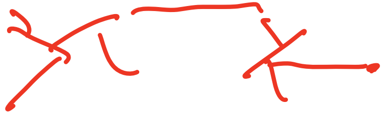
Lecture 16: Minimum Spanning Trees

Michael Dinitz

October 21, 2021
601.433/633 Introduction to Algorithms

Introduction



Definition

A *spanning tree* of an undirected graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ is a set of edges $\mathbf{T} \subseteq \mathbf{E}$ such that $(\mathbf{V}, \mathbf{T})$ is connected and acyclic.

Definition

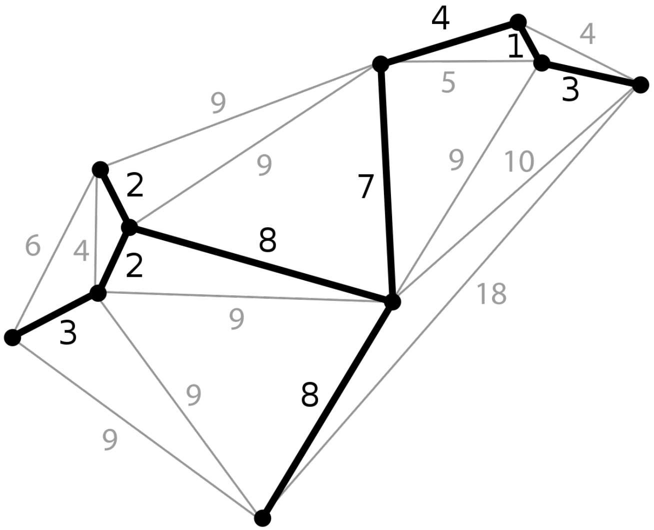
Minimum Spanning Tree problem (MST):

- Input:
  - Undirected graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$
  - Edge weights $\mathbf{w} : \mathbf{E} \to \mathbb{R}_{\geq 0}$
- Output: Spanning tree minimizing $\mathbf{w}(\mathbf{T}) = \sum_{\mathbf{e} \in \mathbf{T}} \mathbf{w}(\mathbf{e})$.

Foundational problem in *network design*. Tons of applications.

Today: one "recipe", two different algorithms from recipe. Main idea: greedy.

Examples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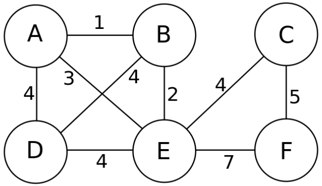
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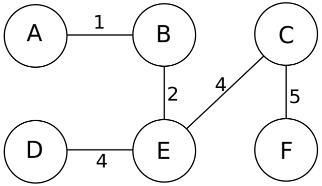
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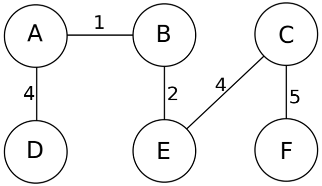
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Generic Algorithm

Generic Greedy

Definition

Suppose that $\mathbf{A}$ is subset of *some* MST. If $\mathbf{A} \cup \{\mathbf{e}\}$ is also a subset of some MST, then $\mathbf{e}$ is *safe* for $\mathbf{A}$.

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Generic-MST {
    A = ∅
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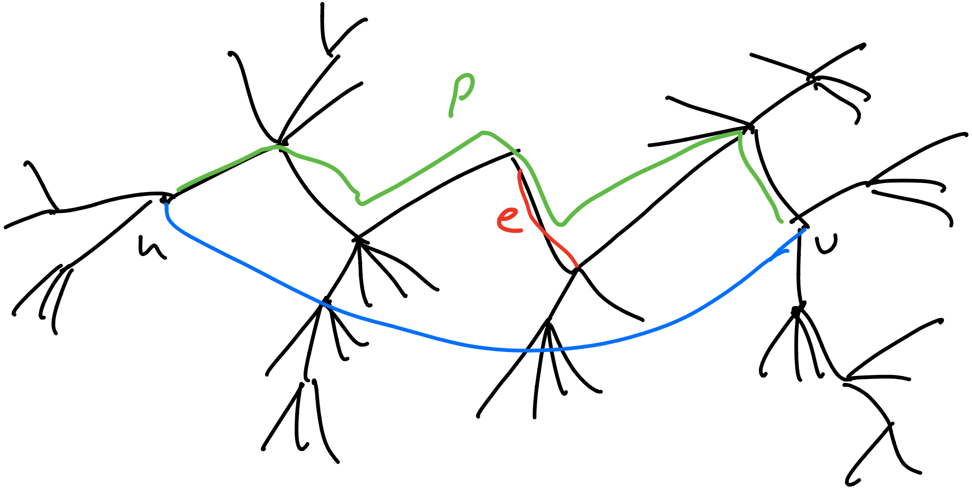
But how to find a safe edge? And which one to add?

