Lecture 17: Matroids and the Greedy Algorithm

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601.433/633 Introduction to Algorithms

Introduction

Last time: somewhat greedy algorithm (Prim's), extremely greedy algorithm (Kruskal's)

Question: when does greedy algorithm return optimal solution?

Want abstraction that includes MSTs, but also works for many other problems.

Weighted Set System:

- Universe $\mathbf{U}$
- Collection $\mathcal{I} \subseteq \mathbf{2^U}$ (so $\mathbf{I} \subseteq \mathbf{U}$ for all $\mathbf{I} \in \mathcal{I}$). Called *independent sets*
- Weights $\mathbf{w} : \mathbf{U} \to \mathbb{R}^+$

Problem: find *max weight* independent set

MST as Weighted Set System

MST: weighted graph $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{w})$. Find MST.

Set system:

- $\mathbf{U} = \mathbf{E}$
- $\mathcal{I} = \{\mathbf{F} \subseteq \mathbf{E} : (\mathbf{V}, \mathbf{F}) \text{ a forest}\}$

What about weights? MST is minimize, but problem we defined is maximize.

- Let $\bar{\mathbf{w}} > \mathbf{w}(\mathbf{e})$ for all $\mathbf{e} \in \mathbf{E}$, let $\mathbf{w}'(\mathbf{e}) = \bar{\mathbf{w}} - \mathbf{w}(\mathbf{e})$ for all $\mathbf{e} \in \mathbf{E}$

For any tree $\mathbf{T}$:

$$\mathbf{w}'(\mathbf{T}) = \sum_{\mathbf{e} \in \mathbf{T}} \mathbf{w}'(\mathbf{e}) = \sum_{\mathbf{e} \in \mathbf{T}} (\bar{\mathbf{w}} - \mathbf{w}(\mathbf{e})) = (\mathbf{n} - 1)\bar{\mathbf{w}} - \sum_{\mathbf{e} \in \mathbf{T}} \mathbf{w}(\mathbf{e})$$

So under weights $\mathbf{w}'$, max-weight IS $=$ max-weight forest $=$ max-weight tree $=$ min-weight tree (weights $\mathbf{w}$)

- So finding max-weight forest $=$ finding min spanning tree.

Useful Properties of Forests

Let $\mathbf{U} = \mathbf{E}$ and $\mathcal{I} = \{\mathbf{F} \subseteq \mathbf{E} : (\mathbf{V}, \mathbf{F}) \text{ a forest}\}$

Useful properties:

1. $\emptyset \in \mathcal{I}$
2. If $\mathbf{F} \in \mathcal{I}$ and $\mathbf{F}' \subseteq \mathbf{F}$, then $\mathbf{F}' \in \mathcal{I}$
3. *Augmentation Property*: If $\mathbf{F_1} \in \mathcal{I}$ and $\mathbf{F_2} \in \mathcal{I}$ with $|\mathbf{F_2}| > |\mathbf{F_1}|$, then there is some edge $\mathbf{e} \in \mathbf{F_2} \setminus \mathbf{F_1}$ such that $\mathbf{F_1} \cup \{\mathbf{e}\} \in \mathcal{I}$.

Proof Sketch that Forests have Augmentation Property.

Suppose false: no edge in $\mathbf{F_2} \setminus \mathbf{F_1}$ can be added to $\mathbf{F_1}$. Let $\mathbf{c_1} = \#$ components in $\mathbf{F_1}$, $\mathbf{c_2} = \#$ components in $\mathbf{F_2}$

$\implies$ every edge of $\mathbf{F_2}$ has both endpoints in same component of $\mathbf{F_1}$

$\implies$ every component of $\mathbf{F_2}$ contained in component of $\mathbf{F_1}$ $\implies$ $\mathbf{c_2} \geq \mathbf{c_1}$

But $\mathbf{c_2} = \mathbf{n} - |\mathbf{F_2}| < \mathbf{n} - |\mathbf{F_1}| = \mathbf{c_1}$.

