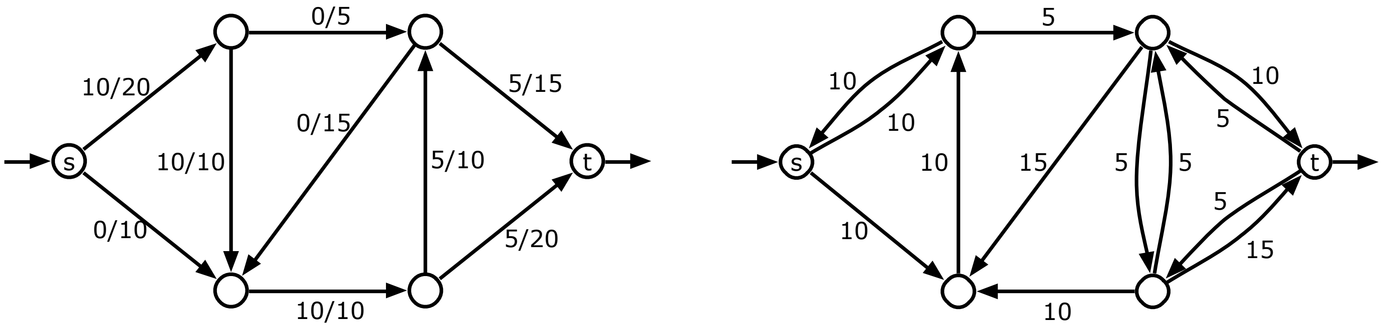
Residual Graph: $\mathbf{G_f = (V, E_f)}$ where $\mathbf{(u,v) \in E_f}$ if $\mathbf{c_f(u,v) > 0}$.



A flow f in a weighted graph G and the corresponding residual graph G_f.

Start of Proof

Let $\mathbf{f}$ be a max $(\mathbf{s}, \mathbf{t})$-flow with residual graph $\mathbf{G_f}$.
Want to Show: There is a cut $(\mathbf{S}, \bar{\mathbf{S}})$ with $\mathbf{cap}(\mathbf{S}, \bar{\mathbf{S}}) = |\mathbf{f}|$.

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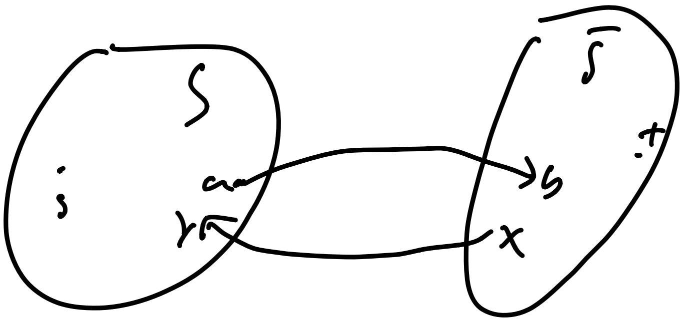

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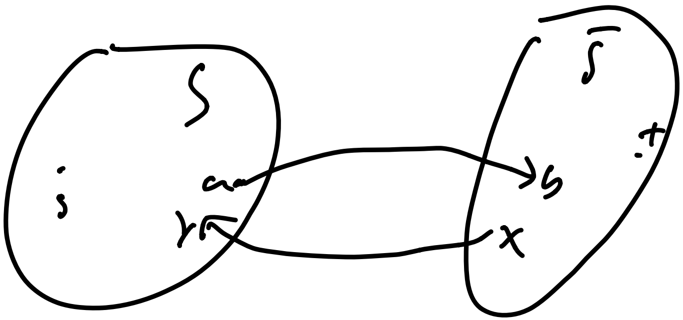

