Introduction

Last time: somewhat greedy algorithm (Prim’s), extremely greedy algorithm (Kruskal’s)
Question: when does greedy algorithm return optimal solution?

Weighted Set System:
- Universe $U$
- Collection $I \subseteq 2^U$ (so $I \subseteq U$ for all $I \in I$). Called independent sets
- Weights $w : U \to \mathbb{R}^+$

Problem: find max weight independent set
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For any tree $T$:

$$w'(T) = \sum_{e \in T} w'(e) = \sum_{e \in T} (\bar{w} - w(e)) = (n - 1)\bar{w} - \sum_{e \in T} w(e) = (n - 1)\bar{w} - w(T)$$

So finding max-weight forest = finding min spanning tree.
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So under weights $w'$, max-weight IS = max-weight forest = max-weight tree = min-weight tree (weights $w$)
- So finding max-weight forest = finding min spanning tree.
Useful Properties of Forests

Let $U = E$ and $I = \{ F \subseteq E : (V, F) \text{ a forest} \}$

Useful properties:

1. $\emptyset \in I$

2. If $F \in I$ and $F' \subseteq F$, then $F' \in I$
Useful Properties of Forests

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1. \( \emptyset \in \mathcal{I} \)
2. If \( F \in \mathcal{I} \) and \( F' \subseteq F \), then \( F' \in \mathcal{I} \)
3. **Augmentation Property:** If \( F_1 \in \mathcal{I} \) and \( F_2 \in \mathcal{I} \) with \( |F_2| > |F_1| \), then there is some edge \( e \in F_2 \setminus F_1 \) such that \( F_1 \cup \{e\} \in \mathcal{I} \).
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**Proof Sketch that Forests have Augmentation Property.**

Suppose false: no edge in $F_2 \setminus F_1$ can be added to $F_1$. Let $c_1 = \#$ components in $F_1$, $c_2 = \#$ components in $F_2$. 

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Contradiction.
Matroids

**Definition**

\((U, \mathcal{I})\) is a *matroid* if the following three properties hold:

1. \(\emptyset \in \mathcal{I}\),
2. If \(F \in \mathcal{I}\) and \(F' \subseteq F\), then \(F' \in \mathcal{I}\), and
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\((U, \mathcal{I})\) is a *hereditary set system* if the first two properties hold.

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**Matroid theory:** Super interesting area of combinatorics! Surprising amount of structure.

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Warmup: In any matroid, the maximal independent sets (called bases) have the same size (called the rank of the matroid).
Examples of Matroids

- Forests in graphs
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- Linearly independent vectors in vector space
  - $U$ a finite set of vectors in $\mathbb{R}^d$
  - $I = \{ F \subseteq U : F \text{ linearly independent} \}$

Matroids: generalize both graph theory and linear algebra!

Originally invented by Whitney as an attempt to generalize the concept of “linear independence”
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  - \( \emptyset \) linearly independent
  - If \( F \) linearly independent and \( F' \subseteq F \), then \( F' \) linearly independent
  - Augmentation: if \( F_1 \) linearly independent, \( F_2 \) linearly independent, and \( |F_2| > |F_1| \) \( \implies \)
    \[ \dim(\text{span}(F_1)) = |F_1| < |F_2| = \dim(\text{span}(F_2)) \]
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We’ll assume we have independence oracle.
Greedy Algorithm

Kruskal, generalized to matroids (and max weight)!
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\[
\begin{align*}
F &= \emptyset \\
\text{Sort } U \text{ by weight (largest to smallest)} \\
\text{For each } u \in U \text{ in sorted order } \{ \\
&\quad \text{If } F \cup \{u\} \in \mathcal{I}, \text{ add } u \text{ to } F \\
\} \\
\text{Return } F
\end{align*}
\]
Correctness

**Theorem**

Let \( F \) be independent set returned by greedy. Then \( w(F) \geq w(F') \) for all \( F' \in \mathcal{I} \).

**Claim:** \( w(f_i) \geq w(e_i) \) for all \( i \).

**Proof:** Suppose false, let \( j \) smallest integer such that \( w(f_j) < w(e_j) \). Let \( F_1 = \{ f_1, \ldots, f_{j-1} \} \) and let \( F_2 = \{ e_1, \ldots, e_j \} \). 

Let \( F_2' \succ F_1' \), so by augmentation there is some \( e_z \in F_2 \setminus F_1 \) such that \( F_1 \cup \{ e_z \} \in \mathcal{I} \).

\( w(e_z) \geq w(e_j) > w(f_j) \)

Contradiction! Greedy would add \( e_z \) next, not \( f_j \).
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**Theorem**

Let $F$ be independent set returned by greedy. Then $w(F) \geq w(F')$ for all $F' \in \mathcal{I}$.