Contradiction. □

Matroids

Definition

$(U, \mathcal{I})$ is a *matroid* if the following three properties hold:

1. $\emptyset \in \mathcal{I}$,
2. If $F \in \mathcal{I}$ and $F' \subseteq F$, then $F' \in \mathcal{I}$, and
3. If $F_1 \in \mathcal{I}$ and $F_2 \in \mathcal{I}$ with $|F_2| > |F_1|$, then there is some element $e \in F_2 \setminus F_1$ such that $F_1 \cup \{e\} \in \mathcal{I}$.

$(U, \mathcal{I})$ is a *hereditary set system* if the first two properties hold.

Matroid theory: super interesting area of combinatorics! Surprising amount of structure.

Warmup: In any matroid, the maximal independent sets (called bases) have the same size (called the rank of the matroid).

Examples of Matroids

- Forests in graphs
- Linearly independent vectors in vector space
  - $\mathbf{U}$ a finite set of vectors in $\mathbb{R}^{\mathbf{d}}$
  - $\mathcal{I} = \{\mathbf{F} \subseteq \mathbf{U} : \mathbf{F} \text{ linearly independent}\}$
  - $\emptyset$ linearly independent
  - If $\mathbf{F}$ linearly independent and $\mathbf{F}' \subseteq \mathbf{F}$, then $\mathbf{F}'$ linearly independent
  - Augmentation: if $\mathbf{F_1}$ linearly independent, $\mathbf{F_2}$ linearly independent, and $|\mathbf{F_2}| > |\mathbf{F_1}| \implies \dim(\text{span}(\mathbf{F_1})) = |\mathbf{F_1}| < |\mathbf{F_2}| = \dim(\text{span}(\mathbf{F_2}))$

Matroids: generalize both graph theory and linear algebra!

- Originally invented by Whitney as an attempt to generalize the concept of “linear independence”

Representation

To do algorithms with matroids, need to figure out how they're represented.

Option 1: list all independent sets

- Too many of them!

What did we need for MST (Kruskal)?

Independence Oracle: algorithm which take $\mathbf{F} \subseteq \mathbf{U}$, returns YES if $\mathbf{F} \in \mathcal{I}$, NO if $\mathbf{F} \notin \mathcal{I}$

For MST: "does **F** have any cycles"? Independence oracle: DFS/BFS

We'll assume we have independence oracle.

Greedy Algorithm

Kruskal, generalized to matroids (and max weight)!

$\mathbf{F} = \emptyset$
Sort $\mathbf{U}$ by weight (largest to smallest)
For each $\mathbf{u} \in \mathbf{U}$ in sorted order $\{$
    If $\mathbf{F} \cup \{\mathbf{u}\} \in \mathcal{I}$, add $\mathbf{u}$ to $\mathbf{F}$
$\}$
Return $\mathbf{F}$

Correctness

Theorem

Let $\mathbf{F}$ *be independent set returned by greedy. Then* $\mathbf{w(F) \geq w(F')}$ *for all* $\mathbf{F'} \in \mathcal{I}$.

- $\mathbf{F} = \{\mathbf{f_1, f_2, \ldots, f_r}\}$, where $\mathbf{w(f_i) \geq w(f_{i+1})}$ for all $\mathbf{i}$ (order added by greedy)
- $\mathbf{F'} = \{\mathbf{e_1, e_2, \ldots, e_r}\}$ where $\mathbf{w(e_i) \geq w(e_{i+1})}$ for all $\mathbf{i}$

Claim: $\mathbf{w(f_i) \geq w(e_i)}$ for all $\mathbf{i}$.

Proof: Suppose false, let $\mathbf{j}$ smallest integer such that $\mathbf{w(f_j) < w(e_j)}$.