- $(\mathbf{S},\bar{\mathbf{S}})$ an $(\mathbf{s},\mathbf{t})$-cut. ✓

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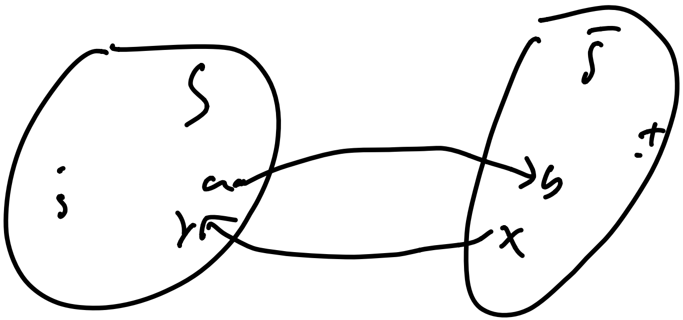

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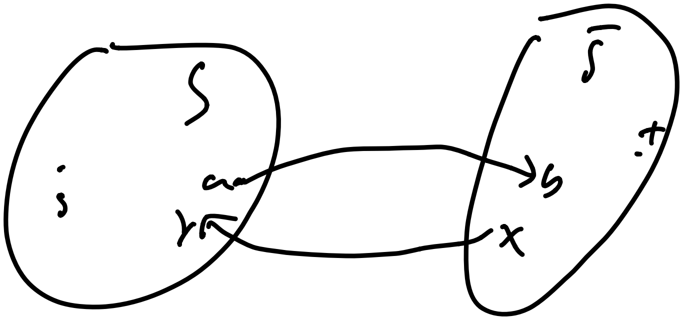

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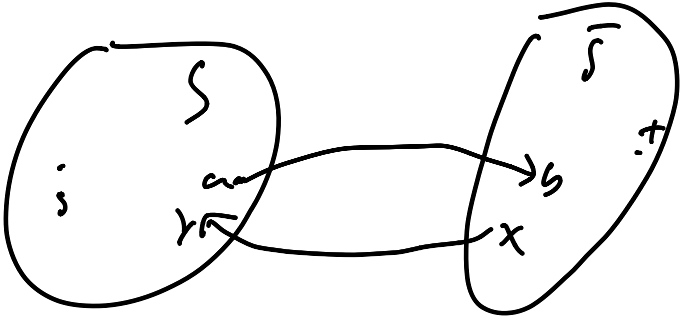

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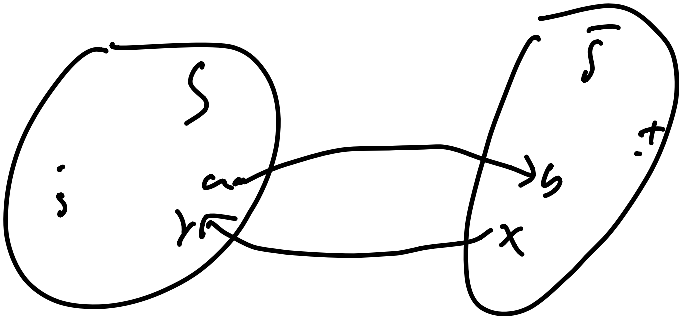

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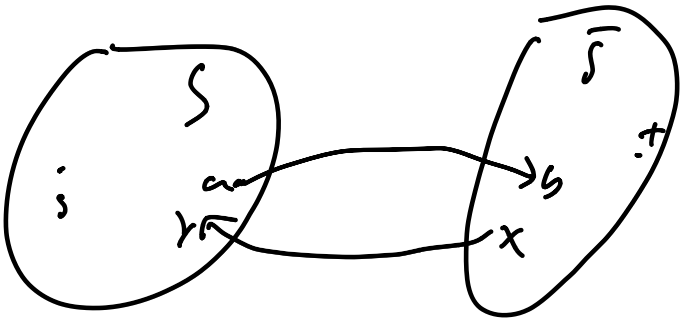