- $F = \{f_1, f_2, \ldots, f_r\}$, where $w(f_i) \geq w(f_{i+1})$ for all $i$ (order added by greedy)
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$F_2 > F_1$, so by augmentation there is some $e_z \in F_2 \setminus F_1$ such that $F_1 \cup \{e_z\} \in \mathcal{I}$. Then $w(e_z) \geq w(e_j) > w(f_j)$, contradiction! Greedy would add $e_z$ next, not $f_j$. 

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Converse

So greedy works on matroids. Amazing fact: if greedy works, set system is a matroid!
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**Theorem**

Let $(U, \mathcal{I})$ be an hereditary set system. If for every weighting $w: U \rightarrow \mathbb{R}_{\geq 0}$ the greedy algorithm returns a maximum weight independent set, then $(U, \mathcal{I})$ is a matroid.
So greedy works on matroids. Amazing fact: if greedy works, set system is a matroid!

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Let \((U, I)\) be an hereditary set system. If for every weighting \(w : U \to \mathbb{R}_{\geq 0}\) the greedy algorithm returns a maximum weight independent set, then \((U, I)\) is a matroid.

So for hereditary set systems, matroids exactly characterize when the greedy algorithm works!
Proof

Contradiction. Suppose false $\implies (U, \mathcal{I})$ hereditary but not matroid.
Proof

Contradiction. Suppose false $\iff (U, \mathcal{I})$ hereditary but not matroid.

$\iff \exists F_1, F_2 \in \mathcal{I}$ such that $|F_1| < |F_2|$ but $F_1 \cup \{e\} \notin \mathcal{I}$ for all $e \in F_2 \setminus F_1$
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Easy facts:

1. $|F_2 \setminus F_1| > |F_1 \setminus F_2|
2. $|F_2 \setminus F_1| \geq 1$
3. $|F_1 \setminus F_2| \geq 1$ (hereditary)
Proof

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$\implies \frac{1}{|F_1 \setminus F_2|} > \frac{1 + \epsilon}{|F_2 \setminus F_1|}$
Proof (cont’d)

Use fact that \( \frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|} \) to define weights.

Greedy:
- Adds all of \( F_1 \cap F_2 \)
- Adds all of \( F_1 \setminus F_2 \)
- Can’t add any of \( F_2 \setminus F_1 \)

\[
\begin{align*}
    w(\text{greedy}) &= 2|F_1 \cap F_2| + |F_1 \setminus F_2| \frac{1}{|F_2 \setminus F_1|} \\
    &= 2|F_1 \cap F_2| + 1
\end{align*}
\]
Proof (cont’d)

Use fact that \( \frac{1}{|F_1 \setminus F_2|} > \frac{1+\varepsilon}{|F_2 \setminus F_1|} \) to define weights.

\[
\begin{align*}
\text{Greedy:} & \quad \text{Adds all of } F_1 \cap F_2 \\
& \quad \text{Adds all of } F_1 \setminus F_2 \\
& \quad \text{Can’t add any of } F_2 \setminus F_1 \\
\end{align*}
\]

\[
\begin{align*}
w(\text{greedy}) &= 2|F_1 \cap F_2| + |F_1 \setminus F_2| \frac{1}{|F_1 \setminus F_2|} \\
&= 2|F_1 \cap F_2| + 1 \\
\end{align*}
\]

\[
\begin{align*}
w(F_2) &= 2|F_1 \cap F_2| + |F_2 \setminus F_1| \frac{1+\varepsilon}{|F_2 \setminus F_1|} \\
&= 2|F_1 \cap F_2| + 1 + \varepsilon \\
\end{align*}
\]
Proof (cont’d)

Use fact that \( \frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|} \) to define weights.

\[ w(x) = \frac{1}{|F_1 \setminus F_2|}, \quad w(x) = 2 \]

\[ w(greedy) = 2|F_1 \cap F_2| + |F_1 \setminus F_2| \cdot \frac{1}{|F_1 \setminus F_2|} = 2|F_1 \cap F_2| + 1 \]

\[ w(F_2) = 2|F_1 \cap F_2| + |F_2 \setminus F_1| \cdot \frac{1+\epsilon}{|F_2 \setminus F_1|} = 2|F_1 \cap F_2| + 1 + \epsilon \]

Greedy:  
- Adds all of \( F_1 \cap F_2 \)  
- Adds all of \( F_1 \setminus F_2 \)  
- Can’t add any of \( F_2 \setminus F_1 \)

Greedy not optimal: contradiction!