Structural Properties

Lemma

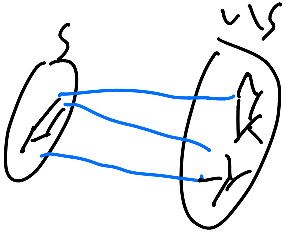
Let $\mathbf{T}$ *be a spanning tree, let* $\mathbf{u}, \mathbf{v} \in \mathbf{V}$, *and let* $\mathbf{P}$ *be the* $\mathbf{u} - \mathbf{v}$ *path in* $\mathbf{T}$. *If* $\{\mathbf{u}, \mathbf{v}\} \notin \mathbf{T}$, *then* $\mathbf{T}' = (\mathbf{T} \cup \{\{\mathbf{u}, \mathbf{v}\}\}) \setminus \{\mathbf{e}\}$ *is a spanning tree for all* $\mathbf{e} \in \mathbf{P}$.



Structural Properties

Definition

A *cut* $(S, V \setminus S)$ (or $(S, \bar{S})$ or just S) is a partition of V into two parts. Edge e *crosses* cut $(S, \bar{S})$ if e has one endpoint in S and one endpoint in $\bar{S}$.



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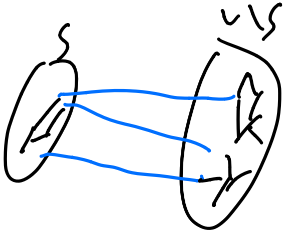
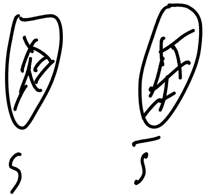
Cut $(\mathbf{S}, \bar{\mathbf{S}})$ *respects* $\mathbf{A} \subseteq \mathbf{E}$ if no edge in $\mathbf{A}$ crosses $(\mathbf{S}, \bar{\mathbf{S}})$





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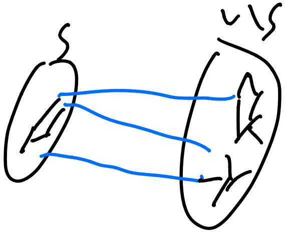
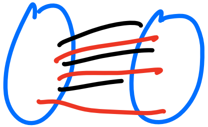
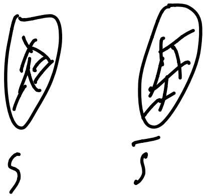






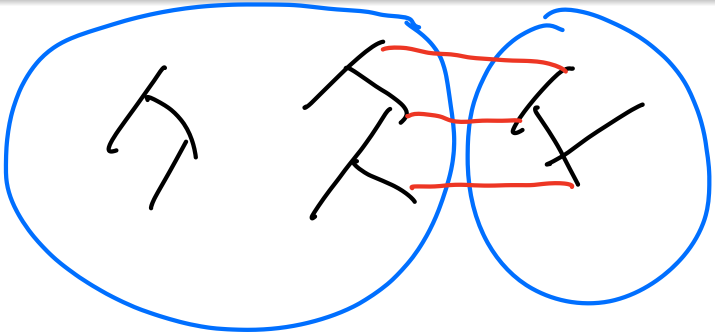
Definition

$\mathbf{e}$ is a *light edge* for $(\mathbf{S}, \bar{\mathbf{S}})$ if $\mathbf{e}$ crosses $(\mathbf{S}, \bar{\mathbf{S}})$ and $\mathbf{w}(\mathbf{e}) = \min_{\mathbf{e}' \text{ crossing } (\mathbf{S}, \bar{\mathbf{S}})} \mathbf{w}(\mathbf{e}')$

Main Structural Theorem

Theorem

Let $\mathbf{A} \subseteq \mathbf{E}$ *be a subset of some MST* $\mathbf{T}$, *let* $(\mathbf{S}, \bar{\mathbf{S}})$ *be a cut respecting* $\mathbf{A}$, *and let* $\mathbf{e} = \{\mathbf{u}, \mathbf{v}\}$ *be a light edge for* $(\mathbf{S}, \bar{\mathbf{S}})$. *Then* $\mathbf{e}$ *is safe for* $\mathbf{A}$.