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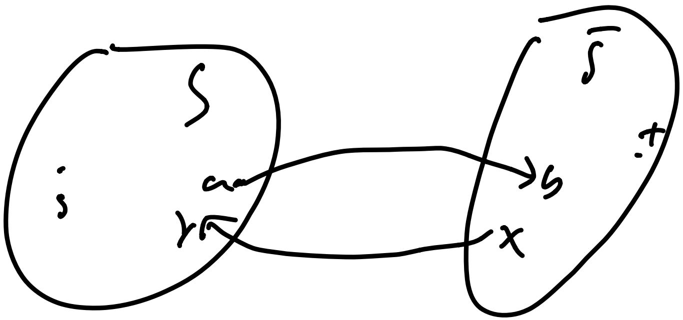

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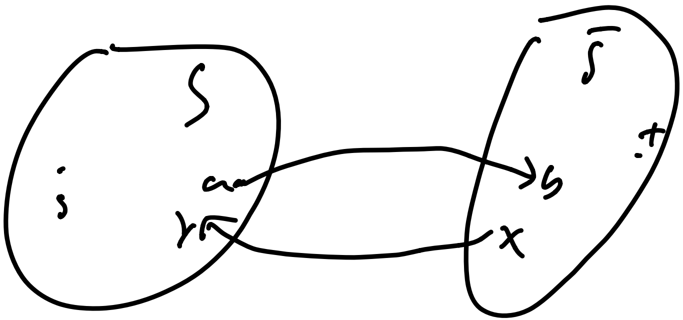

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$\mathbf{f}$ saturates $\mathbf{S} \to \bar{\mathbf{S}}$ edges, avoids $\bar{\mathbf{S}} \to \mathbf{S}$ edges $\implies$ $\mathbf{cap}(\mathbf{S},\bar{\mathbf{S}}) = |\mathbf{f}|$ by corollary

Case 2

Suppose $\exists$ an $\mathbf{s} \rightarrow \mathbf{t}$ path $\mathbf{P}$ in $\mathbf{G_f}$.

- Called an *augmenting path*

Idea: show that we can "push" more flow along $\mathbf{P}$, so $\mathbf{f}$ not a max flow. Contradiction, can't be in this case.

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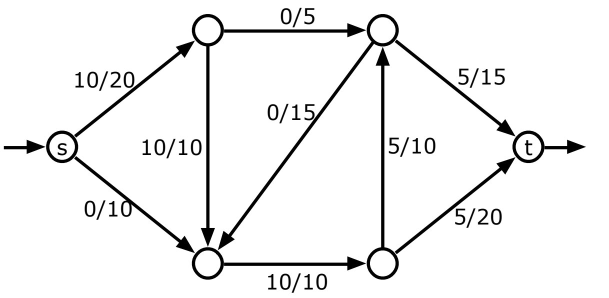
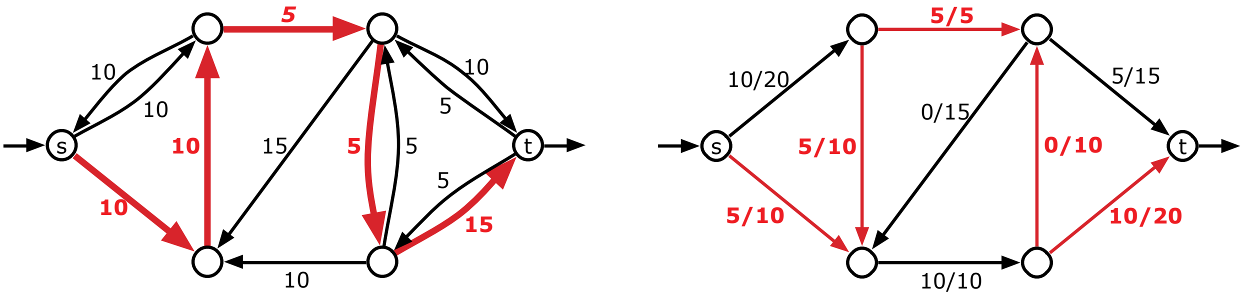
- Foreshadowing: augmenting path allows us to send more flow. Algorithm to increase flow!