Let $\mathbf{F_1} = \{\mathbf{f_1, \ldots, f_{j-1}}\}$ and let $\mathbf{F_2} = \{\mathbf{e_1, \ldots, e_j}\}$

$|\mathbf{F_2}| > |\mathbf{F_1}|$, so by augmentation there is some $\mathbf{e_z} \in \mathbf{F_2} \setminus \mathbf{F_1}$ such that $\mathbf{F_1} \cup \{\mathbf{e_z}\} \in \mathcal{I}$.

$$\mathbf{w(e_z) \geq w(e_j) > w(f_j)}$$

Contradiction! Greedy would add $\mathbf{e_z}$ next, not $\mathbf{f_j}$.

Converse

So greedy works on matroids. Amazing fact: if greedy works, set system is a matroid!

Theorem

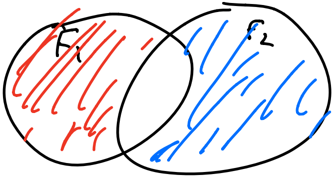
Let $(\mathbf{U}, \mathcal{I})$ *be an hereditary set system. If for every weighting* $\mathbf{w} : \mathbf{U} \to \mathbb{R}_{\geq 0}$ *the greedy algorithm returns a maximum weight independent set, then* $(\mathbf{U}, \mathcal{I})$ *is a matroid.*

So for hereditary set systems, matroids exactly characterize when the greedy algorithm works!

Proof

Contradiction. Suppose false $\implies$ $(\mathbf{U}, \mathcal{I})$ hereditary but not matroid.

$\implies \exists \mathbf{F_1}, \mathbf{F_2} \in \mathcal{I}$ such that $|\mathbf{F_1}| < |\mathbf{F_2}|$ but $\mathbf{F_1} \cup \{\mathbf{e}\} \notin \mathcal{I}$ for all $\mathbf{e} \in \mathbf{F_2} \setminus \mathbf{F_1}$



Easy facts:

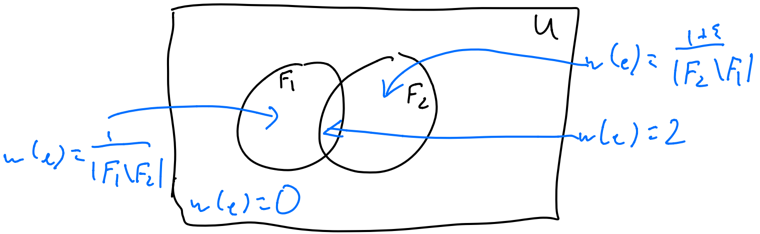
1. $|\mathbf{F_2} \setminus \mathbf{F_1}| > |\mathbf{F_1} \setminus \mathbf{F_2}|$
2. $|\mathbf{F_2} \setminus \mathbf{F_1}| \geq \mathbf{1}$
3. $|\mathbf{F_1} \setminus \mathbf{F_2}| \geq \mathbf{1}$ (hereditary)

$\implies \exists \epsilon > \mathbf{0}$ such that $\mathbf{0} < (\mathbf{1} + \epsilon)|\mathbf{F_1} \setminus \mathbf{F_2}| < |\mathbf{F_2} \setminus \mathbf{F_1}|$

$$\implies \frac{\mathbf{1}}{|\mathbf{F_1} \setminus \mathbf{F_2}|} > \frac{\mathbf{1} + \epsilon}{|\mathbf{F_2} \setminus \mathbf{F_1}|}$$

Proof (cont'd)

Use fact that $\frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$ to define weights.



Greedy:

- Adds all of $F_1 \cap F_2$
- Adds all of $F_1 \setminus F_2$
- Can't add any of $F_2 \setminus F_1$

$$\mathbf{w}(\text{greedy}) = 2|F_1 \cap F_2| + |F_1 \setminus F_2| \frac{1}{|F_1 \setminus F_2|} = 2|F_1 \cap F_2| + 1$$

$$\mathbf{w}(F_2) = 2|F_1 \cap F_2| + |F_2 \setminus F_1| \frac{1+\epsilon}{|F_2 \setminus F_1|} = 2|F_1 \cap F_2| + 1 + \epsilon$$

Greedy not optimal: contradiction!