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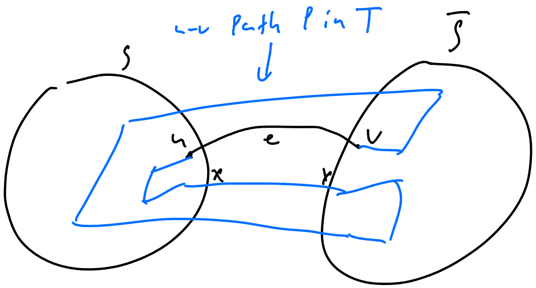
Let $\mathbf{T}' = (\mathbf{T} \cup \{\mathbf{e}\}) \setminus \{\{\mathbf{x}, \mathbf{y}\}\}$

$\implies$ $\mathbf{T}'$ a spanning tree by first lemma



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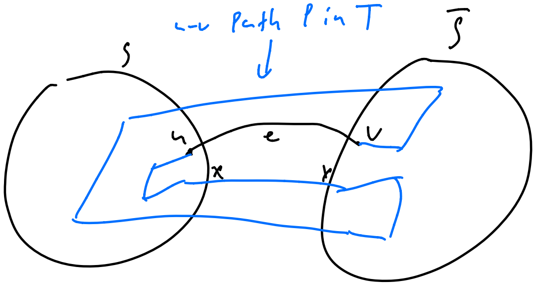

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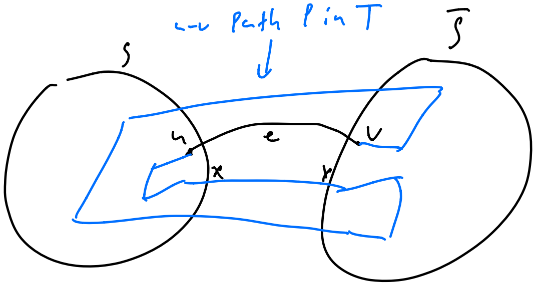
=



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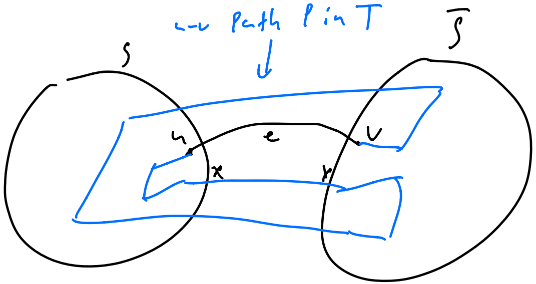

Prim's Algorithm

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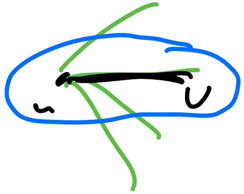

Idea: start at arbitrary node $\mathbf{u}$. Greedily grow MST out of $\mathbf{u}$.

$\mathbf{A} = \varnothing$
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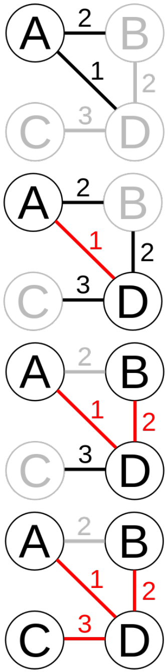

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Proof.

Just Generic-MST!

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Just Generic-MST!

- $(S, \bar{S})$ always respects A (induction).
- If edge e added then light for $(S, \bar{S})$
- Hence e safe for A by main structural theorem.

□

Running Time

Trivial analysis:

- Every spanning tree has $\mathbf{n-1}$ edges $\implies$ $\mathbf{n-1}$ iterations
- In each iteration, look through all edges to find min-weight edge crossing $(\mathbf{S}, \bar{\mathbf{S}})$ $\implies$ $\mathbf{O(m)}$ time
- Total $\mathbf{O(mn)}$

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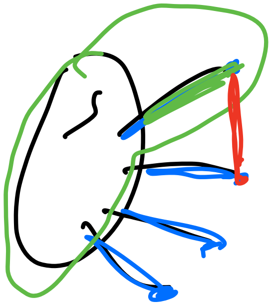

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- Every spanning tree has $n-1$ edges $\Longrightarrow$ $n-1$ iterations
- In each iteration, look through all edges to find min-weight edge crossing $(S, \bar{S})$ $\Longrightarrow$ $O(m)$ time
- Total $O(mn)$

Like Dijkstra's algorithm, do better by using a data structure: heap!