Intuition



A flow f in a weighted graph G and the corresponding residual graph G_f.



An augmenting path in G_f with value $F = 5$ and the augmented flow f'.

Formalities

Let $\mathbf{P}$ be (simple) augmenting path in $\mathbf{G_f}$. Let $\mathbf{F = \min_{e \in P} c_f(e)}$.

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$$\mathbf{f'(u,v)} = \begin{cases} \mathbf{f(u,v) + F} & \text{if } \mathbf{(u,v)} \text{ in } \mathbf{P} \\ \mathbf{f(u,v) - F} & \text{if } \mathbf{(v,u)} \text{ in } \mathbf{P} \\ \mathbf{f(u,v)} & \text{otherwise} \end{cases}$$

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Plan: prove (sketch) each subclaim individually

- $|\mathbf{f'}| > |\mathbf{f}|$
- $\mathbf{f'}$ an $\mathbf{(s,t)}$-flow (flow conservation)
- $\mathbf{f'}$ feasible (obeys capacities)

$|f'| > |f|$

Consider first edge of $\mathbf{P}$ (out of $\mathbf{s}$), say $(\mathbf{s}, \mathbf{v_1})$

- If $(\mathbf{s}, \mathbf{v_1}) \in \mathbf{E}$, then $\mathbf{f}'(\mathbf{s}, \mathbf{v_1}) = \mathbf{f}(\mathbf{s}, \mathbf{v_1}) + \mathbf{F}$
- If $(\mathbf{v_1}, \mathbf{s}) \in \mathbf{E}$ then $\mathbf{f}'(\mathbf{v_1}, \mathbf{s}) = \mathbf{f}(\mathbf{v_1}, \mathbf{s}) - \mathbf{F}$

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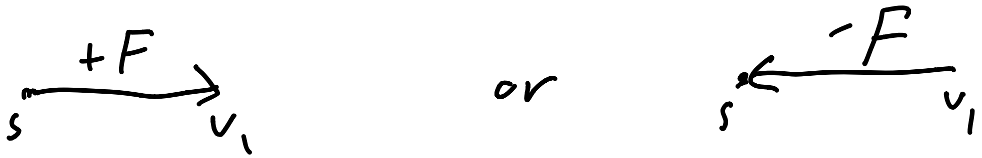

$$|\mathbf{f'}| = \sum_{\mathbf{u}} \mathbf{f'}(\mathbf{s}, \mathbf{u}) - \sum_{\mathbf{u}} \mathbf{f'}(\mathbf{u}, \mathbf{s}) = |\mathbf{f}| + \mathbf{F} > |\mathbf{f}|$$

f' obeys flow conservation

Consider some $u \in V \setminus \{s, t\}$.

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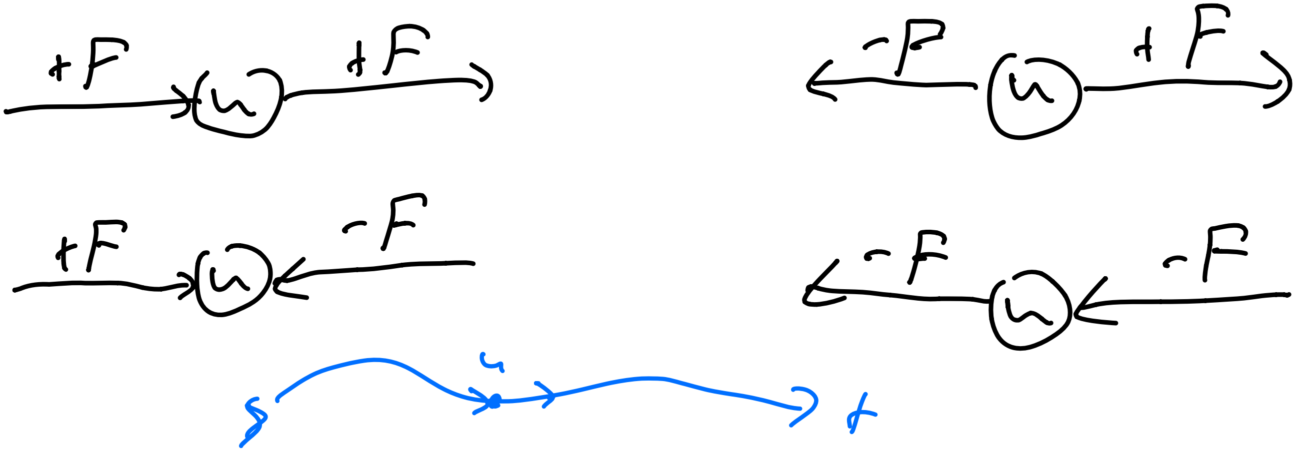
Consider some $\mathbf{u} \in \mathbf{V} \setminus \{\mathbf{s}, \mathbf{t}\}$.

- If $\mathbf{u} \notin \mathbf{P}$, no change in flow at $\mathbf{u} \implies$ still balanced.
- If $\mathbf{u} \in \mathbf{P}$, four possibilities:



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$$\begin{aligned} \mathbf{f'}(\mathbf{u}, \mathbf{v}) &= \mathbf{f}(\mathbf{u}, \mathbf{v}) + \mathbf{F} \\ &\leq \mathbf{f}(\mathbf{u}, \mathbf{v}) + \mathbf{c_f}(\mathbf{u}, \mathbf{v}) \\ &= \mathbf{f}(\mathbf{u}, \mathbf{v}) + \mathbf{c}(\mathbf{u}, \mathbf{v}) - \mathbf{f}(\mathbf{u}, \mathbf{v}) \\ &= \mathbf{c}(\mathbf{u}, \mathbf{v}) \end{aligned}$$

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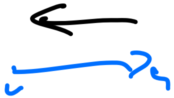

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Ford-Fulkerson Algorithm and Integrality

FF Algorithm

Obvious algorithm from previous proof: keep pushing flow!

$\mathbf{f} = \vec{\mathbf{0}}$
while($\exists \mathbf{s} \to \mathbf{t}$ path $\mathbf{P}$ in $\mathbf{G_f}$) {
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return $\mathbf{f}$

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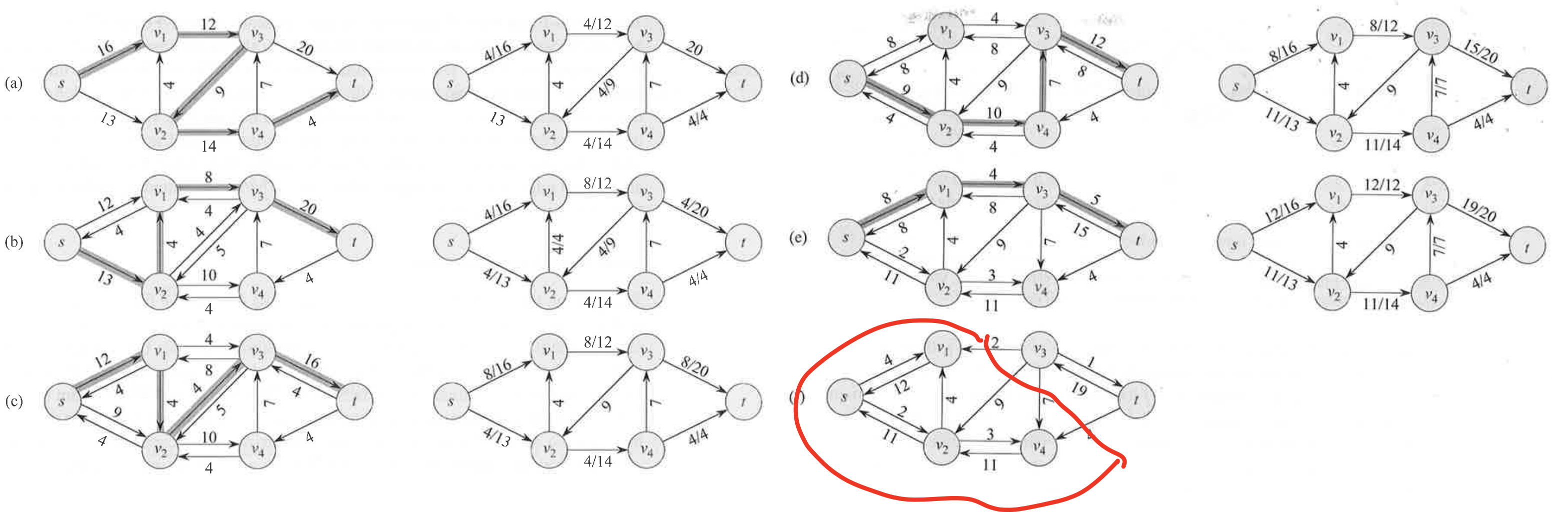
Correctness: directly from previous proof