- Need to be able to get minimum-weight edge across $(S, \bar{S})$

Heap of *vertices* in $\bar{S}$. Key of v is min-weight edge from v to S.

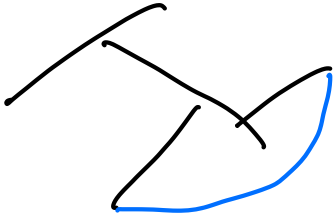
- When new vertex y added to S, need to update keys of nodes adjacent to y
  - Happens at most m times total
- n Inserts, n Extract-Mins, m Decrease-Keys
- Like Dijkstra, $O(m \log n)$ using binary heap. $O(m + n \log n)$ with Fibonacci heap (only Extract-Min is logarithmic)

Kruskal's Algorithm

Algorithm

Intuition: can we be *even greedier* than Prim's Algorithm?

Algorithm



Intuition: can we be *even greedier* than Prim's Algorithm?

$\mathbf{A} = \varnothing$
Sort edges by weight (small to large)
For each edge $\mathbf{e}$ in this order $\{$
    if $\mathbf{A} \cup \{\mathbf{e}\}$ has no cycles, $\mathbf{A} = \mathbf{A} \cup \{\mathbf{e}\}$
$\}$
return $\mathbf{A}$

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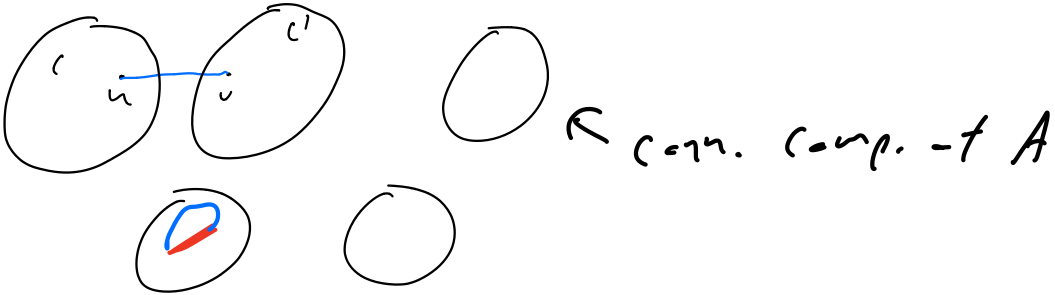

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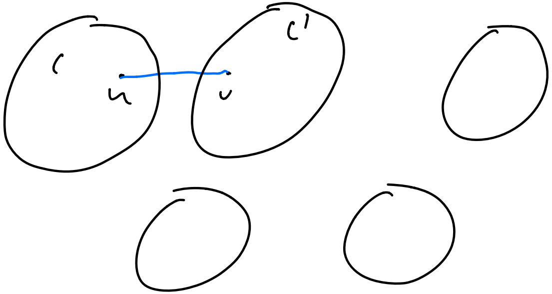

Consider cut $(\mathbf{C}, \bar{\mathbf{C}})$. Respects $\mathbf{A}$, and $\{\mathbf{u}, \mathbf{v}\}$ light for it.

Correctness

Theorem

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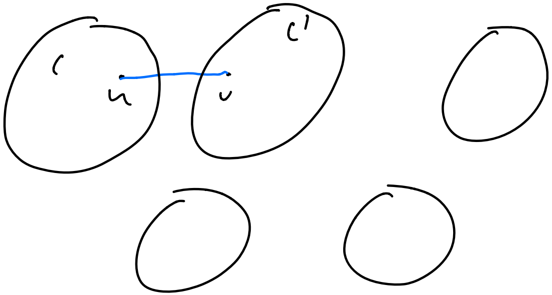

Consider cut $(\mathbf{C},\bar{\mathbf{C}})$. Respects $\mathbf{A}$, and $\{\mathbf{u},\mathbf{v}\}$ light for it.
Main structural theorem $\implies$ $\{\mathbf{u},\mathbf{v}\}$ safe for $\mathbf{A}$

Running Time

Sorting edges: $\mathbf{O(m \log m) = O(m \log n)}$

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- Finds: $2m$
- Unions:

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Union-Find! Connected components of A are disjoint sets.

- Make-Sets: n
- Finds: $2m$
- Unions: $n - 1$

$O(m \log^* n)$ using union-by-rank + path compression

$O(m + n \log n)$ using list data structure

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$\mathbf{O(m + n \log n)}$ using list data structure

Sorting dominates! $\mathbf{O(m \log n)}$ total.