Example



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If all capacities are integers, then there is a max flow such that the flow through every edge is an integer

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Proof.

Induction on iterations of the Ford-Fulkerson algorithm: initially true, stays true $\implies$ true at end. $\square$

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Theorem

If all capacities are integers and the max flow value is $\mathbf{F}$*, Ford-Fulkerson takes time at most* $\mathbf{O(F(m+n))}$

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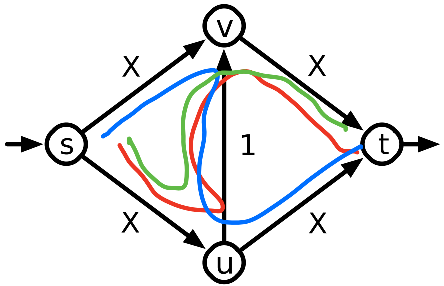

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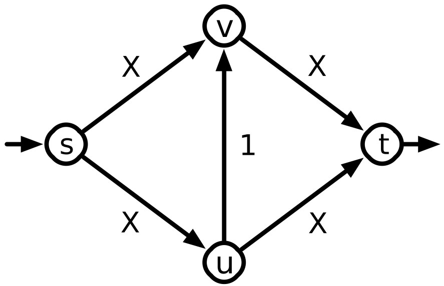

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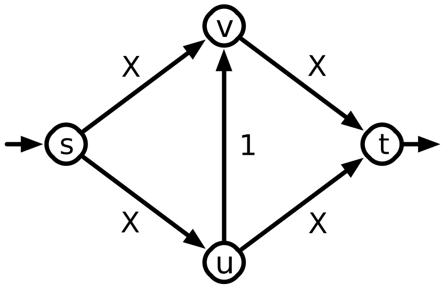

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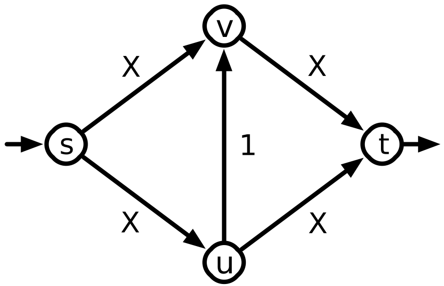

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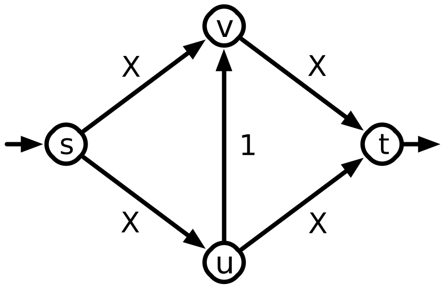

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$\Longrightarrow$ Exponential time!